Game Logic

— Coalgebraic Completeness and Automata

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Introduction

Modal logics:

- Versatile logics for reasoning about state-based systems.
- Good trade-off between expressiveness and decidability.
- Established as the logics of coalgebras.

Aim: Coalgebraic understanding of dynamic modal logics, like Propositional Dynamic Logic (PDL) and Game Logic.

- Identify relevant mathematical structure.
- Framework for developing dynamic coalgebraic logics.
- Transfer insights to other dynamic/game settings.
- Improve our understanding of fragments of fixpoint logics.

Game Logic (GL)

Rohit Parikh, "The Logic of Games and its Applications".

- Strategic ability in determined 2-player games.

  $\langle\alpha\rangle\varphi$ expresses

  "player I has strategy in α to ensure outcome satisfies φ"

- Game version of PDL:
  – PDL: 1-player game (nondeterministic programs)
  – from program constructs to game constructs.

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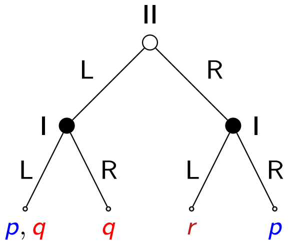
Determined 2-Player Games

Typical examples: 2-player, extensive games with perfect information.

Example: Players: I (black) and II (white), moves: L or R.



Strategic ability	formula	strategy
I can ensure p:	$\langle\alpha\rangle p$	LR
I cannot ensure q:	$\neg\langle\alpha\rangle q$	
I cannot ensure r:	$\neg\langle\alpha\rangle r$	
II can ensure q:	$[\alpha]q$	L
II can ensure $\neg r$:	$[\alpha]r$	L

Note that: $\langle\alpha\rangle(q \vee r)$, but $\neg\langle\alpha\rangle q$ and $\neg\langle\alpha\rangle r$.

Game modalities are not disjunctive.
They are only monotonic: $\langle\alpha\rangle p \to \langle\alpha\rangle(p \vee q)$

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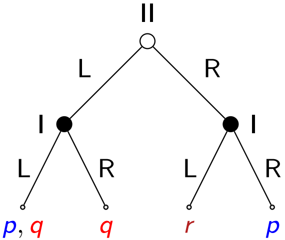


Strategic normal form:

I \ II	L	R
LL	p,q	r
LR	p,q	p
RL	q	r
RR	q	p

$\langle\alpha\rangle p$

$[\alpha]q$

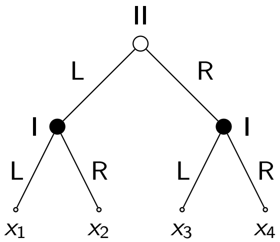
Determinacy: $\langle\alpha\rangle\varphi \leftrightarrow \neg[\alpha]\neg\varphi$

("I can ensure φ iff II cannot avoid φ")

Effectivity in State-Based Game Models

- Games are played in the context of a state space X.
- Game outcomes are associated with states.

Player I is *effective for* $U \subseteq X$ if I has a strategy to ensure the outcome is in U.



- Player I is effective for:

  $\{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}$

  and all supersets of those.
- Player II is effective for:

  $\{x_1, x_2\}$ and $\{x_3, x_4\}$.

  and all supersets of those.

Game Structures

- Let $E_\alpha(x) \subseteq \mathcal{P}(X)$ be defined by:

  $U \in E_\alpha(x)$ iff player 1 is effective for U in α starting in x.

  Then: $U \in E_\alpha(x)$ and $U \subseteq U' \quad \Rightarrow \quad U' \in E_\alpha(x)$.
- Let $\mathcal{M}$ be the monotone neighbourhood functor:

  $\mathcal{M}(X) = \{N \subseteq \mathcal{P}(X) \mid U \in N, U \subseteq U' \Rightarrow U' \in N\}$
  $\mathcal{M}(f) = (f^{-1})^{-1}$
- A game frame is
  - a multi-modal monotonic neighbourhood frame

    $F = (X, \{E_\alpha \colon X \to \mathcal{M}(X) \mid \alpha \in A\})$ or equivalently,
  - a coalgebra $F \colon X \to (\mathcal{M}(X))^A$
- A game model $M = (F, V)$ is a game frame F with a valuation $V \colon X \to \mathcal{P}(P_0)$.

