Categorical Proof Theory of Modal Logics

Dirk Pattinson, Imperial College London

(joint work with Ana Sokolova, Salzburg)

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A Wee Bit of Motivation

Idea. Systems are *spaces* together with *dynamics*, aka T*-coalgebras*

$$(C, \gamma : C \to TC)$$

where $\mathbb{C}$ is a category of *spaces* and $T : \mathbb{C} \to \mathbb{C}$ determines dynamics.

Example.

- $\mathbb{C} = \mathsf{Meas}$ (measurable spaces), $T =$ probability distributions
- $\mathbb{C} = \mathsf{Fuzz}$ (fuzzy sets), $T =$ 'fuzzy powerset'
- $\mathbb{C} = \mathsf{PPos}$ (posets with p-morphisms), $T =$ 'powerset with E/M'
- $\mathbb{C} = \mathsf{Set}$, $T =$ 'subsets with labels'

Goal. *language* and *proofs* over T-coalgebras

Logics for coaglebras: beg, steal or borrow?

General Idea.

- assume that $\mathbb{C}$ comes with a notion of logic
- extend the logic over $\mathbb{C}$ to $\mathsf{Coalg}(T)$

Standard Example. Propositional logic over $\mathbb{C} = \mathsf{Set}$

$$\begin{tikzcd} \mathsf{Set}^{\mathrm{op}} \arrow[r, bend right, "2"'] & \mathsf{BA} \arrow[l, bend right, "\mathsf{Uf}"'] \end{tikzcd}$$

where BA is the category of boolean algebras.

Logics via Lifting (where $T : \mathsf{Set} \to \mathsf{Set}$)

$$\begin{tikzcd} \mathsf{Coalg}(T)^{\mathrm{op}} \arrow[r, bend right] & \mathsf{Alg}(L) \arrow[l, bend right] \end{tikzcd}$$

for $L : \mathsf{BA} \to \mathsf{BA}$

Logics via Lifting

Languages for T-coalgebras: (L, δ) where L : BA $\rightarrow$ BA and $\delta : L2 \rightarrow 2T$

Interpretation via **algebraisation**

$$\frac{\gamma : C \rightarrow TC}{2(\gamma) \circ \delta_C : L2C \rightarrow 2TC \rightarrow 2C}$$

turning a T-coalgebra into an L-algebra.

Interpretation via **initiality**

$$\begin{tikzcd} LF \arrow[r] \arrow[dd] & L2C \arrow[d, "\delta_C"] \\ & 2TC \arrow[d, "2(\gamma)=\gamma^{-1}"] \\ F \arrow[r] & 2C \end{tikzcd}$$

where F is the (carrier of) the initial L-algebra $\approx$ formulas.

General Gist

Example.

$$\begin{tikzcd} \mathsf{Set}^{\mathrm{op}} \arrow[r, bend right, "2"'] & \mathsf{BA} \arrow[l, bend right, "\mathsf{Uf}"'] \end{tikzcd}$$

- $\mathsf{BA} =$ formulas modulo (provable) equations
- $L =$ dynamics modulo (provable) equations

Logics via Representation: Assume that $\mathsf{Alg}(L) \cong \mathsf{Alg}(\Sigma, E)$

- equational logic sound , complete if δ injective, expressive if δ surjective

when interpreted over $\mathsf{Coalg}(T)$. (Kurz, Kupke, Jacobs, Sokolova)

Remark. Equational Logic 'hard-coded' into categrical setup

- makes it hard to show decidability, and not the only way of doing proofs!

Categorical Setup

Dual Adjunction mediates between ’spaces’ and ’logics’

$$
\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[r, bend right, "P"'] & \mathbb{A} \arrow[l, bend right, "S"']
\end{tikzcd}
$$

where $\mathbb{C}$: category of ‘spaces’ and $\mathbb{A}$: category of algebras and $S \dashv P$.

- think of $\mathbb{A}$ as coming with a natural notion of ‘logic’.
- will assume that $\mathbb{A} \subseteq \mathsf{Alg}(\Sigma)$ later

‘Logical’ Adjunctions

Examples.

$$\begin{tikzcd} \mathsf{Set}^{\mathrm{op}} \arrow[r, bend right, "2"'] & \mathsf{BA} \arrow[l, bend right, "\mathsf{Uf}"'] \end{tikzcd}$$

with 2: contravariant powerset and Uf: ultrafilters

$$\begin{tikzcd} \mathsf{PPos}^{\mathrm{op}} \arrow[r, bend right, "\mathsf{Pf}"'] & \mathsf{HA} \arrow[l, bend right, "\mathsf{Up}"'] \end{tikzcd}$$

with Pf: prime filters and Up: upsets

$$\begin{tikzcd} \mathsf{Meas}^{\mathrm{op}} \arrow[r, bend right, "F"'] & \mathsf{MSL} \arrow[l, bend right, "\Sigma"'] \end{tikzcd}$$

with F: filters and Σ : σ-algebras

Different Styles of Proof

Equational Logic over meet-semilattices:

$$a \wedge b = b \wedge a \qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

