

Moessner's Theorem

- an extensive exercise in coinduction -

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Overview

1. Moessner's Theorem
2. Streams and coinduction
3. Formalising Moessner's Theorem
4. Proving Moessner's Theorem
5. The heart of the matter: circularity
6. Paasche's Theorem
7. Discussion

1. Moessner's Theorem

Moessner's Theorem ($k = 2$)

nat	1	2	3	4	5	6	7	8	9	10	11	12	...
<i>Drop₂</i>	1		3		5		7		9		11	...	
Σ	1	4	9	16	25	36	...						
	=												
nat ²	1 ²	2 ²	3 ²	4 ²	5 ²	6 ²	...						

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Moessner's Theorem ($k = 2$)

nat 1 2 3 4 5 6 7 8 9 10 11 12 ...

*Drop*₂ 1 3 5 7 9 11 ...

Σ 1 4 9 16 25 36 ...

=

nat² 1² 2² 3² 4² 5² 6² ...

Moessner's Theorem ($k = 2$)

nat 1 2 3 4 5 6 7 8 9 10 11 12 ...

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nat² 1² 2² 3² 4² 5² 6² ...

Moessner's Theorem ($k = 3$)

nat	1	2	3	4	5	6	7	8	9	10	11	12	...
$Drop_3$	1	2		4	5		7	8		10	11	...	
Σ	1	3	7	12	19	27	37	48	...				
$Drop_2$	1		7		19		37	...					
Σ	1	8	27	64	...								
	=												
nat^3	1^3	2^3	3^3	4^3	...								

Moessner's Theorem ($k = 3$)

nat	1	2	3	4	5	6	7	8	9	10	11	12	...
<i>Drop</i> ₃	1	2		4	5		7	8		10	11	...	
Σ	1	3	7	12	19	27	37	48	...				
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	=												
nat ³	1 ³	2 ³	3 ³	4 ³	...								

Moessner's Theorem ($k = 3$)

nat	1	2	3	4	5	6	7	8	9	10	11	12	...
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Moessner's Theorem ($k = 3$)

nat 1 2 3 4 5 6 7 8 9 10 11 12 ...

$Drop_3$ 1 2 4 5 7 8 10 11 ...

Σ 1 3 7 12 19 27 37 48 ...

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Σ 1 8 27 64 ...

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nat^3 1^3 2^3 3^3 4^3 ...

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	=												
nat^3	1^3	2^3	3^3	4^3	...								

Moessner's Theorem ($k = 4$)

nat	1	2	3	4	5	6	7	8	9	10	11	...
<i>Drop</i> ₄	1	2	3		5	6	7		9	10	11	...
Σ	1	3	6	11	17	24	33	43	54	...		
<i>Drop</i> ₃	1	3		11	17		33	43		67	81	...
Σ	1	4	15	32	65	108	175	...				
<i>Drop</i> ₂	1		15		65		175	...				
Σ	1	16	81	256	...							
=	1^4	2^4	3^4	4^4	...							

Moessner's Theorem: history

- Conjectured by **A. Moessner** (1951), first proved by **O. Perron** (1951), generalised by **I. Paasche** (1952) and **H. Salie** (1952).
- Proof in functional programming by **R. Hinze** (2008, 2011).
- First coinductive proof by **M. Niqui** and **J.R.** (2011).
- New proof using multivariate generating functions, by **D. Kozen** and **A. Silva** (2013).
- Formalisation in COQ of the coinductive proof of **M. Niqui** and **J.R.**, by **R. Krebbers**, **L. Parlant** and **A. Silva** (2016).

Moessner's Theorem: history

- Today: a new coinductive proof (J.R. 2016, unpublished).
- Very simple, a student's exercise.
- We prove that streams **are** the same by showing that they **behave** the same.
- Cf. classical proofs use complicated bookkeeping, involving binomial coefficients and falling factorials.

2. Streams and coinduction

Streams of natural numbers

$$\begin{array}{c} \mathbb{N}^\omega \\ \downarrow \langle \text{head}, \text{tail} \rangle \\ \mathbb{N} \times \mathbb{N}^\omega \end{array}$$

where

$$\text{head}(\sigma) = \sigma(0)$$

$$\text{tail}(\sigma) = (\sigma(1), \sigma(2), \sigma(3), \dots)$$

for any stream $\sigma = (\sigma(0), \sigma(1), \sigma(2), \dots) \in \mathbb{N}^\omega$.