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Game Logic Syntax

$$
\begin{array}{lcl}
\text{formulas } \varphi & ::= & p \in P_0 \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\alpha\rangle\varphi \\
\text{games } \alpha & ::= & a \in A_0 \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \varphi? \mid \alpha^d
\end{array}
$$

Game operations:

- (composition) $\alpha_1;\alpha_2$: play α_1 then α_2,
- (angelic choice) $\alpha_1 \cup \alpha_2$: player 1 chooses between α_1 or α_2,
- (angelic iteration) α^*: α is played repeatedly (possibly 0 times), after each round, player I chooses whether to continue.
- (dual) α^d: players switch roles in α.
- (tests) $\varphi?$: if φ holds then continue, otherwise player I loses.

Standard Game Models

(similar to standard PDL model)

- (composition)

$$U \in E_{\alpha_1;\alpha_2}(x) \text{ iff } \exists V \in E_{\alpha_1}(x) : \forall v \in V : X \in E_{\alpha_2}(v).$$

- (angelic choice) $E_{\alpha_1 \cup \alpha_2}(x) = E_{\alpha_1}(x) \cup E_{\alpha_2}(x)$
- (angelic iteration)

$$U \in E_{\alpha^*}(x) \text{ iff } x \in \widehat{E}_{\alpha^*}(U) \text{ where } \widehat{E}_{\alpha^*}(U) = \mu X.U \cup \widehat{E}_{\alpha}(X).$$

(after each round, player I chooses whether to continue).

- (dual)

$$U \in E_{\alpha^d}(x) \quad \text{iff} \quad X \setminus U \notin E_{\alpha}(x).$$

Axiomatisation and Completeness

- **GL** = monotonic modal logic **M** (ML of mon. nbhd. frames) plus

$$\langle \alpha ; \delta \rangle \varphi \leftrightarrow \langle \alpha \rangle \langle \delta \rangle \varphi \qquad \langle \alpha \cup \delta \rangle \varphi \leftrightarrow \langle \alpha \rangle \varphi \vee \langle \delta \rangle \varphi$$

$$\langle \psi ? \rangle \varphi \leftrightarrow (\psi \wedge \varphi) \qquad \langle \alpha^{d} \rangle \varphi \leftrightarrow \neg \langle \alpha \rangle \neg \varphi$$

$$\varphi \vee \langle \alpha \rangle \langle \alpha^{*} \rangle \varphi \rightarrow \langle \alpha^{*} \rangle \varphi \qquad \frac{\varphi \vee \langle \alpha \rangle \varphi \rightarrow \psi}{\langle \alpha^{*} \rangle \varphi \rightarrow \psi}$$

- Without dual: sound and complete [Parikh 1985].
- Without iteration: sound and strongly complete [Pauly 2001].
- Completeness of full GL still open question.

Coalgebraic Dynamic Logic

Joint work with Clemens Kupke

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A General Picture...?

PDL	GL	Basic set up
Kripke semantics $X \xrightarrow{\alpha} \mathcal{P}X$	Mon. nbhd. semantics $X \xrightarrow{\alpha} \mathcal{M}X$	T-Coalgebra semantics $X \xrightarrow{\alpha} TX$
$X \to (\mathcal{P}X)^A$ $;, \cup, (-)^*, ?$ Normal ML **K** plus reduction axioms	$X \to (\mathcal{M}X)^A$ $;, \cup, (-)^*, ?, (-)^d$ Monotonic ML **M** plus reduction axioms	$X \to (TX)^A$ T-coalg. operations (?) T-Coalgebraic ML plus reduction axioms (?)
$f: X \to Y$ $\mathcal{P}(f) = f[\,]$ (direct image)	$f: X \to Y$ $\mathcal{M}(f) = (f^{-1})^{-1}$ (double-inv. image)	$f: X \to Y$ $T(f)\colon TX \to TY$

Coalgebraic Modal Logic

Basic idea

$$\frac{\text{Basic Modal Logic}}{\text{Kripke frames } X \to \mathcal{P}(X)} = \frac{\text{Coalgebraic Modal Logic}}{T\text{-coalgebras } X \to T(X)}$$

Develop modal logic for T-coalgebras, parametric in $T \colon C \to C$.

