Triangulated Categories in Algebraic Geometry

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Outline

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Issues in Algebraic Geometry

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- This process is functorial.

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- $\underline{AbCat}$ is the (lax 2-)category of abelian categories with exact functors.
- Any abelian category $\underline{A}$ admits an exact full embedding in $\underline{\mathrm{Ab}}$.

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Let A, B and C be complexes and suppose $0 \to A \to B \to C \to 0$ is exact (we say short exact) then there is a canonical map $H^i(C) \to H^{i+1}(A)$ such that the resulting sequence

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to H^{i+1}(B) \to \cdots$$

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- Well developed theory of dimension.
- Schemes come with structure sheaves $\mathcal{O}_X$. Can consider $\mathcal{O}_X-\mathrm{mod}$. Restrict to finitely presented modules: $\mathsf{Coh}(X)$.
- $\mathsf{Coh}(X)$ is abelian.
- There are cohomology functors $\mathsf{Coh}(X) \to \mathrm{Ab}^{\mathbb{N}}$ with $H^0 = \Gamma$, the global sections functor.

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Let X be a 2 dimensional (projective) variety. A curve on X can be viewed as the zero set of an algebraic map $s : \mathcal{O}_X \to L$, where L is a suitable (locally-free) rank 1 $\mathcal{O}_X$-module.

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Estimating the size of $H^0(K)$ is then useful to determine incidence properties.

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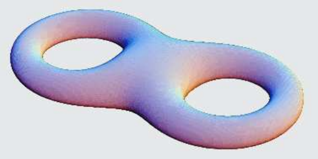
Dimension 1 projective varieties can be classified into type according to a non-negative integer called its genus. Loosely, the genus corresponds to the number of holes in the space:



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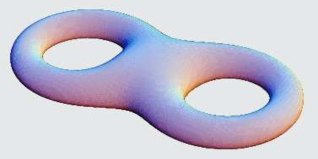

More generally, we can find a range of numerical invariants and, more generally, cohomology classes (characteristic classes) which allow us to crudely classify both schemes and sheaves on schemes.

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- But $\mathcal{M}_{X,c}$ is not representable.
- Partly fix by tweaking the moduli functor (to make $\mathcal{M}$ into a sheaf in a suitable subcanonical topology on $\underline{\mathrm{Sch}}$).

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- This still doesn't exist for $\mathcal{M}_{X,c}$ in general.

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- eg for torsion-free coherent sheaves on a curve, we say that E is stable if $\frac{d(F)}{r(F)} < \frac{d(E)}{r(E)}$ for all proper subsheaves F.

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- Now, stability conditions themselves have other applications (see later)

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- Provides a better setting for generalities about geometric objects related to schemes and is now widely used.
- There are still many technical difficulties (eg need for algebraic stacks).

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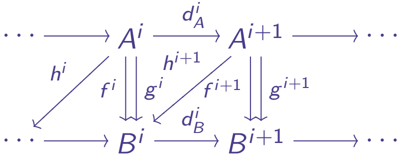
- $\underline{D(A)}$ is additive but is not generally abelian.
- There is a fully faithful additive functor $\underline{A} \to \underline{D(A)}$ given by mapping and object a to the complex $\cdots \to 0 \to a \to 0 \to \cdots$ centred on 0.

- The localization functor factors through $K(A)$, the category of maps up to homotopy.



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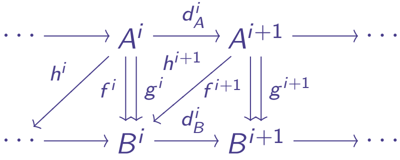

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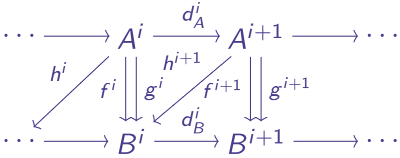

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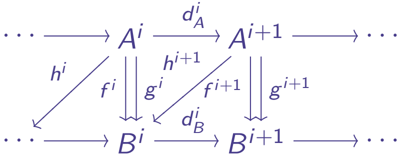

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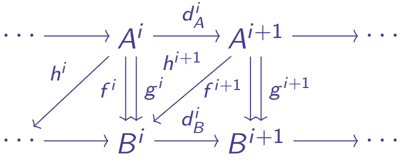

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- We can find a subcategory $\underline{I}$ of $\underline{A}$ such that $\underline{K(I)} \cong \underline{D(A)}$.
- Useful to construct (derived) functors on $\underline{D(A)}$ and to explicitly compute their cohomology.

