Dichotomies in Ontology-Mediated Querying with the Guarded Fragment

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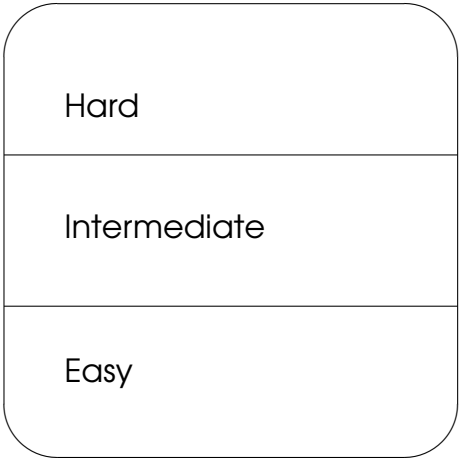
Based on joint work with A. Hernich, C. Lutz and F. Papacchini (PODS 2017)

Dichotomy Theorems

Given a class of problems, we would like to classify them into the hard and the easy problems. Ideally, there shouldn't be any intermediate problems.

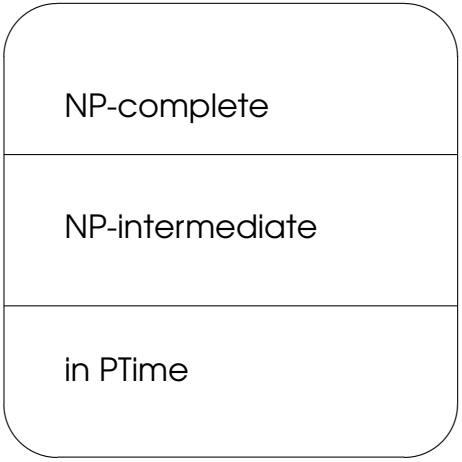


We focus on P/NP Dichotomy Theorems

By Ladner's Theorem, there are NP-intermediate problems (if P≠NP).

Moreover, being in P is undecidable for problems in NP (if P≠NP).

Thus, we can expect P/NP dichotomy theorems only for rather restricted classes of problems.



Homomorphism (or CSP) Problems

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Hell and Nesetril (1990): This problem is in PTime iff H contains a self-loop or is bipartite. Otherwise this problem is NP-complete.

Generalization to Relational Structures (CSP)

Let $\mathcal{H}$ be a finite relational structure (also called template). The constraint satisfaction problem for $\mathcal{H}$,

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Lots of progress over the past 20 years (mainly due to algebraic reformulation):

- Early result: There is a P/NP Dichotomy for CSPs with two elements (Schaefer 1978).
- Example: There is a P/NP Dichotomy for CSPs with three elements (Bulatov 2006).

Ontology Mediated Querying of Data (Example 1)

- **Data** $\mathcal{D}$: finite set of ground atoms (often regarded as finite relational structure); e.g.,

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- One can do this for every CSP($\mathcal{H}$).

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We also consider the **2-variable guarded fragment of FO with counting (GC_2)**:

$$\forall x(x = x \to (\exists^{\geq 200} y\ \textbf{author\_of}(x, y) \to \textbf{ProlificAuthor}(x)))$$

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Slightly misleading!

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For data complexity coNP-hardness is very bad news! It has become a huge industry to determine fragments of GF and GC_2 in PTime (or even better FO or datalog rewritable fragments)

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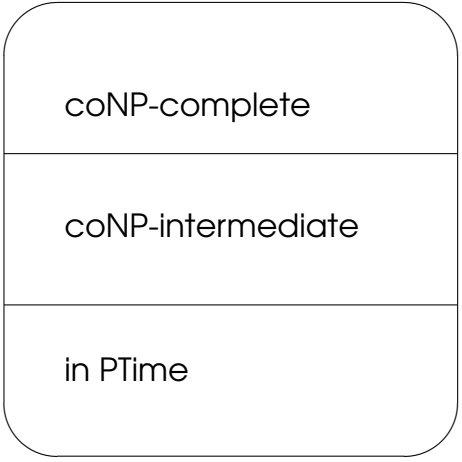
P/coNP Dichotomy Theorems for Ontology Mediated Querying in GF and GC_2

- $\mathcal{O}$ **is in PTime** if for every conjunctive query q deciding $\mathcal{O} \cup \mathcal{D} \models q$ is in PTime in data complexity.
- $\mathcal{O}$ **is coNP-hard** if there exists a conjunctive query q such that deciding $\mathcal{O} \cup \mathcal{D} \models q$ is coNP-hard in data complexity.



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The **depth** of a uGF or uGC_2 sentence is the number of nestings of guarded quantifiers without the outermost universal guarded quantifier. The following sentence has depth 1:

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385 out of 411 ontologies in the **Bioportal** repository are in GC_2^-(depth 1)

Illustration (all in $\mathsf{GC}_2^-(1)$)

- The ontology

$$\mathcal{O}_1 = \{\forall x(\exists^{\geq 200} y\ \textbf{author\_of}(x, y) \rightarrow \textbf{ProlificWriter}(x))\}$$

  is in PTime.