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which we will typically write as

$$\text{head}(\sigma) = \sigma(0)$$

$$\text{tail}(\sigma) = \sigma'$$

(initial value)

(derivative)

Finality of streams

$$\begin{array}{ccc} X & \overset{\exists! h}{\dashrightarrow} & \mathbb{N}^\omega \\ \downarrow \langle \text{out}, \text{tr} \rangle & & \downarrow \langle \text{head}, \text{tail} \rangle \\ \mathbb{N} \times X & \dashrightarrow & \mathbb{N} \times \mathbb{N}^\omega \end{array}$$

The function h , defined by

$$h(x) = (\text{out}(x), \text{out}(\text{tr}(x)), \text{out}(\text{tr}(\text{tr}(x))), \dots)$$

is the *unique* function making the diagram commute.

Streams and bisimulation

A relation $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a **stream bisimulation** if

$$\begin{array}{ccccc} \mathbb{N}^\omega & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & \mathbb{N}^\omega \\ \downarrow & & \downarrow \exists! \gamma & & \downarrow \\ \mathbb{N} \times \mathbb{N}^\omega & \xleftarrow{\quad} & \mathbb{N} \times R & \xrightarrow{\quad} & \mathbb{N} \times \mathbb{N}^\omega \end{array}$$

Equivalently, $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a bisimulation if for all $(\sigma, \tau) \in R$:

- (i) $\sigma(0) = \tau(0)$ and
- (ii) $(\sigma', \tau') \in R$

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Let $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ be a bisimulation. For all streams $\sigma, \tau \in \mathbb{N}^\omega$,

$$(\sigma, \tau) \in R \Rightarrow \sigma = \tau$$

Proof: straightforward, by showing that $\sigma(n) = \tau(n)$, for all $n \geq 0$, by induction on n . □

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Example

Define

$$\text{zip} : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$$

$$\text{even} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$$

$$\text{odd} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$$

by

$$\text{zip}(\sigma, \tau) = (\sigma(0), \tau(0), \sigma(1), \tau(1), \sigma(2), \tau(2), \dots)$$

$$\text{even}(\sigma) = (\sigma(0), \sigma(2), \sigma(4), \dots)$$

$$\text{odd}(\sigma) = (\sigma(1), \sigma(3), \sigma(5), \dots)$$

Their initial values and derivatives satisfy:

$$\text{zip}(\sigma, \tau)(0) = \sigma(0)$$

$$\text{zip}(\sigma, \tau)' = \text{zip}(\tau, \sigma')$$

$$\text{even}(\sigma)(0) = \sigma(0)$$

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A quick aside: **definitions** by coinduction

Equivalently: let the functions

$$\text{zip} : \mathbb{N}^\omega \times \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$$

$$\text{even} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$$

$$\text{odd} : \mathbb{N}^\omega \rightarrow \mathbb{N}^\omega$$

be defined by the following **stream differential equations**:

$$\text{zip}(\sigma, \tau)(0) = \sigma(0)$$

$$\text{zip}(\sigma, \tau)' = \text{zip}(\tau, \sigma')$$

$$\text{even}(\sigma)(0) = \sigma(0)$$

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Then we can show that

$$\text{zip}(\sigma, \tau) = (\sigma(0), \tau(0), \sigma(1), \tau(1), \sigma(2), \tau(2), \dots)$$

$$\text{even}(\sigma) = (\sigma(0), \sigma(2), \sigma(4), \dots)$$

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Example: a proof by coinduction

Proposition: for all $\sigma, \tau \in \mathbb{N}^\omega$, $\text{even}(\text{zip}(\sigma, \tau)) = \sigma$

Proof: we define

$$R = \{ \langle \text{even}(\text{zip}(\sigma, \tau)), \sigma \rangle \mid \sigma, \tau \in \mathbb{N}^\omega \}$$

and prove that R is a **bisimulation**. First note that

$$(i) \quad \text{even}(\text{zip}(\sigma, \tau))(0) = \text{zip}(\sigma, \tau)(0) = \sigma(0)$$

