Expressivity of Many-valued Modal Logics, Coalgebraically

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The abstract theory of coalgebras has recently become one of the most important bridges connecting modal logic and computer science: from a logician's point of view it provides techniques and a new level of generality for studying various modal logics, while from a computer-scientist's point of view it provides a general framework for designing expressive modal languages describing behavior of abstract transition systems modeled as coalgebras. It is natural to ask what benefits a coalgebraic approach brings to study of many-valued modal logics: from a logician's point of view we can generalize logics of many-valued Kripke-style relational semantics [1, 4], where valuations and the accessibility relation take values in a given algebra $\mathscr{V}$, to the coalgebraic level, while from a computer-scientist's point of view we generalize coalgebraic logics to the many-valued setting, allowing for many-valued observable phenomena to be captured by modal languages with genuinely many-valued semantics. The notion of behavioral equivalence is central in studying coalgebras, and it can often be captured by bisimilarity, a central notion in model theory of modal logics. In particular, for coalgebras with a finitary type of behaviour, we are interested in finitary modal languages being expressive for bisimilarity, i.e., logics satisfying the Hennessy-Milner property.

We adopt the approach based on understanding modalities as predicate liftings and apply it in a many-valued setting. Such languages for classical coalgebraic logics were developed and their expressivity investigated by Pattinson in [6, 5] and further by Schröder in [7]. In particular, a sufficient condition on a set of predicate liftings, namely being separating, is given to ensure that the resulting modal logic is expressive for behavioral equivalence, respectively for bisimilarity, depending on the setting. We address the limitative results of Metcalfe and Martí [4] providing a sufficient and necessary condition on the algebra of truth values $\mathscr{V}$ ensuring the Hennessy-Milner property for the $\mathscr{V}$-valued modal language with box and diamond over image-finite Kripke frames with two-valued accessibility relation, where $\mathscr{V}$ is an MTL-chain. The condition they provide says that we can distinguish truth values in $\mathscr{V}$ with propositional formulas. Therefore, in contrast to the boolean case, also expressivity of the purely propositional part of the language matters. If we want to avoid including constants for all truth values of $\mathscr{V}$ in the language, the condition rules out many interesting fuzzy modal logics: for $\mathscr{V}$ being a complete BL-chain with finite universe or $[0, 1]$, this yields expressivity if and *only if* $\mathscr{V}$ is a MV-chain or the ordinal sum of two (hoop reducts of) MV-chains, leaving out most Gödel modal logics.

We shall apply the approach of [7] to generalize the results of [4]. In particular, we also address logics of Kripke frames with many-valued accessibility relation, probabilistic Kripke frames and extend the negative results on Gödel logics. On a positive side, we can provide a countable expressive language for Kripke frames and any MTL chain as the algebra $\mathscr{V}$ of truth values.

The setting: We fix a *standard finitary* endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ and consider coalgebras for T as many-valued models. As a propositional base we assume for simplicity and sake of our examples the language of Full Lambek calculus (with exchange and weakening) [2]. As the algebra of truth values we therefore fix a commutative integral residuated lattice $\mathscr{V}$. The semantics is computed in the lattice $\mathscr{V}$[1]: Given a coalgebra $c : X \to TX$ and a valuation of atoms $\|.\|_c : At \to [X, \mathscr{V}]$, the value $\| * (a_1, \ldots, a_n)\|_c$ is computed inductively for each n-ary connective $*$, as

$$X \xrightarrow{\overrightarrow{\|a\|_c}} \mathscr{V}^n \xrightarrow{*_{\mathscr{V}}} \mathscr{V}.$$

Locally, the semantics can be seen as a $\mathscr{V}$-valued relation $x \Vdash_c a \ = \ \|a\|_c(x)$. A relation $B \subseteq X \times Y$ is a T-*bisimulation* between $c : X \to TX$ and $d : Y \to TY$ iff there is a coalgebra structure $b : B \to TB$

