crude but effective stratification

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Hotel Erica, Berg en Dal, 2002

Peter Aczel reminds us of the joyful innocence of

$Set : Set$

$ID : Set$

$ID = (X : Set) \to X \to X$

$id : ID$

$id = \lambda X \to \lambda x \to x$

$id\ ID\ id : ID$

uniformity is the virtue purchased by involution

- use same machinery for Sets and values

zipWith : (Ss : List Set) → (T : Set) →
 (Ss ->> T) →
 (map List Ss ->> List T)

_->>_ : List Set → Set → Set

[] ->> T = T
(S :: Ss) ->> T = S → (Ss ->> T)

uniformity is the virtue purchased by involution

- use same machinery for Sets and values

```
         big
zipWith : (Ss : List Set) → (T : Set) →
          (Ss ->> T) →
           big    wee         wee
          (map List Ss ->> List T)

          big
_->>_ : List Set → Set → Set
[] ->> T        = T
(S :: Ss) ->> T = S → (Ss ->> T)
```

- can we pay for uniformity with less?

'Levitation' negotiates with the serpent

$$[\![\_]\!] : \mathsf{Desc}\ I \to (I \to \mathsf{Set}) \to \mathsf{Set}$$

$$\mu : (F : I \to \mathsf{Desc}\ I) \to I \to \mathsf{Set}$$

$$\langle\_\rangle : [\![F\ i]\!]\ (\mu F) \to \mu\ F\ i$$

$$\mathsf{DESC} : \mathsf{Set} \to \mathsf{Desc}\ 1$$

$$\mathsf{Desc} : \mathsf{Set} \to \mathsf{Set}$$

$$\mathsf{Desc} = \lambda I \to \mu\ (\lambda\_ \to \mathsf{DESC}\ I)\ \_$$

'Levitation' negotiates with the serpent

$$[\![\_]\!]^{\text{wee}} : \mathrm{Desc}^{\text{wee}}\ I \to (I \to \mathrm{Set}^{\text{wee}}) \to \mathrm{Set}^{\text{wee}}$$

$$\mu^{\text{wee}} : (F : I \to \mathrm{Desc}^{\text{wee}}\ I) \to I \to \mathrm{Set}^{\text{wee}}$$

$$\langle\_\rangle^{\text{wee}} : [\![F\ i]\!]^{\text{wee}}\ (\mu^{\text{wee}} F) \to \mu^{\text{wee}} F\ i$$

$$\mathrm{DESC}^{\text{wee}} : \mathrm{Set}^{\text{wee}} \to \mathrm{Desc}^{\text{big}}\ 1$$

$$\mathrm{Desc}^{\text{wee}} : \mathrm{Set}^{\text{wee}} \to \mathrm{Set}^{\text{big}}$$

$$\mathrm{Desc}^{\text{wee}} = \lambda I \to \mu^{\text{big}}\ (\lambda\_ \to \mathrm{DESC}^{\text{wee}}\ I)\ \_$$

- the circle is a coil

requirements

- $\mathsf{Set}^i : \mathsf{Set}^j$ when $i < j$
- $\Uparrow_i^j : \mathsf{Set}^i \hookrightarrow \mathsf{Set}^j$ when $i \le j$ 'embed'
- $$\frac{t : T : \mathsf{Set}^i}{\Uparrow_i^j t : \Uparrow_i^j T : \mathsf{Set}^j}$$ when $i \le j$ 'embiggen'
- embedding keeps small things small but views them from further away
- embiggening makes the big version of a small thing

embedding is rife

$ID = (X : Set) \to \uparrow X \to \uparrow X$

shift X to Set's level

- how to hide $\uparrow$?
  - cumulativity $Set^i \subseteq Set^j$ if $i \leq j$?
  - PTS-flexibility $(i, j, i \sqcup j) \in \Pi\text{-RULES}$

a bidirectional approach

- ask not the type of a canonical thing
- ask which canonical things a type has

- observations have types
- subtyping answers
  'is the type we've got good
  for the type we want?'