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- The ontology

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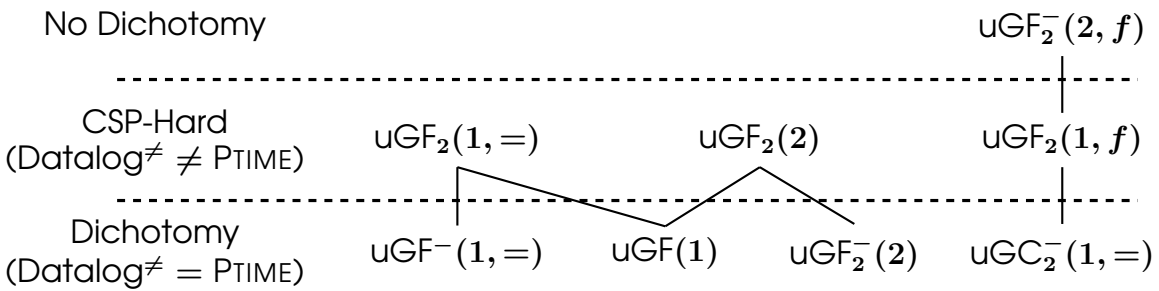
  is again in PTime.

Summary of Results



Number in brackets indicates

- depth,
- f presence of partial functions,
- $\cdot_2$ restriction to two variables,
- $\cdot^-$ restricts outermost guards to be equality.

Necessary condition for PTime: Materializability

An ontology $\mathcal{O}$ is **materializable** if for every $\mathcal{D}$ there exists a model $\mathcal{A}$ of $\mathcal{O} \cup \mathcal{D}$ such that for all conjunctive queries q:

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Let $\mathcal{O}$ be an FO ontology invariant under disjoint unions. If $\mathcal{O}$ is not materializable, then $\mathcal{O}$ is coNP-hard.

Materializability is not a sufficient for PTime

We construct a materializable ontology from the ontology $\mathcal{O}$ encoding three-colorability:

- $\forall x\ (\mathbf{red}(x) \vee \mathbf{blue}(x) \vee \mathbf{green}(x))$.
- $\forall x\ (\mathbf{red}(x) \wedge \boldsymbol{E}(x, y) \wedge \mathbf{red}(y) \rightarrow \mathbf{clash}(x))$ (same for **blue, green**).

Clearly $\mathcal{O}$ itself is not materializable.

Replace $\mathbf{red}(x)$, $\mathbf{blue}(x)$, and $\mathbf{green}(x)$ by complex formulas that are not directly visible to conjunctive queries, e.g.

$$\exists y(\boldsymbol{R}_{\mathbf{red}}(x, y) \wedge \forall z(\boldsymbol{S}_{\mathbf{red}}(y, z) \rightarrow \mathbf{red}(z)))$$

Then the resulting ontology is still coNP-hard but materializable.

A sufficient condition for PTime (even datalog-rewritability)

Every relational structure $\mathcal{D}$ can be unravelled into a

guarded tree-decomposable

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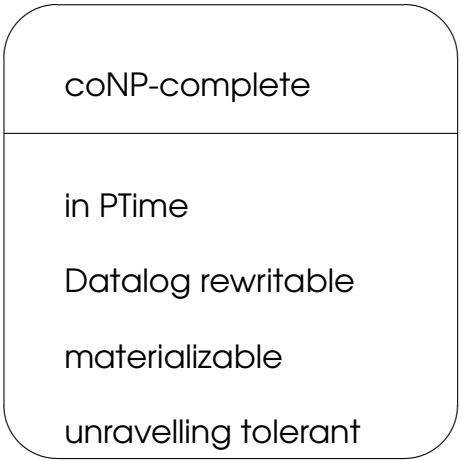
Let $\mathcal{O}$ be a uGF or uGC_2 ontology. If $\mathcal{O}$ is unravelling tolerant, then $\mathcal{O}$ is in PTime (actually datalog-rewritable).

The Dichotomy Theorem

Let $\mathcal{O}$ be in any of the languages uGF$^-(1,=)$, uGF(1), uGF$_2^-(2)$, uGC$_2^-(1,=)$. Then we have the following classification:



Undecidability and Non-Dichotomy

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For $\mathrm{uGF}_2^-(2, f)$ ontologies it is undecidable whether they are in PTime and whether they are coNP-hard (unless P=NP). Materializability and datalog rewritability are undecidable.

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Using this result we show:

For $\mathrm{uGF}_2^-(2, f)$ ontologies there is no P/coNP dichotomy (unless P=NP).

Problems

- Is there a P/coNP dichotomy for uGF? Many smaller steps...

- Decidability: assume we have a dichotomy for a class of ontologies. Is it decidable whether an ontology $\mathcal{O}$ from the class is in PTime?