Syntax

Given a collection of modal operators Λ and a set P_0 of propositional variables. The set $\mathcal{F}(\Lambda)$ of formulas over Λ is defined a follows:

$$\mathcal{F}(\Lambda) \ni \varphi ::= p \in P_0 \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \heartsuit\varphi, \quad \heartsuit \in \Lambda$$

(We only consider unary modalities $\heartsuit$)

Coalgebraic Modal Logic: Semantics

cf. [Pattinson, Roessiger]

T-coalgebraic semantics consists of:

- a functor $T : \mathsf{Set} \to \mathsf{Set}$
- for every modal operator $\heartsuit \in \Lambda$, a natural transformation

  $$\heartsuit : \mathcal{Q} \Rightarrow \mathcal{Q}T \quad \text{(predicate lifting)}$$

  where $\mathcal{Q}$ denotes the contravariant power set functor ($\mathcal{Q}X = 2^X$, $\mathcal{Q}(f) = f^{-1}$), so $\heartsuit_X : 2^X \to 2^{TX}$.

Truth in T-model $(X, \gamma : X \to TX, V : P_0 \to \mathcal{P}X)$

$$\begin{aligned} [\![p]\!] &= V(p) \qquad \text{for } p \in P_0 \\ &\vdots \\ [\![\heartsuit\varphi]\!] &= \gamma^{-1}(\heartsuit_X([\![\varphi]\!])) \end{aligned}$$

Equivalently...

There is a well-known one-to-one correspondence between:

- $\heartsuit : \mathcal{Q} \Rightarrow \mathcal{Q}T \quad (\heartsuit_X : 2^X \to 2^{TX})$
- $\widehat{\heartsuit} : T \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q} \quad$ (transpose $\widehat{\heartsuit}_X : TX \to 2^{2^X}$)
- $\check{\heartsuit} : T2 \to 2 \quad$ (Yoneda) ("allowed 0-1 patterns")

Examples:

- Kripke box:
$$\begin{aligned} \heartsuit_X(U) &= \{V \subseteq X \mid V \subseteq U\}, \\ \widehat{\heartsuit}_X(V) &= \{U \subseteq X \mid V \subseteq U\} \text{ and} \\ \check{\heartsuit}(V \in \mathcal{P}2) &= 1 \quad \text{iff} \quad 0 \notin V \end{aligned}$$

- Mon. nbhd. diamond:
$$\begin{aligned} \heartsuit_X(U) &= \{N \in \mathcal{M}X \mid U \in N\} \\ \widehat{\heartsuit}_X(N) &= N \\ \check{\heartsuit}(N \in \mathcal{M}2) &= 1 \quad \text{iff} \quad \{1\} \in N \end{aligned}$$

3.1 Dynamic Syntax and Semantics

Coalgebra-Algebra

Two perspectives:

$$\frac{\xi\colon X \to (TX)^A \quad T^A\text{-coalgebra, modalities}}{\widehat{\xi}\colon A \to (TX)^X \quad \text{algebra homomorphism, program operations}}$$

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Dynamic Syntax

Given

- Σ, a signature (functor).
- P_0, a countable set of atomic propositions.
- A_0, a countable set of atomic programs.

we define

$$\begin{array}{llcl} \text{formulas} & \mathcal{F} \ni \varphi & ::= & p \in P_0 \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\alpha\rangle\varphi \\ \text{programs} & A \ni \alpha & ::= & a \in A_0 \mid \alpha;\alpha \mid \underline{\sigma}(\alpha_1, \ldots, \alpha_n) \mid \alpha^* \mid \varphi? \end{array}$$

where $\underline{\sigma} \in \Sigma$ is n-ary operation symbol.

3.2 Operations on T-coalgebras

Program Operations from Monads

(cf. Moggi, and many others)

- Monad T encodes computational effects (non-determinism, exceptions, continuations, input/output,...)
- Kleisli arrows $X \to TY$ are functional programs.

- A monad is functor $T\colon \mathsf{Set} \to \mathsf{Set}$ together with natural transformations

  $$\eta\colon \mathsf{Id} \Rightarrow T \text{ (unit) and } \mu\colon T \circ T \Rightarrow T \text{ (multiplication)}$$

  satisfying certain axioms...
- Sequential composition is Kleisli composition.

  $$(X \xrightarrow{\alpha} TX) *_T (X \xrightarrow{\beta} TX) =$$
  $$X \xrightarrow{\alpha} TX \xrightarrow{T\beta} TTX \xrightarrow{\mu_X} TX$$
- Skip program is unit: $\eta_X\colon X \to TX$.

$\mathcal{P}$ and $\mathcal{M}$ are monads

- $\mathcal{P}$ is monad $(\mathcal{P}, \eta, \mu)$ with:
  $\eta_X(x) = \{x\}$,

  $\mu_X(\{U_i \mid i \in I\}) = \bigcup_{i \in I} U_i$.