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- Giving $0 \longrightarrow \ker f \longrightarrow a \xrightarrow{f} b \longrightarrow \operatorname{coker} f \longrightarrow 0$.

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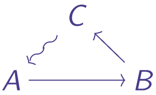
repeating with shifts in both directions. We call such diagrams triangles:



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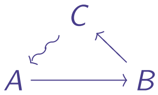

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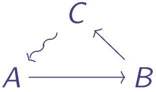

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- Observe that our special triangles have the property that if we are given maps $A \to A'$ and $B \to B'$ commuting with f and f' then we have a map $C \to C'$ which gives a map of triangles.

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  5. any triangle isomorphic to a triangle in Δ is in Δ.

(6) Given two triangles in Δ with a common vertex

$$\begin{array}{ccccccc} & & a & & & & \\ & & \downarrow & & & & \\ b & \longrightarrow & c & \longrightarrow & d & \longrightarrow & b[1] \\ & & \downarrow & & & & \\ & & e & & & & \\ & & \downarrow & & & & \\ & & a[1] & & & & \end{array}$$

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$$
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\downarrow & & \downarrow & & \| & & \downarrow \\
b & \longrightarrow & c & \longrightarrow & d & \longrightarrow & b[1] \\
\downarrow & & \downarrow & & & & \\
e & = & e & & & & \\
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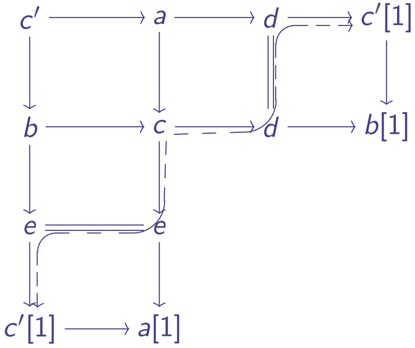

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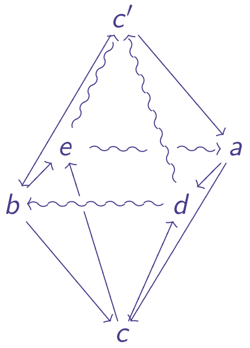

we can complete it to a commuting diagram with all rows and columns in Δ, for which the composites $c \to e \to c'[1]$ and $c \to d \to c'[1]$ agree. Called the octahedral axiom: "the bottom of such an octahedron can be completed to an octahedron".

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- Axiom (6) relates to the image/coimage of a map:

$$
\begin{array}{ccccccc}
a & \longrightarrow & \operatorname{Im} f & \longrightarrow & \ker f[1] & \longrightarrow & a[1] \\
\| & & \downarrow & & \downarrow & & \| \\
a & \xrightarrow{f} & b & \longrightarrow & c & \longrightarrow & a[1] \\
 & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} f & = & \operatorname{coker} f & &
\end{array}
$$

for a map $f : a \longrightarrow b$ of $\underline{A} \subset \underline{D(A)}$.

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- Then $\underline{A} = \underline{D}^{\leq 0} \cap \underline{D}^{\geq 0}$.

More generally:

- Given an exact subcategory $\underline{D} \subset \underline{T}$ such that $\underline{D}[1] \subset \underline{D}$ and for each object a of $\underline{T}$ there is a distinguished triangle $a' \to a \to a''$ with a' in $\underline{D}$ and a'' in $\underline{D}^{\perp}$. We call this a t-structure on $\underline{T}$.

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- For example, $\times 2 : \mathbb{Z} \to \mathbb{Z}$ injects in $\underline{\mathrm{Ab}}$ but there is a core in $\underline{D(\mathrm{Ab})}$ for which is does not inject but surjects with kernel $\mathbb{Z}_2[-1]$.

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- We then have a continuously varying family of abelian subcategories of $\underline{T}$.
- For example, on a curve we can choose $Z(a) = -d(a) + ir(a)$ and then the standard t-structure is $P((0,1])$.

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