Then observe that

$$\begin{aligned} \text{even}(\text{zip}(\sigma, \tau))' &= \text{even}(\text{zip}(\sigma, \tau)') = \\ \text{even}(\text{zip}(\tau, \sigma'))' &= \text{even}(\text{zip}(\sigma', \tau')) \end{aligned}$$

which implies: $(ii) \quad \langle \text{even}(\text{zip}(\sigma, \tau))', \sigma' \rangle \in R.$ □

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3. Formalising Moessner's Theorem

Moessner's theorem ($k = 3$)

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Formalising Moessner's theorem ($k = 3$)

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 (\text{nat})$$

On the left, we have:

$$\text{nat} = (1, 2, 3, \dots)$$

$$\text{nat}^3 = (1^3, 2^3, 3^3, \dots) = \text{nat} \odot \text{nat} \odot \text{nat}$$

with

$$\sigma \odot \tau = (\sigma(0) \cdot \tau(0), \sigma(1) \cdot \tau(1), \sigma(2) \cdot \tau(2), \dots)$$

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On the right, we have:

$$\Sigma \sigma = (\sigma(0), \sigma(0) + \sigma(1), \sigma(0) + \sigma(1) + \sigma(2), \dots)$$

$$D_2 \sigma = (\sigma(0), \sigma(2), \sigma(4), \dots)$$

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A more convenient formulation

$$\begin{aligned}\text{nat}^3 &= \Sigma \circ D_2 \circ \Sigma \circ D_3 (\text{nat}) \\ &= \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4 (\bar{1})\end{aligned}$$

where

$$\bar{1} = (1, 1, 1, \dots)$$

since

$$\begin{aligned}\Sigma \circ D_4 (\bar{1}) &= \Sigma \circ D_4 (1, 1, 1, \dots) \\ &= \Sigma (1, 1, 1, \dots) \\ &= (1, 1 + 1, 1 + 1 + 1, \dots) \\ &= (1, 2, 3, \dots) \\ &= \text{nat}\end{aligned}$$

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4. Proving Moessner's Theorem

A proof by coinduction

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4 (\bar{1})$$

The aim is to construct a **bisimulation** relation containing the pair

$$\langle \text{nat}^3, \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4 (\bar{1}) \rangle$$

Towards that end, let us investigate the **derivatives** of the streams and operators above.

(**Initial values** will all be straightforward.)

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Inspecting derivatives

For the stream $\text{nat} = (1, 2, 3, \dots)$, we have

$$\begin{aligned}\text{nat}' &= (2, 3, 4, \dots) \\ &= (1 + 1, 1 + 2, 1 + 3, \dots) \\ &= (1, 1, 1, \dots) \oplus (1, 2, 3, \dots) \\ &= \bar{1} \oplus \text{nat}\end{aligned}$$

where \oplus denotes the elementwise sum of streams.

Inspecting derivatives

For the product $\sigma \odot \tau$, we have

$$\begin{aligned}(\sigma \odot \tau)' &= (\sigma(0) \cdot \tau(0), \sigma(1) \cdot \tau(1), \sigma(2) \cdot \tau(2), \dots)' \\ &= (\sigma(1) \cdot \tau(1), \sigma(2) \cdot \tau(2), \sigma(3) \cdot \tau(3), \dots) \\ &= \sigma' \odot \tau'\end{aligned}$$

Inspecting derivatives

These properties of nat' and $(\sigma \odot \tau)'$ imply:

$$\begin{aligned}(\text{nat}^3)' &= (\text{nat} \odot \text{nat} \odot \text{nat})' \\ &= \text{nat}' \odot \text{nat}' \odot \text{nat}' \\ &= (\bar{1} \oplus \text{nat}) \odot (\bar{1} \oplus \text{nat}) \odot (\bar{1} \oplus \text{nat}) \\ &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cdot \bar{1} \oplus \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \text{nat} \oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \text{nat}^2 \oplus \begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdot \text{nat}^3\end{aligned}$$

using some elementary properties of \oplus and \odot ,
and defining $k \cdot \sigma$ by

$$k \cdot \sigma = (k \cdot \sigma(0), k \cdot \sigma(1), k \cdot \sigma(2), \dots)$$

Inspecting derivatives

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4 (\bar{1})$$

So for the stream on the left, we have:

$$(\text{nat}^3)' = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cdot \bar{1} \oplus \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \text{nat} \oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \text{nat}^2 \oplus \begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdot \text{nat}^3$$