  ($*_{\mathcal{P}}$ is relation composition, $\eta_X = \mathsf{Id}_X$)

- $\mathcal{M}$ is a monad $(\mathcal{M}, \eta, \mu)$ with:

  $$\begin{aligned} \eta_X(x) &= \{U \subseteq X \mid x \in U\} \\ \mu_X(W) &= \{U \subseteq X \mid \eta_{\mathcal{P}(X)}(U) \in W\} \end{aligned}$$

  ($*_{\mathcal{M}}$ is composition of effectivity functions)

- $\mathcal{N} = \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ is a monad (η and μ as for $\mathcal{M}$):

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Dynamic Monads

Iteration requires extra structure.

A monad (T, μ, η) is called dynamic if

- For all sets X, TX can be equipped with a sup-lattice structure $(TX, \bigvee)$ (i.e., a complete join semilattice).

  (We denote the empty join in TX by $\bot_{TX}$.)
- Lift $\bigvee$ pointwise to the Kleisli Hom-sets $\mathcal{K}\ell(T)(X, X)$, then Kleisli-composition is monotone:

$$\forall f, g_1, g_2 \colon X \to TX : g_1 \leq g_2 \quad \Rightarrow \quad f * g_1 \leq f * g_2.$$

In FICS 2015 paper: we assumed that $*$ left-distributes over join.

- Bad news: Doesn't seem to hold for $\mathcal{M}$.
- Good news: We don't need it! (Monotonicity suffices)

Iteration and Tests

Let (T, η, μ) be a dynamic monad.

Iteration: For a map $\alpha : X \to TX$, we define $\alpha^* = LFP\,.\,\Phi_\alpha$ where

$$\begin{aligned} \Phi_\alpha : \mathcal{K}\ell(T) &\to \mathcal{K}\ell(T) \\ g &\mapsto \eta_X \vee (\alpha * g) \end{aligned}$$

Tests: For a formula φ, we define $\alpha = \varphi?$ via Kleisli identity and empty join $\bot = \bigvee \emptyset \in TX$:

$$\varphi?(x) = \eta_X(x) \quad \text{if } x \in [\![\varphi]\!]^{\mathfrak{M}}, \qquad \text{else } \bot.$$

Pointwise Operations

- An *n-ary natural operation on T* is a natural transformation

$$\sigma\colon T^n \Rightarrow T \qquad (Tf \text{ preserves } \sigma)$$

- $\sigma\colon T^n \Rightarrow T$ yields pointwise operation on $(TX)^X$, e.g.,

$$\sigma_X^X(c_1, c_2)(x) = \sigma_X(c_1(x), c_2(x))$$

- Given finitary signature functor Σ,
a natural Σ-algebra is natural transformation $\theta\colon \Sigma T \Rightarrow T$
and yields pointwise Σ-algebra on $(TX)^X$:

$$\theta_X^X\colon \Sigma((TX)^X) \to (TX)^X$$

Natural Pointwise Operations: Examples

Natural operations on $\mathcal{P}$:

- Union $\cup\colon \mathcal{P} \times \mathcal{P} \Rightarrow \mathcal{P}$ is natural operation, since

  $$f[U \cup U'] = f[U] \cup f[U'] \quad (\mathcal{P}f(U) = f[U])$$

  The pointwise extension of $\cup\colon \mathcal{P} \times \mathcal{P} \Rightarrow \mathcal{P}$ is union of relations $(R_1 \cup R_2)(x) = R_1(x) \cup R_2(x)$.
- Note: Intersection, complement are not natural on $\mathcal{P}$.

Natural operations on $\mathcal{M}$:

- All Boolean operations (since preserved by f^{-1}).
- Dual operation $d\colon \mathcal{M} \Rightarrow \mathcal{M}$ where for all $N \in \mathcal{M}(X)$, and $U \subseteq X$, $U \in d_X(N)$ iff $X \setminus U \notin N$.
  Game operation $(-)^d$ is the pointwise extension.

Summary of Requirements for Coalgebraic Dynamic Semantics

We assume given:

- set A_0 of atomic programs.
- set P_0 of atomic propositions.

