

# Lindenbaum lemma for infinitary logics

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**Abstract.** Lindenbaum lemma is one of the crucial results in algebraic logic. It is however limited to logics with Hilbert style axiomatization with finitary rules only. In this contribution we prove a (slightly weaker) variant of this lemma for a wide class of infinitary logics and demonstrate its application for a variety of such logics studied in the literature.

Lindenbaum lemma says that for any finitary logic  $\vdash$  (i.e., a finitary substitution-invariant consequence relation over the set of formulas of a given language) each theory (i.e., a set of formulas closed under  $\vdash$ ) not containing a formula  $\varphi$  can be extended into a maximal theory not containing  $\varphi$ . The lemma is crucial for the proof of completeness theorem with respect to more meaningful algebraic semantics (e.g., that based on the two-valued Boolean algebra in classical logic, that based on linearly ordered algebras in semilinear logics, and that based on subdirectly irreducible logical matrices in general).

It is well known that the set  $\text{Th}(\vdash)$  of all theories of a given (finitary) logic  $\vdash$  forms an (algebraic) closure system, in which the theories which are maximal among those not containing some formula are exactly the meet-irreducible ones. Thus the Lindenbaum lemma can be formally stated as follows (actually formulated like this it holds for an arbitrary algebraic closure system):

**Theorem 1 (Lindenbaum lemma).** *Let  $\vdash$  be a finitary logic. Then meet-irreducible theories form a basis of  $\text{Th}(\vdash)$ .*

Finitarity is crucial for the proof of the lemma: it entails that the union of a chain of sets of formulae not proving a formula  $\varphi$  still does not prove  $\varphi$ , a crucial step in the proof based on Zorn's lemma or the 'direct constructive' proof.

There are however important examples of logics which are not finitary, e.g., logics with intended many-valued semantics such as Łukasiewicz logic, logics of common knowledge over substructural logics, logics with truth constants crucial for dealing with modalities in non-contractive logics, or logic with infinitary operators such as conjunction and disjunction.

In this contribution we prove a slightly weaker variant of Lindenbaum lemma for any, non-necessarily finitary, logic  $\vdash$  that as:

- *Countably axiomatizable*, i.e., there is a countable set of rules closed under substitution, such that  $\vdash$  is the weakest logic containing that set (note that all finitary logics with countable sets of formulae are countably axiomatizable).
- *Strongly disjunctive*, i.e., there is a set of formulas  $\nabla(p, q)$  in two variables  $p$  and  $q$  such that the following metarule (called strong proof by cases property) is valid for all sets of formulae  $\Gamma \cup \Pi \cup \Delta \cup \{\chi\}$ :<sup>1</sup>

$$\frac{\Gamma, \Delta \vdash \chi \quad \Gamma, \Pi \vdash \chi}{\Gamma \cup \bigcup_{\delta \in \Delta, \pi \in \Pi} \nabla(\delta, \pi) \vdash \chi}$$

**Theorem 2.** *Let  $\vdash$  be a countably axiomatizable strongly disjunctive logic. Then finitely meet-irreducible theories form a basis of  $\text{Th}(\vdash)$ .*

<sup>1</sup> If  $\Delta$  and  $\Pi$  are singletons, we obtain the well known *proof by cases property* which in finitary logics coincides with its strong form but not in general it differs, see [2].

The proof is based on the following trick: assume that  $\varphi \notin T$ , enumerate all the rules, and in each step expand  $T$  in a way that makes sure that either it already contains the conclusion of the given rule or the rule is not applicable to any of its extensions not proving certain special formula. We use the existing disjunction connective to combine and ‘remember’ these special formulas. The resulting theory does not prove any of those special formulas and thus is of a ‘finitary’ nature. Then, with a bit more of work, the theorem follows.

One may object that the new lemma gives less information than its original variant, but one should note that 1) in classical logic consistent finitely meet-irreducible and meet-irreducible theories coincide, 2) in non-classical logics the finitely meet-irreducible theories are usually ‘the interesting ones’, e.g. in (strongly) disjunctive logics they are exactly the  $\nabla$ -prime ones (i.e., if  $\varphi \nabla \psi \subseteq T$ , then  $\varphi \in T$  or  $\psi \in T$ ) and in semilinear logics (i.e., logics complete w.r.t. matrices linearly ordered by their generalized implication  $\Rightarrow$ ) they are the  $\Rightarrow$ -linear ones (i.e.,  $\varphi \Rightarrow \psi \subseteq T$  or  $\psi \Rightarrow \varphi \subseteq T$ ).

The lemma has numerous general and particular corollaries, let us present one example for each kind. First we strengthen (a part of) [3, Theorem 4], showing that to prove that a logic is complete w.r.t. its linearly ordered matrices one only has to check the provability of certain formulas.

**Corollary 1 (Disjunction-based characterization of semilinear logics).** *Let  $\vdash$  be a countably axiomatizable strongly disjunctive logic with a weak  $p$ -implication  $\Rightarrow$ . Then  $\vdash$  is semilinear w.r.t.  $\Rightarrow$  iff  $\vdash (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi)$ .*

Second we prove the folklore result (formally obtainable as a consequence of [4]) about the axiomatization of the infinitely-valued Łukasiewicz logic.

**Corollary 2.** *The logic  $\vdash_{L^\infty}$  of the standard MV-algebra is the extension of Łukasiewicz logic  $\vdash_L$  (of all MV-algebras) by the rule*

$$\{\neg\varphi \rightarrow \varphi^n \mid n \geq 0\} \vdash \varphi. \quad (*)$$

*Proof.* Let us by  $\vdash'$  denote the logic defined above. If we show that  $\{\neg\varphi \rightarrow \varphi^n \mid n \geq 0\} \vee \chi \vdash' \varphi \vee \chi$  then  $\vee$  is its strong disjunction due to [2, Theorem 5.1]. First we observe that  $\neg\varphi \rightarrow \varphi^n \vdash_L \neg(\varphi \vee \chi) \rightarrow (\varphi \vee \chi)^n$  and also  $\chi \vdash_L \neg(\varphi \vee \chi) \rightarrow (\varphi \vee \chi)^n$ , thus, using the proof by cases property of  $\vee$  in  $\vdash_L$  we obtain  $(\neg\varphi \rightarrow \varphi^n) \vee \chi \vdash_L \neg(\varphi \vee \chi) \rightarrow (\varphi \vee \chi)^n$ . Then clearly  $\{\neg\varphi \rightarrow \varphi^n \mid n \geq 0\} \vee \chi \vdash_L \{\neg(\varphi \vee \chi) \rightarrow (\varphi \vee \chi)^n \mid n \geq 0\}$  and so the claim follows using (\*).

Thus  $\vdash'$  is semilinear (using the previous corollary). Assume that  $\Gamma \not\vdash' \varphi$  and consider a linearly ordered  $\vdash'$ -algebra  $\mathbf{A}$ . Obviously  $\mathbf{A}$  is an MV-chain, if we show that  $\mathbf{A}$  is simple the proof is complete by the known fact that every simple MV-chain is embeddable into the standard one [1]. Assume that it is not the case: i.e.,  $0 < x < 1$ , and  $x^n > 0$  for each  $n$ , then clearly  $\neg x < x^n$  for each  $n$  (because  $x^{n-1} \geq \neg x$  implies  $x^n = 0$ ), a contradiction with the validity of (\*) in  $\mathbf{A}$ .

In the talk I would present the sketch of the proof of the infinitary Lindebaum lemma, demonstrate that the condition of countable axiomatizability cannot be omitted, and illustrate its consequences on other notable examples, both general and particular.

## References

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