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1 We would like to stress that we do not include constants for elements of $\mathscr{V}$ in the language

which makes the projections $p_0 : B \to X$ and $p_1 : B \to Y$ into coalgebra morphisms. Two states $x \in X$ and $y \in Y$ in coalgebras c and d are T-*bisimilar* if there exists a T-bisimulation $B \subseteq X \times Y$ such that $B(x, y)$ holds, and moreover the atomic harmony, $x' \Vdash_c p = y' \Vdash_d p$, holds for all atoms $p \in At$, and all $x'By'$ The $\mathscr{V}$-valued modal language for T-coalgebras is given via extending the propositional language by a set of modalities. Modalities can arise semantically as an abstract way of lifting predicates on X (maps in $[X, \mathscr{V}]$) to predicates on TX (maps in $[TX, \mathscr{V}]$). Following ideas of [6, 7] we define $\mathscr{V}$-valued n-ary *predicate liftings* to be maps: $\hat{\heartsuit}_X : [X, \mathscr{V}^n] \to [TX, \mathscr{V}]$, natural in X. n-ary predicate liftings are essentially the same things as the maps $\heartsuit : T\mathscr{V}^n \to \mathscr{V}$, which we will call the n-ary modalities. We extend the propositional language with a set of such modalities (possibly all), stating that whenever $a_1, \ldots, a_n$ are formulas and $\heartsuit$ is an n-ary modality, $\heartsuit(a_1, \ldots, a_n)$ is a formula. For a set Λ of modalities, we denote by $\mathscr{L}(\Lambda)$ the resulting modal language. On a coalgebra $c : X \to TX$ with valuations $\|a_i\|_c : X \to \mathscr{V}$ the formula $\heartsuit(a_1, \ldots, a_n)$ is interpreted as follows (cf. [3]):

$$X \xrightarrow{c} TX \xrightarrow{T\|\vec{a}\|} T(\mathscr{V}^n) \xrightarrow{\heartsuit} \mathscr{V}$$

By a routine induction we can see that the resulting language is invariant under bisimulations, it is *adequate*. To prove *expressivity* of such languages, we have to ensure first that we have enough modalities to separate different behaviours of coalgebras in question[2]. Second, we have to see that also the propositional language is expressive enough to handle the modalities We therefore need to generalize the separation condition of [7] to take into account also the algebra $\mathscr{V}$. For simplicity, we concentrate on unary (monadic) predicate liftings.

Definition 1 ($\mathscr{V}$-separation). *We call a function $f : \mathscr{V}^n \to \mathscr{V}$* expressible*, if there is a term σ in n variables in the language of $\mathscr{V}$, such that $\sigma[x_1, \ldots, x_n/v_1, \ldots, v_n] = f(v_1, \ldots, v_n)$.*

We call a set of predicate liftings Λ $\mathscr{V}$-separating*, if the collection of expressible functions separates values in $T\mathscr{V}^n$, i.e., the following condition holds: $t \neq t'$ in $T\mathscr{V}^n$ implies there exists $f : \mathscr{V}^n \to \mathscr{V}$ expressible, and $\heartsuit \in \Lambda$ such that*

$$\heartsuit(Tf)(t) \neq \heartsuit(Tf)(t').$$

Theorem 1 (Expressivity). *Let T be finitary, w.p.p., and Λ a $\mathscr{V}$-separating set of predicate liftings. Then $\mathscr{L}(\Lambda)$ is expressive for bisimilarity.*

In particular, the modal logic of P_ω coalgebras based on a $\mathscr{V}$-separating Λ is expressive for bisimilarity. In particular, modal logic of $\{\Box, \Diamond\}$ for $\mathscr{V}$ being an MV algebra, $\mathscr{V} = 2$, or $\mathscr{V} = G_3$ is expressive. The condition on $\mathscr{V}$-separation in this case is not only sufficient, but also necessary. To apply the theory to a few particular examples, we can e.g. show that (i) there is no monadic modal logic expressive for P_ω coalgebras and $\mathscr{V}$ being a Gödel chain bigger then 3, but there is an expressive language for such chains (or any MTL chain) if one allows for modalities of unbounded arities; (ii) the results of [4] about expressivity of Łukasziewicz's logics with box and diamond extend to the $\mathscr{V}$-valued finitary powerset functor; and (iii) we can present an expressive language for D_ω coalgebras based on a finite set of modalities and the standard Łukasziewicz algebra.

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[2] Such sets of modalities are in the boolean case of $\mathscr{V} = 2$ called *separating* [7]. In case that $\mathscr{V} = 2$ separation is in fact sufficient for expressivity.