Inspecting derivatives

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4 (\bar{1})$$

Turning to the right hand side, we observe:

$$\bar{1}' = \bar{1}$$

Inspecting derivatives

For the drop operators, we have

$$\begin{aligned}(D_2 \sigma)' &= (\sigma(0), \sigma(2), \sigma(4), \dots)' \\ &= (\sigma(2), \sigma(4), \sigma(6), \dots) \\ &= D_2 \sigma''\end{aligned}$$

And, similarly,

$$\begin{aligned}(D_3 \sigma)^{(2)} &= D_3 \sigma^{(3)} \\ (D_4 \sigma)^{(3)} &= D_4 \sigma^{(4)}\end{aligned}$$

where the repeated derivatives are defined as usual:

$$\begin{aligned}\sigma^{(0)} &= \sigma \\ \sigma^{(k+1)} &= (\sigma^{(k)})'\end{aligned}$$

Inspecting derivatives

For the drop operators, we have

$$\begin{aligned}(D_2 \sigma)' &= (\sigma(0), \sigma(2), \sigma(4), \dots)' \\ &= (\sigma(2), \sigma(4), \sigma(6), \dots) \\ &= D_2 \sigma''\end{aligned}$$

And, similarly,

$$\begin{aligned}(D_3 \sigma)^{(2)} &= D_3 \sigma^{(3)} \\ (D_4 \sigma)^{(3)} &= D_4 \sigma^{(4)}\end{aligned}$$

where the repeated derivatives are defined as usual:

$$\begin{aligned}\sigma^{(0)} &= \sigma \\ \sigma^{(k+1)} &= (\sigma^{(k)})'\end{aligned}$$

Inspecting derivatives

$$\begin{aligned}(\Sigma \sigma)' &= (\sigma(0), \sigma(0) + \sigma(1), \sigma(0) + \sigma(1) + \sigma(2), \dots)' \\ &= (\sigma(0) + \sigma(1), \sigma(0) + \sigma(1) + \sigma(2), \dots) \\ &= (\sigma(0), \sigma(0), \sigma(0), \dots) \oplus \\ &\quad (\sigma(1), \sigma(1) + \sigma(2), \sigma(1) + \sigma(2) + \sigma(3), \dots) \\ &= \overline{\sigma(0)} \oplus \Sigma(\sigma')\end{aligned}$$

where

$$\overline{\sigma(0)} = (\sigma(0), \sigma(0), \sigma(0), \dots)$$

Inspecting derivatives

Together, these properties imply:

$$\begin{aligned} & (\Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1}))' \\ &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \cdot \bar{1} \\ &\oplus \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \Sigma \circ D_2(\bar{1}) \\ &\oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3(\bar{1}) \\ &\oplus \begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1}) \end{aligned}$$

(The details would fill 1 or 2 additional slides.)

Inspecting derivatives

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Proving **Moessner's** theorem ($k = 3$)

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1})$$

All in all, we have found:

$$\begin{array}{ll} (\text{nat}^3)' & (\Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1}))' \\ = \binom{3}{0} \cdot \bar{1} & = \binom{3}{0} \cdot \bar{1} \\ \oplus \binom{3}{1} \cdot \text{nat} & \oplus \binom{3}{1} \cdot \Sigma \circ D_2(\bar{1}) \\ \oplus \binom{3}{2} \cdot \text{nat}^2 & \oplus \binom{3}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3(\bar{1}) \\ \oplus \binom{3}{3} \cdot \text{nat}^3 & \oplus \binom{3}{3} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1}) \end{array}$$

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All in all, we have found:

$(\text{nat}^3)'$	$(\Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1}))'$	M3'
$= \binom{3}{0} \cdot \bar{1}$	$= \binom{3}{0} \cdot \bar{1}$	M0
$\oplus \binom{3}{1} \cdot \text{nat}$	$\oplus \binom{3}{1} \cdot \Sigma \circ D_2(\bar{1})$	M1
$\oplus \binom{3}{2} \cdot \text{nat}^2$	$\oplus \binom{3}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3(\bar{1})$	M2
$\oplus \binom{3}{3} \cdot \text{nat}^3$	$\oplus \binom{3}{3} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4(\bar{1})$	M3