Sequent Calculi over Heyting algebras:

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

Calculus of Structures over Boolean algebras:

$$\frac{S\{\top\}}{S\{p \vee \overline{p}\}} \qquad \frac{S\{([R \vee U] \wedge T)\}}{S\{[(R \wedge T) \vee U]\}}$$

Mathematical Philosophy on Proofs in General

Proofs operate *inductively* on *structures* over *formulas*

- Equational Logic: pairs of formulas $A = B$
- Sequent Calculus: pairs of finite (multi) sets of formulas $\Gamma \Rightarrow \Delta$
- Deep Inference: structures generated by $[\ldots]$ and $(\ldots)$

$$\boxed{\textbf{Slogan: } \textit{Syntax} \text{ and } \textit{Proof} \text{ live over } \mathsf{Set}.}$$

Algebraic Setup. Σ algebraic signature that induces

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "F_\Sigma"] & \mathsf{Alg}(\Sigma) \arrow[l, bend left, "U_\Sigma"] \end{tikzcd}$$

where we write $H = U_\Sigma \circ F_\Sigma$ for Σ-formulas.

Proof Systems

Proof Systems over a signature Σ are triples $\mathsf{Pf} = (\mathsf{St}, \eta, M)$ where

- $\mathsf{St} : \mathsf{Set} \longrightarrow \mathsf{Set}$ (defining the *structures*)
- $\rho : H^2 \longrightarrow \mathsf{St} \circ H$ (giving a *representation* of $A \leq B$)
- $M : \mathcal{P} \circ \mathsf{St} \circ H \rightarrow \mathcal{P} \circ \mathsf{St} \circ H$ monotone (defining *provability*)

Provable Judgements. Let $A, B \in H(V)$

$$\mathsf{Pf} \vdash A \leq B \iff \rho(A, B) \in \mu M_V$$

where μM_V is the least fixpoint of $M_V : \mathcal{P} \circ \mathsf{St} \circ H(V) \rightarrow \mathcal{P} \circ \mathsf{St} \circ H(V)$.

Examples

Equational Logic.

- structures are $\mathsf{St}X = X \times X$: pairs of formulas
- provability given by equational laws and axioms

Sequent Calculus.

- structures are $\mathsf{St}X = \mathcal{P}_f(X) \times \mathcal{P}_f(X)$: finite sets of formulas
- provability given by sequent rules

Deep Inference.

- structures are inductive types (generated by $(\ldots)$ and $[\ldots]$)
- provability by applying rules deeply

Interpretation

Blanket Assumption. we have an inclusion functor

$$I : \mathbb{A} \hookrightarrow \mathsf{Alg}(\Sigma)$$

so that $\mathbb{A}$ is a category of algebras, and Σ contains $\wedge, \top$.

Interpretation. Given $\Omega \in \mathbb{A}$ and $\pi : V \to U_\Sigma \Omega$

$$[\![A]\!]_{(\Omega,\pi)} \in U_\Sigma \Omega \text{ given by adjoint transpose}$$

$$\frac{\pi : V \to U_\Sigma \Omega}{[\![\cdot]\!]_{(\Omega,\pi)} : F_\Sigma(V) \to \Omega}$$

induced by $F_\Sigma \dashv U_\Sigma$.

(Inductive Extension of π to formulas $A \in H(V) = U_\Sigma \circ F_\Sigma(V)$)

In relation to proof systems

Derived Notions. Let $\Omega \in \mathbb{A}$ and $A, B \in H(V)$.

- $\Omega \models A \leq B \iff \forall \pi : V \to U_\Sigma\Omega\,([\![A]\!]_\pi \wedge [\![B]\!]_\pi = [\![A]\!]_\pi)$
- $\mathbb{A} \models A \leq B \iff \forall \Omega \in \mathbb{A}\,(\Omega \models A \leq B)$

Semantics vs Proofs. Let $\mathsf{Pf} = (\mathsf{St}, \rho, M)$ be a proof system over Σ

Soundness of Pf over $\mathbb{A}$

$$\mathsf{Pf} \vdash A \leq B \text{ only if } \mathbb{A} \models A \leq B$$

Completeness of Pf over $\mathbb{A}$

$$\mathbb{A} \models A \leq B \text{ only if } \mathsf{Pf} \vdash A \leq B$$

In the Examples. Soundness and completeness is well-known – we piggy-back.

