Ordered Groups and Proof Theory

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Uniform algebraic completeness proofs for analytic sequent and hypersequent calculi with respect to classes of residuated lattices, the algebras of substructural logics, have been provided in [12, 2, 3]. However, "ordered group-like" structures, such as algebras with a group reduct like lattice-ordered groups (ℓ-groups) [1, 8] or others like MV-algebras [4], GBL-algebras [7], and cancellative residuated lattices [11] admitting representations via ordered groups, are not covered by these methods. Proof calculi have been defined for some of these classes in [9, 10, 6] but the completeness proofs are mainly syntactic. In this work we exploit ordering theorems for groups to generate hypersequent calculi for varieties of ℓ-groups, thereby taking a first step towards a general algebraic proof theory for ordered group-like structures. These calculi are also used to tackle some algebraic problems arising in the theory of ordered groups.

Taking advantage of the strong distributivity properties of ℓ-groups, we identify sequents with group terms and hypersequents with joins of group terms. We then obtain an analytic hypersequent calculus GA for the variety $\mathcal{A}$ of abelian ℓ-groups (Fig. 1), a close relative of the calculus introduced in [9], as a direct consequence of a theorem of Fuchs [5] for extending partial orders of abelian groups to total orders. The completeness of GA is justified by the following result, where the class of abelian groups is denoted by $\mathcal{A}b$ and partial orders of groups are identified with their positive cones:

Theorem 1. *The following are equivalent for group terms $t_1, \ldots, t_n$ over k variables:*

(1) $\mathcal{A} \models e \leq t_1 \vee \ldots \vee t_n$.

(2) $\{t_1, \ldots, t_n\}$ *does not extend to a total order of the free abelian group over k generators.*

(3) $\mathcal{A}b \models e \approx t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ *for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.*

A hypersequent calculus GLG* for the variety $\mathcal{LG}$ of ℓ-groups (Fig. 2) is obtained from a theorem of Kopytov and Medvedev [8] for extending partial right orders of groups to total right orders. This calculus is used to obtain the following relationship between the problem of checking validity in $\mathcal{LG}$ to the problem of extending finite subsets of the free group $\mathbf{F}(k)$ over k generators to right orders.

Theorem 2. *The following are equivalent for group terms $t_1, \ldots, t_n$ over k variables:*

(1) $\mathcal{LG} \models e \leq t_1 \vee \ldots \vee t_n$.

(2) $\{t_1, \ldots, t_n\}$ *does not extend to a right order of $\mathbf{F}(k)$.*

This result is then used to obtained new proofs of decidability for the two problems.

A calculus for the variety of representable ℓ-groups (equivalently, ordered groups) is obtained via a theorem of Fuchs [5] for extending partial orders of groups to total orders as an extension of GLG* with the rule

$$\frac{\mathcal{G} \mid \Delta, \Gamma}{\mathcal{G} \mid \Gamma, \Delta} \ (\text{CYCLE})$$

An analogue of Theorem 2 for representable ℓ-groups is proved and used to provide a new proof that free groups are orderable.

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$$\frac{}{\mathcal{G} \mid \Delta, \overline{\Delta}} \ (\textsc{id}) \qquad \frac{\mathcal{G} \mid \Pi, \Delta, \Gamma}{\mathcal{G} \mid \Pi, \Gamma, \Delta} \ (\textsc{ex}) \qquad \frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma \mid \Delta} \ (\textsc{split})$$

Figure 1: The hypersequent calculus GA

$$\frac{}{\mathcal{G} \mid \Gamma} \ (\textsc{gv}) \qquad \frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma \mid \Delta} \ (\textsc{split}) \qquad \frac{\mathcal{G} \mid \Delta \quad \mathcal{G} \mid \overline{\Delta}}{\mathcal{G}} \ (*)$$

Γ group valid $\qquad$ Δ not group valid

Figure 2: The hypersequent calculus GLG*

References

[1] M. E. Anderson and T. H. Feil. *Lattice-Ordered Groups: An Introduction.* Springer, 1988.

[2] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory for substructural logics: Cut-elimination and completions. *Ann. Pure Appl. Logic*, 163(3):266–290, 2012.

[3] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory: Hypersequents and hypercompletions. *Ann. Pure Appl. Logic*, 168(3):693–737, 2017.

[4] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning.* Kluwer, 1999.

[5] L. Fuchs. *Partially Ordered Algebraic Systems.* Pergamon Press, 1963.

[6] N. Galatos and G. Metcalfe. Proof theory for lattice-ordered groups. *Ann. Pure Appl. Logic*, 8(167):707–724, 2016.

[7] P. Jipsen and F. Montagna. Embedding theorems for classes of GBL-algebras. *J. Pure Appl. Algebra*, 214(9):1559–1575, 2010.

[8] V. M. Kopytov and N. Y. Medvedev. *The theory of lattice-ordered groups.* Kluwer, 1994.

[9] G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Trans. Comput. Log.*, 6(3):578–613, 2005.

[10] G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics.* Springer, 2008.

[11] F. Montagna and C. Tsinakis. Ordered groups with a conucleus. *J. Pure Appl. Algebra*, 214(1):71–88, 2010.

[12] K. Terui. Which structural rules admit cut elimination? — an algebraic criterion. *J. Symbolic Logic*, 72(3):738–754, 2007.