$$\frac{j > i}{\mathsf{Set}^j \ni \mathsf{Set}^i}$$

$$\frac{\mathsf{Set}^j \ni S \quad S \Downarrow S' \quad S' \to \mathsf{Set}^j \ni T}{\mathsf{Set}^j \ni \Pi\ S\ T}$$

$$\frac{x : S \vdash \{T \cdot \underline{x} \Rightarrow T' ;\ T' \ni t\}}{\Pi\ S\ T \ni \lambda x \to t}$$

$$\frac{t \in S \quad S \leq T}{T \ni \underline{t}}$$

$$\frac{x : S}{x \in S}$$

$$\frac{\begin{array}{l} f \in \Pi\ S\ T \\ S \ni s \\ s \Downarrow s' \\ T \cdot s' \Rightarrow T' \end{array}}{f\ s \in T'}$$

'bigger than' not 'suc of'

$$\frac{j > i}{\mathrm{Set}^j \ni \mathrm{Set}^i}$$

evaluation

$$\frac{\mathrm{Set}^j \ni S \quad S \Downarrow S' \quad S' \to \mathrm{Set}^j \ni T}{\mathrm{Set}^j \ni \Pi\, S\, T}$$

no ⊔-ing

$$\frac{x : S \vdash \{T \cdot \underline{x} \Rightarrow T' ;\ T' \ni t\}}{\Pi\, S\, T \ni \lambda x \to t}$$

$$\frac{t \in S \quad S \le T}{T \ni \underline{t}}$$

value application

$$\frac{x : S}{x \in S}$$

$$\frac{f \in \Pi\, S\, T \quad S \ni s \quad s \Downarrow s' \quad T \cdot s' \Rightarrow T'}{f\ s \in T'}$$

$$\frac{i \leq j}{Set^i \leq Set^j}$$

$$\frac{S' \leq S \qquad x : S' \vdash \left\{ \begin{array}{l} T \cdot x \Rightarrow U \\ T' \cdot x \Rightarrow U' \\ U \leq U' \end{array} \right\}}{\Pi\, S\, T \leq \Pi\, S'\, T'}$$

$$\frac{S \equiv T \in Set^i}{\underline{S} \leq \underline{T}}$$

$$\frac{i \leq j}{\mathrm{Set}^i \leq \mathrm{Set}^j}$$

nonstrict order induces embedding

$$\frac{S' \leq S \qquad x : S' \vdash \{ T \cdot x \Rightarrow U \quad T' \cdot x \Rightarrow U' \quad U \leq U' \}}{\Pi\, S\, T \leq \Pi\, S'\, T'}$$

contravariant

ok, because of

covariant

$$\frac{S \equiv T \in \mathrm{Set}^i}{\underline{S} \leq \underline{T}}$$

for noncanonical types, degenerate to equality

definitional equality is also bidirectional

$$\frac{j \geq i}{\mathsf{Set}^j \ni \mathsf{Set}^i \equiv \mathsf{Set}^i}$$

$$\frac{\mathsf{Set}^j \ni S \equiv S' \qquad x : S' \vdash \{ T \cdot x \Rightarrow U \quad T' \cdot x \Rightarrow U' \quad \mathsf{Set}^j \ni U \equiv U' \}}{\mathsf{Set}^j \ni \Pi\, S\, T \equiv \Pi\, S'\, T'}$$

$$\frac{x : S \vdash \{ T \cdot x \Rightarrow U \quad f \cdot x \Rightarrow t \quad f' \cdot x \Rightarrow t' \quad U \ni t \equiv t' \}}{\Pi\, S\, T \ni f \equiv f'}$$