We require for dynamic T-coalgebra semantics:

- $\heartsuit\colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ is predicate lifting (for modalities)
- $\mathbb{T} = (T, \eta, \mu)$, a monad (for sequential comp.)
- $\mathbb{T} = (T, \eta, \mu)$, is a dynamic monad (for iteration and tests)
- $\theta\colon \Sigma T \Rightarrow T$, a natural Σ-algebra (for pointwise ops)

Standard Dynamic Models

Def. A θ-dynamic $\mathbb{T}$-model is a triple $\mathfrak{M} = (X, \gamma\colon X \to (TX)^A, V)$ where

- $\widehat{\gamma}|_{A_0} = \widehat{\gamma}_0\colon A_0 \to (TX)^X$ interprets atomic programs,
- sequential composition, iteration, tests and pointwise operations are defined compositionally from γ_0 as described.
- $V\colon P_0 \to \mathcal{P}(X)$ is a valuation of atomic propositions.
- Modalities are interpreted by $\heartsuit$:

$$[\![\langle\alpha\rangle\varphi]\!] = \widehat{\gamma}(\alpha)^{-1}(\heartsuit_X([\![\varphi]\!]))$$

3.3 Axiomatising Standard Dynamic Models

Axiomatising Sequential Composition

Sequential composition axiom: $\langle \alpha ; \beta \rangle p \leftrightarrow \langle \alpha \rangle \langle \beta \rangle p$.

Recall: $\heartsuit : \mathcal{Q} \Rightarrow \mathcal{Q} \circ T \quad \overset{1-1}{\leftrightarrow} \quad \widehat{\heartsuit} : T \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$

Lemma (Soundness for sequential composition)

If $\widehat{\heartsuit} \colon T \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ is a monad morphism, and $\gamma \colon X \to (TX)^A$ is ;-standard, then $\langle \alpha ; \beta \rangle p \leftrightarrow \langle \alpha \rangle \langle \beta \rangle p$ is valid in γ.

Note: Holds for $\heartsuit$ iff holds for $\neg \heartsuit \neg$.

Examples

Remark: $\dfrac{\text{Monad morphism } T \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}}{\text{Eilenberg-Moore algebra } T2 \to 2}$ [Kelly & Power, 1993]

- **Kripke diamond** ($T = \mathcal{P}$):

  $\Diamond \colon \mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{P}$ corr. to $\check{\Diamond} \colon \mathcal{P}\mathcal{P}(1) \to \mathcal{P}(1)$ (free $\mathcal{P}$-algebra)

  so $\widehat{\Diamond} \colon \mathcal{P} \to \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ is a monad morphism.

- **Monotonic nbhd diamond** ($T = \mathcal{M}$):

  $\heartsuit \colon \mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{M}$ corr. to $\widehat{\heartsuit} \colon \mathcal{M} \Rightarrow \mathcal{Q}^{\mathrm{op}}\mathcal{Q}$ (inclusion)

  hence a monad morphism.

Axiomatising Pointwise Operations

- Example: PDL axiom for choice $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$.
- Idea: $\widehat{\heartsuit} \colon T \Rightarrow \mathcal{N}$ turns operation σ on T into operation χ on $\mathcal{N}$.

$$\begin{array}{ccc} T^n & \overset{\widehat{\heartsuit}^n}{\Longrightarrow} & \mathcal{N}^n \\ \Downarrow \sigma & & \Downarrow \chi \\ T & \overset{\widehat{\heartsuit}}{\Longrightarrow} & \mathcal{N} \end{array}$$

For example:

$$\begin{array}{ccc} \mathcal{P} \times \mathcal{P} & \overset{\widehat{\Box} \times \widehat{\Box}}{\Longrightarrow} & \mathcal{N} \times \mathcal{N} \\ \Downarrow \cup & & \Downarrow \cap \\ \mathcal{P} & \overset{\widehat{\Box}}{\Longrightarrow} & \mathcal{N} \end{array}$$

- Need: $\chi \colon \mathcal{N}^n \Rightarrow \mathcal{N}$ such that the diagram commutes.
- From $\chi \colon \mathcal{N}^n \Rightarrow \mathcal{N}$, we get rank-1 formula $\varphi(\chi, \alpha_1, \ldots, \alpha_n, p)$ (details in paper).
- Def. φ is rank 1 if $\varphi \in Prop(\Lambda(Prop(P_0)))$.
  Example: $\Box(p \to q) \to (\Box p \to \Box q)$

Axiomatising Pointwise Operations

Lemma (Soundness)

If $\gamma \colon X \to (TX)^A$ is θ-standard, and there exists $\chi \colon \mathcal{N}^n \Rightarrow \mathcal{N}$ such that $\widehat{\heartsuit} \circ \sigma = \chi \circ \widehat{\heartsuit}^n$, then the rank-1 formula $\langle \underline{\sigma}(\alpha_1, \ldots, \alpha_n) \rangle p \leftrightarrow \varphi(\chi, \alpha_1, \ldots, \alpha_n, p)$ is valid in γ.