Moessner's theorem: the general case

$$\text{nat}^k = \Sigma \circ D_2 \circ \dots \circ \Sigma \circ D_{k+1}(\bar{1})$$

$(\text{nat}^k)'$	$(\Sigma \circ D_2 \circ \dots \circ \Sigma \circ D_{k+1}(\bar{1}))'$	Mk'
$= \binom{k}{0} \cdot \bar{1}$	$= \binom{k}{0} \cdot \bar{1}$	M0
$\oplus \binom{k}{1} \cdot \text{nat}^1$	$\oplus \binom{k}{1} \cdot \Sigma \circ D_2(\bar{1})$	M1
$\oplus \binom{k}{2} \cdot \text{nat}^2$	$\oplus \binom{k}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3(\bar{1})$	M2
$\oplus \dots$	$\oplus \dots$...
$\oplus \binom{k}{k} \cdot \text{nat}^k$	$\oplus \binom{k}{k} \cdot \Sigma \circ D_2 \circ \dots \circ \Sigma \circ D_{k+1}(\bar{1})$	Mk

Moessner's theorem: the general case

And so we define $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ by

$$R = \{ \langle \text{nat}^k, \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}(\bar{1}) \rangle \mid k \geq 0 \}$$

Is R a **bisimulation relation**?

No, but almost: R is a bisimulation relation **up to sum**!

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Bisimulations up to sum

A relation $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a bisimulation relation **up to sum** if, for all $(\sigma, \tau) \in R$,

- (i) if $(\sigma, \tau) \in R$ then $\sigma(0) = \tau(0)$
- (ii) there are $n_1, \dots, n_l \in \mathbb{N}$ and $\sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_l \in \mathbb{N}^\omega$ such that

$$\sigma' = n_1 \cdot \sigma_1 \oplus \dots \oplus n_l \cdot \sigma_l$$

$$\tau' = n_1 \cdot \tau_1 \oplus \dots \oplus n_l \cdot \tau_l$$

and

$$(\sigma_1, \tau_1) \in R, \dots, (\sigma_l, \tau_l) \in R$$

Coinduction up to sum

Theorem

Let $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ be a bisimulation **up to sum**.

$$\forall \sigma, \tau \in \mathbb{N}^\omega : (\sigma, \tau) \in R \Rightarrow \sigma = \tau$$

Proof: We define $R^c \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ as the smallest relation s.t.

1. $R \subseteq R^c$
2. if $(\sigma, \tau) \in R^c$ then $(n \cdot \sigma, n \cdot \tau) \in R^c$ (all $n \in \mathbb{N}$)
3. if $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in R^c$ then $(\sigma_1 \oplus \sigma_2, \tau_1 \oplus \tau_2) \in R^c$

It is easy to see that R^c is an (ordinary) bisimulation.

Now the theorem follows by (ordinary) coinduction. □

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$$R = \{ \langle \text{nat}^k, \Sigma \circ D_2 \circ \dots \circ \Sigma \circ D_{k+1}(\bar{1}) \rangle \mid k \geq 0 \}$$

is a bisimulation up to sum:

$(\text{nat}^k)'$	$(\Sigma \circ D_2 \circ \dots \circ \Sigma \circ D_{k+1}(\bar{1}))'$	Mk'
$= \binom{k}{0} \cdot \bar{1}$	$= \binom{k}{0} \cdot \bar{1}$	M0
$\oplus \binom{k}{1} \cdot \text{nat}^1$	$\oplus \binom{k}{1} \cdot \Sigma \circ D_2(\bar{1})$	M1
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It follows by coinduction up to sum that

$$\text{nat}^k = \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}(\bar{1})$$

for all $k \geq 0$.