Enter Coalgebra . . .

Semantically. Add dynamics to the original picture

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend right, "P"'] & \mathbb{A} \arrow[l, bend right, "S"'] \end{tikzcd}$$

where $S \dashv P$ as before.

Coagebras for $T : \mathbb{C} \to \mathbb{C}$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ TC \arrow[r, "Tf"'] & D \end{tikzcd}$$

giving rise to the category $\mathsf{Coalg}(T)$.

Intuition. $(C, \gamma : C \to TC)$ adds 'dynamics' γ on top of a 'space' C.

Examples.

On $\mathbb{C} = \mathsf{Set}$**.**

$$TX = \mathcal{P}(A \times X)$$

- $(C, \gamma : C \to TC)$ are labelled transition systems

On $\mathbb{C} = \mathsf{Meas}$**.**

$$TX = \mathcal{M}X = \{\text{ probability measures on } X\}$$

- $(C, \gamma : C \to \mathcal{M}C)$ are *Markov Processes*

On $\mathbb{C} = \mathsf{PPos}$**.**

$$T(X, \leq) = \mathcal{K}(X, \leq) = (\mathcal{P}(X), \leq^{EM})$$

- $(C, \gamma : C \to \mathcal{K}C)$ are frames for intuitionistic modal logic

Lifting of Syntax and Proofs

Goal. Lift the logic over $\mathbb{A}$ to T-coalgebras.

Presentation Approach (Alexander Kurz et.al.) Find ‘dual’ $L : \mathbb{A} \longrightarrow \mathbb{A}$

$$\begin{tikzcd} \mathbb{C}^{op} \arrow[loop left, "T"] \arrow[rr, bend right, "P"'] & & \mathbb{A} \arrow[ll, bend right, "S"'] \arrow[loop right, "L"] \end{tikzcd}$$

Syntax given by $L : \mathbb{A} \longrightarrow \mathbb{A}$, **Semantics** given by $\delta : LP \longrightarrow PT$

$$\begin{tikzcd} LF \arrow[rr] \arrow[dd] & & LPC \arrow[d, "\delta_C"] \\ & & PTC \arrow[d, "2(\gamma)=\gamma^{-1}"] \\ F \arrow[rr] & & PC \end{tikzcd}$$

where $(C, \gamma) \in \mathsf{Coalg}(T)$ and F is the initial algebra of formulas.

Equational Proofs

Assumptions. $\mathbb{A} = \mathsf{Alg}(\Sigma, E)$ and $\mathsf{Alg}(L) \cong \mathsf{Alg}(\Sigma_L, E_L)$ where

- $\Sigma_L = \Sigma +$ modal operators
- $E = E +$ modal identities

Example. $\mathsf{BA} \cong \mathsf{Alg}(\Sigma, E)$ and we take

$$L : \mathsf{BA} \to BA$$
$$\Omega \mapsto F_{(\Sigma,E)}\{\Box a \mid a \in \Omega\}/\sim$$

where $\sim$ generated by $\Box\top = \top$, $\Box(a \wedge b) = \Box a \wedge \Box b$.

If $TX = \mathcal{P}(X) : \mathsf{Set} \to \mathsf{Set}$, then

$$\delta_X(\Box a) = \{b \subseteq X \mid b \subseteq a\}$$

defines the semantics of modal logic K.

Proof-Theoretic Approach

Observation. If $L : \mathbb{A} \longrightarrow \mathbb{A}$, then

- no *a priori* notion of 'free syntax' and semantics
- equational reasoning 'built into' categorical / algebraic structure

Syntax in the Front Seat (assuming that Pf sound & complete over $\mathbb{A}$)

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend right, "P"'] & \mathbb{A} \arrow[l, bend right, "S"'] \arrow[r, bend right, "I"'] & \mathsf{Alg}(\Sigma) \arrow[l, bend right, "Q"'] \arrow[loop right, "L"] \end{tikzcd}$$

together with an adjoint situation

$$\begin{tikzcd} \mathsf{Alg}(L) \arrow[r, bend right, "U_L"'] & \mathsf{Alg}(\Sigma) \arrow[l, bend right, "F_L"'] \arrow[r, bend right, "U_\Sigma"'] & \mathsf{Set} \arrow[l, bend right, "F_\Sigma"'] \end{tikzcd}$$

where $\mathsf{Alg}(L) \cong \mathsf{Alg}(\Sigma + \Sigma_L)$: syntax *freely generated*.

Languages for Coalgebras

Languge given by $L : \mathsf{Alg}(\Sigma) \to \mathsf{Alg}(\Sigma)$ as before

Interpretation given by

$$\delta : L \circ I \circ P \to I \circ P \circ T$$

where $I : \mathbb{A} \hookrightarrow \mathsf{Alg}(\Sigma)$ is the inclusion functor.