Positive operations

- $\chi \colon \mathcal{N}^n \Rightarrow \mathcal{N}$ corr. to $\check{\chi} \in \mathcal{N}(n \cdot \mathcal{Q}(2))$, the free Boolean algebra on $n \cdot \mathcal{Q}(2)$, i.e. n copies of elements of $\mathcal{Q}(2)$.
- Def. σ is *positive*, if corresponding $\check{\chi}$ can be expressed without $\neg$.

Axiomatising Tests

In PDL: $[\varphi?]p \leftrightarrow (\varphi \to p)$ or $\langle\varphi?\rangle p \leftrightarrow (\varphi \wedge p)$

In GL: $\langle\varphi?\rangle p \leftrightarrow (\varphi \wedge p)$

Axioms (when $\langle\alpha\rangle$ interpreted by $\heartsuit$):

- If $\heartsuit$ is "box-like",
  then add $\langle\varphi?\rangle p \leftrightarrow (\varphi \to p)$ as frame condition.
- If $\heartsuit$ is "diamond-like",
  then add $\langle\varphi?\rangle p \leftrightarrow (\varphi \wedge p)$ as frame condition.

Axiomatising Tests: $\heartsuit$ is diamond/box-like

Let $\heartsuit\colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ be a predicate lifting. We say that

- $\heartsuit$ is diamond-like if for all sets X, all $U \subseteq X$, and all $\{t_i \mid i \in I\} \subseteq TX$:

$$\bigvee_{i \in I} t_i \in \heartsuit_X(U) \quad \text{iff} \quad \exists i \in I:\ t_i \in \heartsuit_X(U).$$

- $\heartsuit$ is box-like if for all sets X, all $U \subseteq X$, and all $\{t_i \mid i \in I\} \subseteq TX$:

$$\bigvee_{i \in I} t_i \in \heartsuit_X(U) \quad \text{iff} \quad \forall i \in I:\ t_i \in \heartsuit_X(U).$$

Note: this has nothing to do with requiring the modality to preserve disjunctions or conjunctions!

Axiomatising Iteration (diamond-like ♡)

(Following GL axiomatisation)

For $\alpha \in A$:

axiom: $\langle \alpha^* \rangle p \leftrightarrow p \vee \langle \alpha \rangle \langle \alpha^* \rangle p$ — $\langle \alpha^* \rangle p$ is fixed point

rule: $$\frac{p \vee \langle \alpha \rangle q \to q}{\langle \alpha^* \rangle p \to q}$$ $\langle \alpha^* \rangle p$ is least prefixed point

Soundness over standard models: ✓

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Logic and Derivability

Def. A modal logic $\mathcal{L} = (\Lambda, \mathrm{Ax}, \mathrm{Fr}, \mathrm{Ru})$ consists of

- a modal signature Λ,
- a set of rank-1 axioms $\mathrm{Ax} \subseteq \mathsf{Prop}(\Lambda(\mathsf{Prop}(P_0)))$,
- a set of frame conditions $\mathrm{Fr} \subseteq \mathcal{F}(\Lambda, P_0)$,
- a set of inference rules $\mathrm{Ru} \subseteq \mathcal{F}(\Lambda, P_0) \times \mathcal{F}(\Lambda, P_0)$.

Def. (Hilbert system derivability)

- $\vdash_{\mathcal{L}} \varphi$ if φ is derivable from $\mathrm{Ax} \cup \mathrm{Fr}$ using propositional reasoning, uniform substitution, rules in Ru, and the congruence rule:

$$\frac{\varphi \leftrightarrow \psi}{\heartsuit\varphi \leftrightarrow \heartsuit\psi} \; (\heartsuit \in \Lambda)$$

Coalgebraic Dynamic Logic (diamond-like ♡)

Given

- "base" logic $\mathcal{L} = (\{\Diamond\}, \mathrm{Ax}(\Diamond, T), \emptyset, \emptyset)$ for T
- $\theta\colon \Sigma T \Rightarrow T$ and set A_0 of atomic actions.

