POWERSET POSSIBILIZATION AND VIETORIS POSSIBILIZATION

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It is well known that every Kripke frame can be represented as a coalgebra for the powerset functor on Set and that every descriptive frame (topological Kripke frame) can be represented as a coalgebra for the Vietoris functor on the category Stone of Stone spaces (compact Hausdorff spaces with a basis of clopen sets) and continuous maps [1, 2, 3, 7]. For applications of this approach to modal and coalgebraic logics we refer to, e.g., [9, 10]. From this point of view, the Vietoris functor is the topological analogue of the powerset functor. In our work, we will highlight yet another logical export of this connection between the powerset and Vietoris functors. In particular, we will investigate the possibility semantics for modal logic and the construction of the powerset possibilization and that of the Vietoris possibilization.

Possibility semantics for modal logic [6, 4, 5, 8, 11] is a generalization of Kripke semantics, based on partially ordered sets of region-like "possibilities" instead of only point-like "worlds." Any Kripke frame can be turned into a semantically equivalent possibility frame in two basic ways. First, every Kripke frame can be viewed as a degenerate possibility frame based on a discrete partial order. Second and more interestingly, every Kripke frame can be turned into a possibility frame by an operation of *powerset possibilization* that builds possibilities as sets of worlds and lifts accessibility relations between worlds to relations or partial functions on the powerset. In [8], this powerset possibilization construction was used to prove syntactical results on translations between standard modal logics and dynamic topological logics.

In this paper, we address two questions (see Figure 1). First, what is the possibility-semantic analogue of Esakia's [2] *topological Kripke frames*? Second, what is the operation from topological Kripke frames to their possibility-semantic analogues, which is analogous to powerset possibilization? We answer these questions with the notions of *topological possibility frames* and *Vietoris possibilization* (see Figure 2).

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$$
\begin{array}{ccc}
\text{Kripke frames} & \xrightarrow{\wp\text{-possibilization}} & \text{possibility frames} \\
\text{topological Kripke frames} & \xrightarrow{?} & ?
\end{array}
$$

FIGURE 1

In addition, we characterize the possibility frames that arise from Kripke frames by powerset possibilization and the topological possibility frames that arise from topological Kripke frames by Vietoris possibilization.

$$
\text{topological Kripke frame} \xrightarrow{V\text{-possibilization}} \text{topological possibility frame}
$$

FIGURE 2

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