Semantics over $(C, \gamma : C \to TC) \in \mathsf{Coalg}(T)$ relative to $\pi : V \to U_\Sigma PC$:

Algebraisation $\Omega : LIPC \xrightarrow{\delta_C} IPTC \xrightarrow{IP\gamma} IPC \in \mathsf{Alg}(L)$

$[\![A]\!]_{(C,\pi)} \in U_\Sigma PC$ via adoint transpose

$$\frac{\pi : V \to U_\Sigma PC = U_\Sigma U_L \Omega}{[\![\cdot]\!]_{(C,\pi)} : F_L F_\Sigma V \to \Omega}$$

induced by $F_L \circ F_\Sigma \dashv U_\Sigma \circ U_L$ (standard inductive extension)

Extensions of Proof Systems

Conservative Extensions of $\mathsf{Pf}_\Sigma = (\mathsf{St}, \rho, M_\Sigma)$ are naturality squares

$$\begin{tikzcd}
\mathcal{P} \circ \mathsf{St} \circ H_\Sigma \arrow[r, "M_\Sigma"] \arrow[d] & \mathcal{P} \circ \mathsf{St} \circ H_\Sigma \arrow[d] \\
\mathcal{P} \circ \mathsf{St} \circ H_L \arrow[r, "M_\Sigma"] & \mathcal{P} \circ \mathsf{St} \circ H_L
\end{tikzcd}$$

where H_Σ and H_L are the Σ and L-formulas, respectively.

Call $\mathsf{Pf}_L = (\mathsf{St}, \rho, M_L)$ a *conservative extension* of Pf_Σ if diagram commutes.

Notes.

- same notion of structures (sequents, equations) in both proof systems
- M_L incorporates the reasoning over base category
- intuitvely: M_L arises from M_Σ by adding equations / rules.

Soundness and Completeness

Recall. Algebraic Characterisation of soundness an completeness

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend right, "P"'] & \mathbb{A} \arrow[l, bend right, "S"'] \arrow[r, bend right, "I"'] & \mathsf{Alg}(\Sigma) \arrow[l, bend right, "Q"'] \arrow[loop right, "L"] \end{tikzcd}$$

where soundness is automatic, and completeness if $\delta : LP \to PT$ injective.

Assumption. Pf_L conservative extension of Pf_Σ and

$$\mathsf{Alg}(L) \supseteq \mathsf{Alg}(\mathsf{Pf}_L) \cong \mathsf{Alg}(L^*) \qquad \text{for } L^* : \mathbb{A} \to \mathbb{A}$$

where $\mathsf{Alg}(\mathsf{Pf}_L)$ is the (full) subcategory of L-algebras validating all provable judgements.

Intuition. Proof rules only act on one-step level, e.g. $\frac{A \to B}{\Box A \to \Box B}$.

Soundness and Completeness

Assume that

$$\mathsf{Alg}(L) \supseteq \mathsf{Alg}(\mathsf{Pf}_L) \cong \mathsf{Alg}(L^*) \qquad \text{for } L^* : \mathbb{A} \to \mathbb{A}$$

Soundness of Pf_L over $\mathsf{Coalg}(T)$ if δ factors

$$\begin{tikzcd} LIP \arrow[r] \arrow[rr, bend right, "\delta"'] & IL^*P \arrow[r, "I\delta^*"] & IPT \end{tikzcd}$$

for some $\delta^* : L^*P \to PT$.

Completeness of Pf_L over $\mathsf{Coalg}(T)$ if δ^* is injective.

(By importing corresponding results from equational setting)

Applications.

On $\mathbb{C} = \mathsf{Set}$.

$$TX = \mathcal{P}(A \times X)$$

- Completeness of equational and sequent proofs over transition systems (well-known)

On $\mathbb{C} = \mathsf{Meas}$.

$$TX = \mathcal{M}X = \{\text{ probability measures on } X\}$$

- Completeness of probabilistic modal logic over measurable spaces (expected and partially known)

On $\mathbb{C} = \mathsf{PPos}$.

$$T(X, \leq) = \mathcal{K}(X, \leq) = (\mathcal{P}(X), \leq^{EM})$$

- Conjectured: completeness for intuitionistic modal logic

Conclusions

Syntax and Proof

- standard, formalised in set theory via inductive definitions

Semantics

- categorical over a space equipped with ‘logical adjunction’

Glue provided by notion of proof system

- crucial: predicates over spaces carry algebraic structure

Element of Novelty.

- ’old’ results but with respect to ’new’ proof systems
- conjectured: new completeness of IK over intuitionistic frames