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5. The heart of the matter: circularity

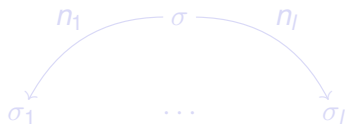
Derivatives in a picture

$$\sigma \longrightarrow \sigma' \longrightarrow \sigma^{(2)} \longrightarrow \sigma^{(3)} \longrightarrow \dots$$

More generally, if

$$\sigma' = n_1 \cdot \sigma_1 \oplus \dots \oplus n_l \cdot \sigma_l$$

then we will write



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then we will write

$$\begin{array}{ccc} & \sigma & \\ n_1 \swarrow & & \searrow n_l \\ \sigma_1 & \dots & \sigma_l \end{array}$$

Circularity

Since

$$\bar{1}' = (1, 1, 1, \dots)' = \bar{1}$$

we write:

$$\bar{1} \longrightarrow \bar{1} \longrightarrow \bar{1} \longrightarrow \dots$$

or, equivalently,

$$\bar{1} \curvearrowright$$

Circularity

Since for the stream $\text{nat} = (1, 2, 3, \dots)$, we have

$$\begin{aligned}\text{nat}' &= (2, 3, 4, \dots) \\ &= (1 + 1, 1 + 2, 1 + 3, \dots) \\ &= (1, 1, 1, \dots) \oplus (1, 2, 3, \dots) \\ &= \bar{1} \oplus \text{nat} \quad (\text{algebra and coalgebra!})\end{aligned}$$

we have

$$\text{nat} \longrightarrow \bar{1} \oplus \text{nat} \longrightarrow \bar{1} \oplus \bar{1} \oplus \text{nat} \longrightarrow \dots$$

and, equivalently,

$$1 \begin{array}{c} \curvearrowright \\ \text{nat} \end{array} \xrightarrow{1} \bar{1} \begin{array}{c} \curvearrowleft \\ \end{array} 1$$

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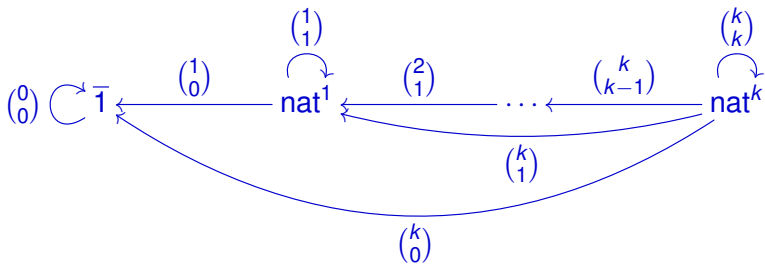
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Circularity

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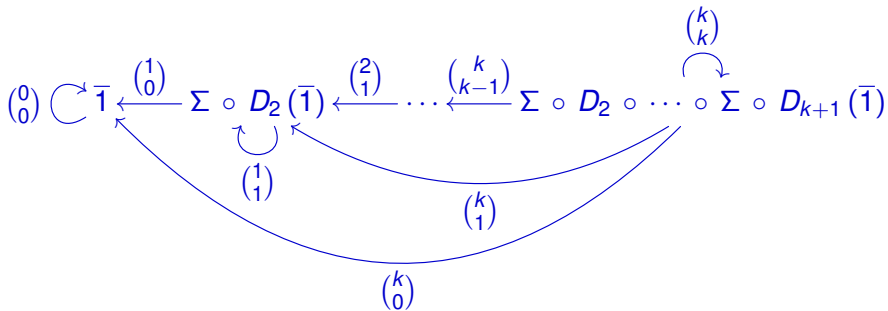
$$(\text{nat}^k)' = \binom{k}{0} \cdot \bar{1} \oplus \binom{k}{1} \cdot \text{nat}^1 \oplus \dots \oplus \binom{k}{k} \cdot \text{nat}^k$$

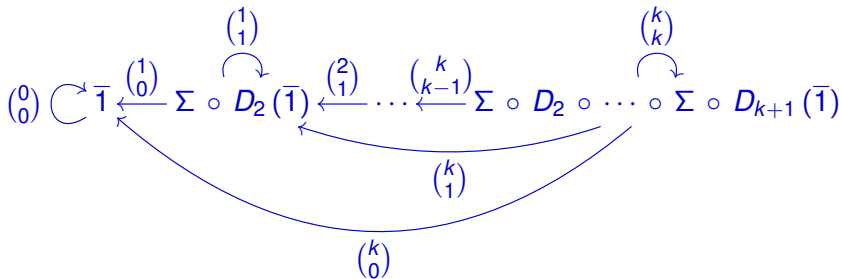
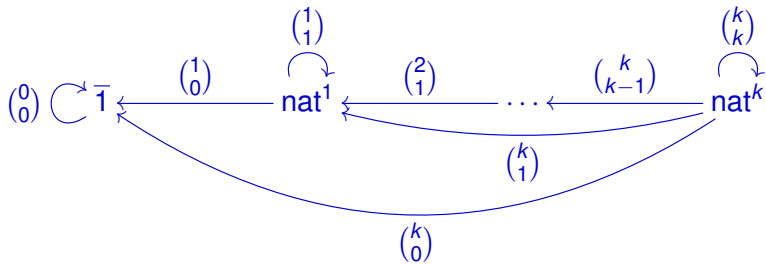
we have