we define the dynamic logic $\mathcal{L}(\theta, ;, ^*, ?) = (\Lambda, \mathrm{Ax}, \mathrm{Fr}, \mathrm{Ru})$ by taking

$$\begin{array}{rcl}
\Lambda &=& \{\langle\alpha\rangle \mid \alpha \in A\}, \\
\mathrm{Ax} &=& \mathrm{Ax}(\Diamond, T)_A \cup \text{"}\theta\text{-axioms"} \\
\mathrm{Fr} &=& \quad \{\langle\alpha;\beta\rangle p \leftrightarrow \langle\alpha\rangle\langle\beta\rangle p \mid \alpha, \beta \in A\} \\
&& \cup \{\langle\alpha^*\rangle p \leftrightarrow p \vee \langle\alpha\rangle\langle\alpha^*\rangle p \mid \alpha \in A\} \\
&& \cup \{\langle q?\rangle p \leftrightarrow (q \wedge p)\} \\
\mathrm{Ru} &=& \left\{ \dfrac{p \vee \langle\alpha\rangle q \to q}{\langle\alpha^*\rangle p \to q} \right\}
\end{array}$$

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Strong Completeness for Iteration-Free Logics

Theorem

If base logic $\mathcal{L}$ satisfies conditions for quasi-canonical T-model [Schröder&Pattinson, 2009], then dynamic logic $\mathcal{L}(\theta, ;, ?)$ is sound and strongly complete wrt θ-dynamic T-models.

- Strong completeness of PDL^{-*} and GL^{-*} recovered.
- Modest new result for "lift" monad $L(X) = 1 + X$.
- cf. HH, C. Kupke, R. A. Leal, IFIP-TCS 2014.

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Weak Completeness Needs Strong Coherence

- Standard weak completeness argument:
  1. build model on collection S of finite maximally consistent sets of "relevant" formulas (needs the definition of closure - a generalised notion of a subformula)
  2. prove truth lemma for this finite model
- any subset $U \subseteq S$ of the constructed finite model can be characterised with a formula

$$\xi_U = \bigvee_{\Delta \in U} \bigwedge \Delta$$

Key for completeness: "strong coherence" property

We say that $\gamma \colon S \to (TS)^A$ is strongly coherent for $\alpha \in A$ if for all $\Gamma \in S$ and all $U \subseteq S$:

$$\widehat{\gamma}(\alpha)(\Gamma) \in \heartsuit_S(U) \quad \text{iff} \quad \langle \alpha \rangle \xi_U \wedge \Gamma \text{ is } \mathcal{L}\text{-consistent.}$$

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Weak Completeness with Iteration and Positive Operations

(Recall: PDL is not compact, hence not strongly complete.)

Theorem

If base logic $\mathcal{L}$ is one-step complete for T, and θ consists of positive operations, then $\mathcal{L}(\theta, ;, ^*, ?)$ is complete wrt standard, dynamic T-models.

- Completeness of PDL and dual-free GL recovered.
- Modest new result for dual-free GL with intersection.
- cf. HH, C. Kupke, FICS 2015.

Summary of Requirements

We require for semantics:

- $\mathbb{T} = (T, \eta, \mu)$, a Set-monad.
- TX carries sup-lattice structure.
- $\heartsuit \colon \mathcal{Q} \Rightarrow \mathcal{Q} \circ T$ is predicate lifting for T.
- $\theta \colon \Sigma T \Rightarrow T$, a natural Σ-algebra.

We require for soundness and completeness:

- for each $\sigma : T^n \Rightarrow T$, a $\chi : \mathcal{N}^n \Rightarrow \mathcal{N}$ s.t. $\widehat{\heartsuit} \circ \sigma = \chi \circ \widehat{\heartsuit}^n$.
- $\widehat{\heartsuit}$ is monad morphism.
- $\heartsuit$ is monotone.
- Kleisli composition is monotone.
- θ consists of "positive operations" ($\breve{\chi}$ negation-free)

Game Logic Automata

Joint work with:
Clemens Kupke, Johannes Marti, Yde Venema

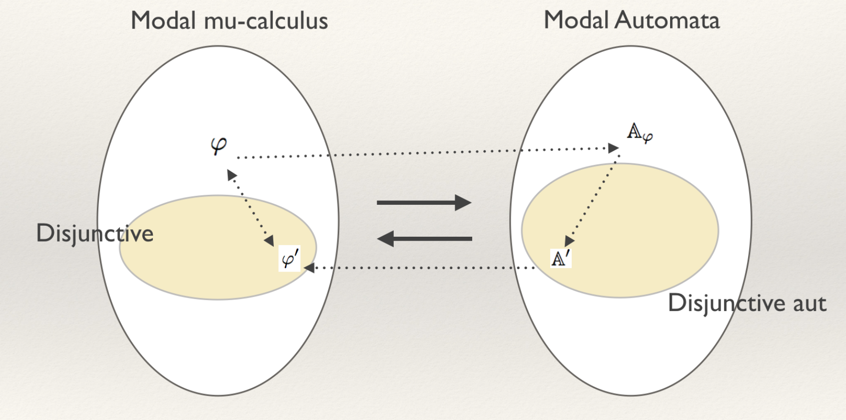
Work in progress...