$$\frac{t \equiv t' \in S \qquad S \leq T}{T \ni t \equiv t'}$$

reconstructing types for observations

$$\frac{x : S}{x \equiv x \in S}$$

$$\frac{f \equiv f' \in \Pi\, S\, T \quad S \ni s \equiv s' \quad T \cdot s \Rightarrow U}{f\, s \equiv f'\, s' \in U}$$

$$\frac{t \Downarrow_{[x \mapsto s]} v}{(\lambda x \to t) \cdot s \Rightarrow v}$$

$$\frac{}{f \cdot s \Rightarrow \underline{f\, s}}$$

$$\frac{}{\mathsf{Set}^i \Downarrow_\sigma \mathsf{Set}^i}$$

$$\frac{S \Downarrow_\sigma S' \quad T \Downarrow_\sigma T'}{\Pi\, S\, T \Downarrow_\sigma \Pi\, S'\, T'}$$

$$\frac{x \vdash t \Downarrow_{\sigma[x \mapsto \underline{x}]} t'}{\lambda x \to t \Downarrow_\sigma \lambda x \to t'}$$

$$\frac{t \Downarrow_\sigma v}{\underline{t} \Downarrow_\sigma v}$$

$$\frac{}{x \Downarrow_\sigma \sigma(x)}$$

$$\frac{f \Downarrow_\sigma f' \quad s \Downarrow_\sigma s' \quad f' \cdot s' \Rightarrow v}{f\, s \Downarrow_\sigma v}$$

pause for thought (I)

- we have cumulativity for embedding ↑
- levels are explicit — no typical ambiguity
- checking is decidable if ⇓, ⇒ terminate (and they should, if < is well-founded)
- so far, no story about universe polymorphism or embiggening ↑

pause for thought (II)

- so far, the only explicit levels are superscripts on Set
- levels are compared only with $<$, $\leq$ and are not specified absolutely in any rule
- derivability preserved by any strictly monotone operator on levels
- that gives us a way to manage embiggening

global definitions

- let us choose a bunch of strictly monotone operators on levels, LOp, closed under identity & composition (e.g. $\{(n+) \mid n \in \mathbb{N}\}$ if levels are in $\mathbb{N}$)
- let us allow top level (unfinished) definitions

$$\left.\begin{array}{l} f =_i s : S \\ h =_i ? : S \end{array}\right\} \text{where } s, S \text{ closed} \quad \begin{array}{l} Set^i \ni S \\ S \ni s \end{array}$$

- let us invoke definitions, f°, h°, with $\circ \in LOp$ (a blank $\circ$ is the identity)

monotone shifting

- $\dfrac{}{f^\circ \in S^\circ} \quad \dfrac{}{f^\circ \Downarrow_\circ s^\circ}$ if $f =_i s : S$

  $\dfrac{}{h^\circ \in S^\circ} \quad \dfrac{}{h^\circ \Downarrow \underline{h}^\circ}$ if $h =_i ? : S$

- the action of $\circ$ on terms is structural, except

  $$(\mathsf{Set}^i)^\circ = \mathsf{Set}^{\circ i}$$
  $$(f^{\circ'})^\circ = f^{\circ \cdot \circ'}$$
  $$(h^{\circ'})^\circ = h^{\circ \cdot \circ'}$$

- build stuff on the ground, then shift it high!

$\mathrm{ID} : \mathrm{Set}$

$\mathrm{ID} = (X : \mathrm{Set}) \to X \to X$

$\mathrm{id} : \mathrm{ID}$

$\mathrm{id} = \lambda X \to \lambda x \to x$

$\mathrm{id}^{+1}\ \mathrm{ID}\ \mathrm{id} : \mathrm{ID}$

discussion

- yes cumulativity (silent)
- yes level polymorphism (explicit)
- no typical ambiguity (levels explicit)
- no $\sqcup$
- no constraint-solving (and before you get tempted to add some, think about what $\leq$ does to argument synthesis)
- need to prove some theorems (build model?)
- need to roll out to more canonical types