Circularity

And similarly, we have found





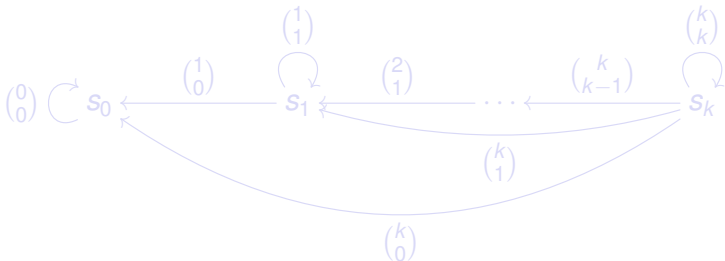
The proof of Moessner, in other words

$$\text{nat}^k = \Sigma \circ D_2 \circ \dots \circ \Sigma \circ D_{k+1} (\bar{1})$$

Both streams **are** the same ...

because they **behave** the same ...

because they are represented by:



the same **weighted automaton**.

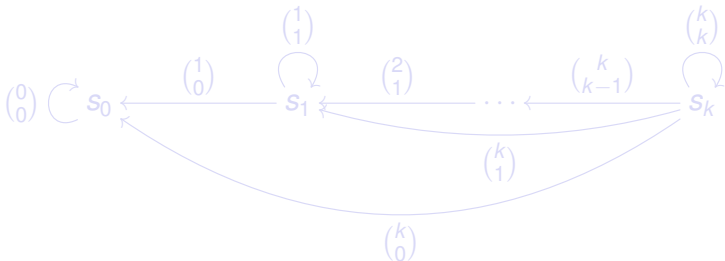
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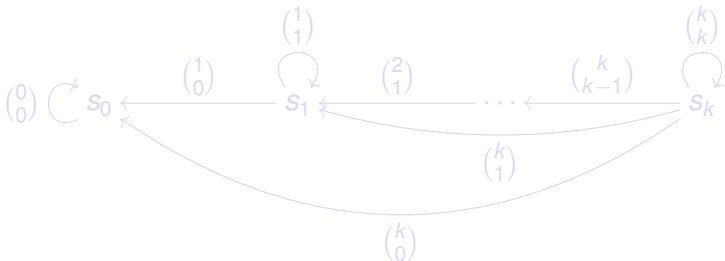
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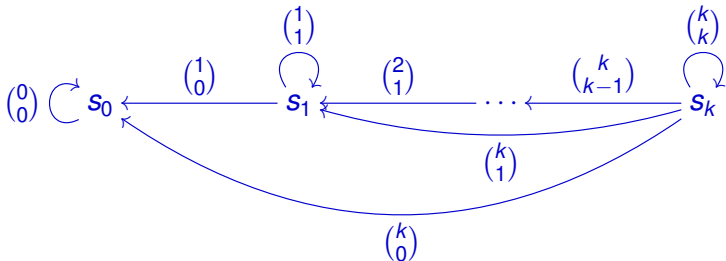
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6. Paasche's Theorem

Dropping at *increasing* distance as in

$$\text{drop}(1, 2, 3, 4, 5, 6, 7, 8, 9, \dots) = (1, \cancel{2}, 3, 4, \cancel{5}, 6, 7, 8, \cancel{9}, \dots)$$

	1	2	3	4	5	6	7	8	9	10	11	...
<i>der</i>	2	3	4	5	6	7	8	9	10	11	12	...
<i>drop</i>	2	4	5	7	8	9	11	12	13	14	16	...
Σ	2	6	11	18	26	35	46	58	71	85	101	...
<i>der</i>	6	11	18	26	35	46	58	71	85	101	...	
<i>drop</i>	6	18	26	46	58	...						
Σ	6	24	50	96	...							
<i>der</i>	24	50	96	...								
<i>drop</i>	24	96	...									
Σ	24	120	...									
<i>der</i>	120	...										
	⋮											