Game Logic, Completeness and μ-Calculus

- Game Logic can be translated into (monotonic) μML.
- Game Logic spans all levels of the alternation hierarchy μML [Berwanger, 2003] (by interleaving * and ^{d})
- μML completeness is hard....but automata can help.

Completeness via Modal Parity Automata

Completeness of the (coalgebraic) modal μ-calculus
(Enqvist, Seifan, Venema)



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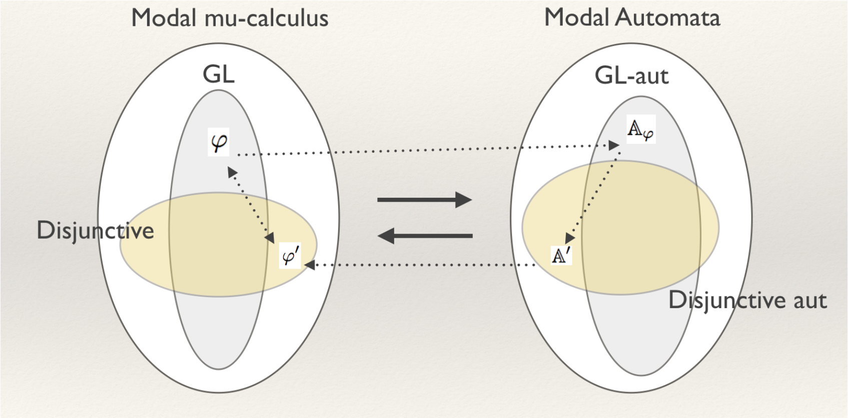

Modal Parity Automata for μML

- Let P be a set of atomic propositions, and
- $\mathrm{Lit} = \{p, \neg p \mid p \in P\}$.
- For a set S, $1ML(P, S) = Prop(\mathrm{Lit} \cup \{\Box t, \Diamond t \mid t \in Latt(S)\})$

Def. A modal parity automaton $\mathbb{A} = (S, \Theta, \Omega, s_I)$ consists of

- a set S of states,
- a 1-step transition structure $\Theta\colon S \to 1ML(P, S)$,
- a priority function $\Omega\colon S \to \omega$,
- an initial state $s_I \in S$.

Acceptance of Kripke structures defined via parity game.

Pre-Automata for Game Logic

Game Logic

- Both formulas and games must be "decomposed" (use reduction axioms).
- Assigning priorities to all vertices is problematic.

We work with pre-automata.

Def. A pre-automaton (or modal graph) $\mathbb{G} = (V, E, L, \Omega)$ consists of

- a finite graph (V, E),
- a labelling $L\colon V \to \mathrm{Lit} \cup \{\vee, \wedge\} \cup \{\langle\alpha\rangle, \langle\alpha^d\rangle \mid \alpha \in A_0\}$,
- a priority function $\Omega\colon S \to \omega$ where $S \subseteq V$

such that

- Arities match: if $L(v) \in \{\vee, \wedge\}$ then $|E(v)| \leq 2$, etc.
- On every cycle there is at least one state from S.

Pre-Automata for Game Logic

We have (to be checked and possibly tweaked):

- Construction from pre-automaton to automaton.
- Correspondence between GL formulas and game pre-automata.
- Conditions that characterise game pre-automata
  (conditions on cycles, sharing paths and maximal priorities)
- Evaluation game for pre-automata.

Conclusion

Summary:

- General coalgberaic completeness of PDL and dual-free Game Logic.
- Modest "new" results as instantiations.
- Need more examples...

Future Work:

- Extend to quantitative setting.
  Problem: for $T = \mathcal{D}_\omega$ the distribution monad, the only EM-algebras $\mathcal{D}_\omega(2) \to 2$ seem to be $\Diamond_{>0}$ and $\Diamond_{=0}$.
  $\rightsquigarrow$ Switch to multi-valued logic.
- Extend to other types of operations (e.g. Coalition Logic).
- Completeness of full GL and coalgebraic CML.

THANKS!