First-order logic properly displayed

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The existing sequent calculi for first-order logic [14] contain special rules for the introduction of quantification and for substitution. The application of these rules depends on the unbounded and bounded variables occurring in formulas. For example, in the standard Gentzen calculus for first-order logic the rules

$$\frac{\Gamma \vdash \Delta, A[x]}{\Gamma \vdash \Delta, \forall x A} \qquad \frac{\Gamma, A[x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}$$

are sound only when x does not appear free in the conclusions of the rules.

A proposal for a display calculus for fragments of first-order logic was first presented in [17, 16]. The key idea of this approach is that existential quantification can be viewed as a diamond-like operator of modal logic, and universal quantification can be seen as a box-like operator as discussed in [10, 15]. The underlying reason for these similarities which have been observed and exploited in [10, 15, 17, 16] is order-theoretic and pertains to the phenomenon of adjunction: indeed the set theoretic semantic interpretation of the existential and universal quantification are the left and right adjoint respectively of the inverse projection map and more generally, in categorical semantics, the left and right adjoint of the pullbacks along projections [11],[6, Chapter 15]. However, the display calculus of [17] contains rules with side conditions restricting on the free and bounded variables of formulas similarly to the ones presented above. This implies that the rules are not closed under uniform substitution, that is, the display calculus is not *proper* [16, Section 4.1].

We present results based on ongoing work in [7] on a proper display calculus for first-order logic. The design of our calculus is based on the multi-type methodology first presented in [4, 2] for DEL and PDL and further developed in [3, 1, 5, 8, 9]. The multi-type approach allows for the co-existence of different types of terms bridged by heterogeneous connectives. The requirement for the calculus is that in a derivable sequent $x \vdash y$ the structures x and y must be of the same type. In this framework properness means uniform substitution within each type.

Using insights from [11, 12, 13] we introduce a proper display calculus for first-order logic. The conditions on rules are internalised in the calculus by the use of appropriate types. The language of first-order logic is expanded with a unary heterogeneous connective that serves as the right adjoint of the existential quantifier and the left adjoint of the universal quantifier. In the context of the calculus this connective signifies the introduction of a fresh variable to a formula.

In my talk I will present the calculus and discuss results on completeness, soundness, cut-elimination and conservativity.

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