	1	2	3	4	5	6	7	8	9	10	11	...
<i>der</i>	2	3	4	5	6	7	8	9	10	11	12	...
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<i>der</i>	6	11	18	26	35	46	58	71	85	101	...	
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Σ	6	24	50	96	...							
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Σ	24	120	...									
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	⋮											

	1	2	3	4	5	6	7	8	9	10	11	...
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Σ	24	120	...									
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<i>der</i>	120	...										
	⋮											

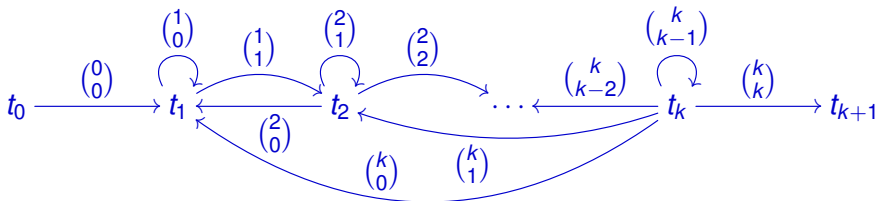
	1	2	3	4	5	6	7	8	9	10	11	...
<i>der</i>	2	3	4	5	6	7	8	9	10	11	12	...
<i>drop</i>	2	4	5	7	8	9	11	12	13	14	16	...
Σ	2	6	11	18	26	35	46	58	71	85	101	...
<i>der</i>	6	11	18	26	35	46	58	71	85	101	...	
<i>drop</i>	6	18	26	46	58	...						
Σ	6	24	50	96	...							
<i>der</i>	24	50	96	...								
<i>drop</i>	24	96	...									
Σ	24	120	...									
<i>der</i>	120	...										
	⋮											

	1	2	3	4	5	6	7	8	9	10	11	...
<i>der</i>	2	3	4	5	6	7	8	9	10	11	12	...
<i>drop</i>	2	4	5	7	8	9	11	12	13	14	16	...
Σ	2	6	11	18	26	35	46	58	71	85	101	...
<i>der</i>	6	11	18	26	35	46	58	71	85	101	...	
<i>drop</i>	6	18	26	46	58	...						
Σ	6	24	50	96	...							
<i>der</i>	24	50	96	...								
<i>drop</i>	24	96	...									
Σ	24	120	...									
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	⋮											

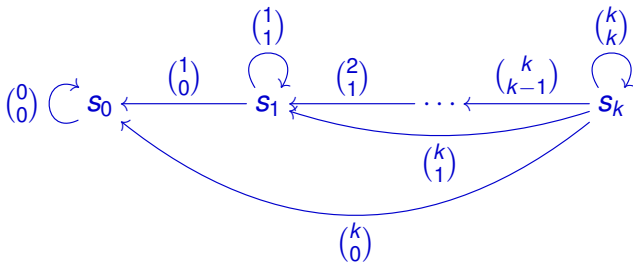
	1	2	3	4	5	6	7	8	9	10	11	...
<i>der</i>	2	3	4	5	6	7	8	9	10	11	12	...
<i>drop</i>	2	4	5	7	8	9	11	12	13	14	16	...
Σ	2	6	11	18	26	35	46	58	71	85	101	...
<i>der</i>	6	11	18	26	35	46	58	71	85	101	...	
<i>drop</i>	6	18	26	46	58	...						
Σ	6	24	50	96	...							
<i>der</i>	24	50	96	...								
<i>drop</i>	24	96	...									
Σ	24	120	...									
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	1	2	3	4	5	6	7	8	9	10	11	...
<i>der</i>	2	3	4	5	6	7	8	9	10	11	12	...
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	⋮											

The proof of Paasche



Cf. the picture for Moessner:



7. Discussion

- We take streams σ as **basic entities**, instead of focussing on their individual **elements** $\sigma(n)$.
- This prevents lots of unnecessary bookkeeping (cf. binomial coefficients).
- The (final) coalgebra structure of the set of streams has a natural interpretation in terms of a **calculus**, in analogy to classical calculus.
- There is initial evidence that this leads to efficient proofs that can be easily automated.

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