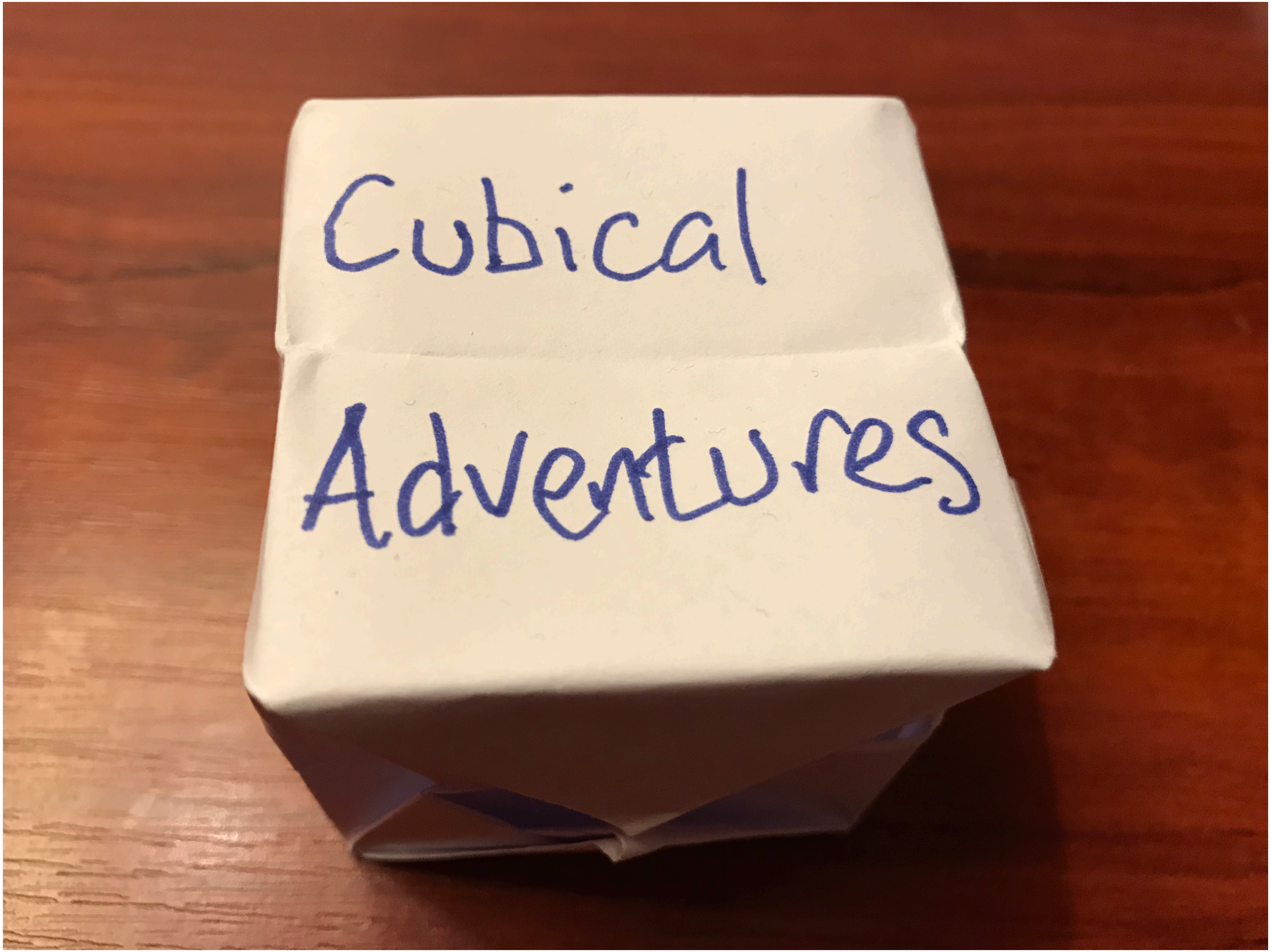
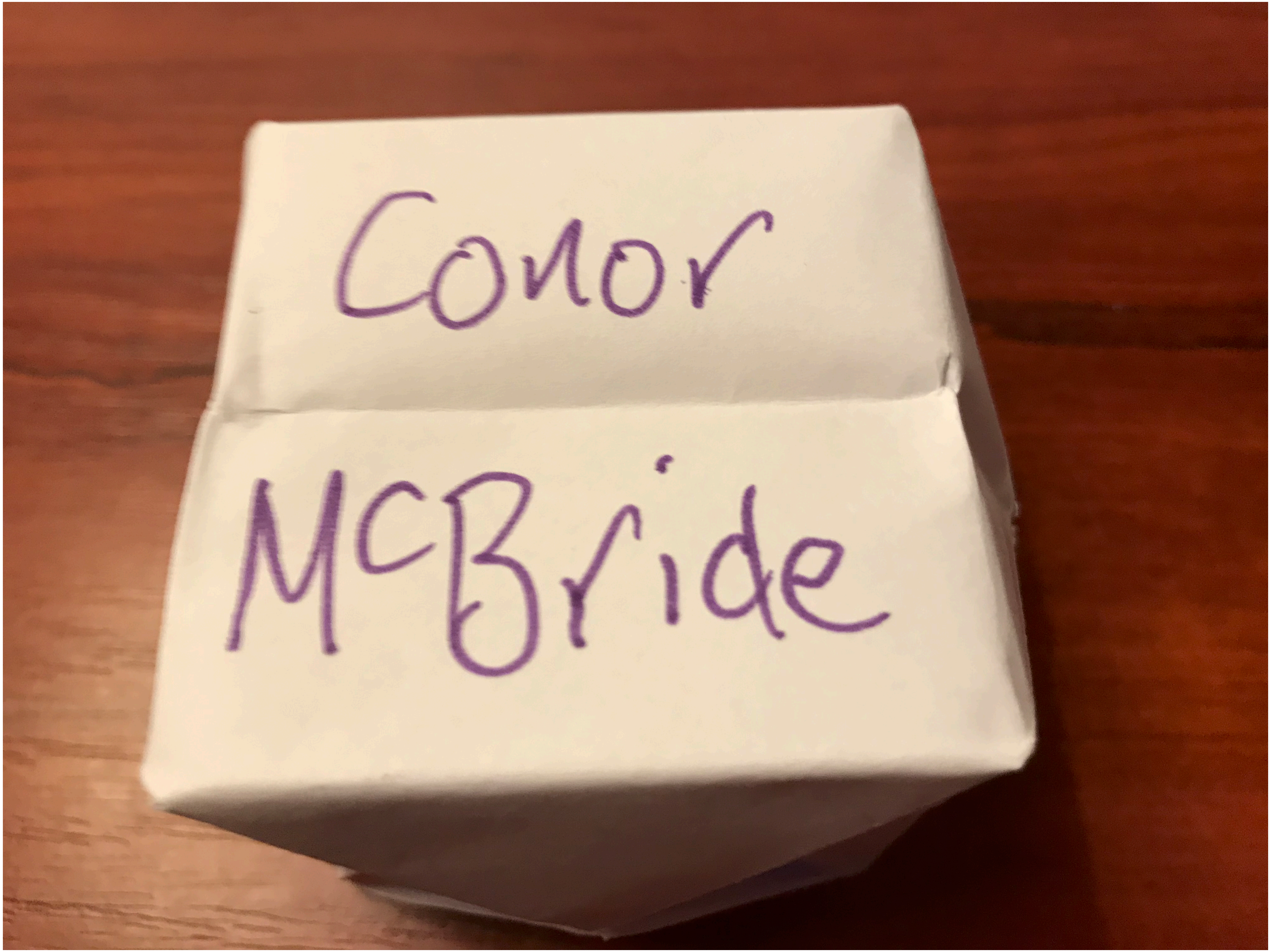

Cubical
Adventures


Conor
McBride

expectation management

- cubical, regular, extensional, Canonical
- but not yet univalent
- 'proof-relevant Observational Type Theory'

mission — neutrality
between U.I.P. (and more aggressive proof-irrelevance)
and Univalence (and more aggressive proof-relevance)

bidirectional ~~religion~~ distraction

The places in a judgment are

inputs

subjects

outputs

bidirectional ~~religion~~ distraction

The places in a judgment are

inputs ⟵ client makes promises
subjects ⟵ rule establishes trust
outputs ⟵ server makes promises

A rule is a server for its conclusions
and a client for its premises.

sine qua non

$$\boxed{\Gamma \vdash}$$

$$\frac{}{\vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash * \ni S}{\Gamma, x : S \vdash}$$

henceforth, write $\dashv x : S$, etc, for lookup and write only context extensions in rules

$$\Gamma \vdash \quad \Gamma \vdash * \ni T \;\rightarrow\; \boxed{\Gamma \vdash T \ni t}$$

$$\frac{}{* \ni *} \qquad \frac{e \in S \quad * \ni S = T}{T \ni \underline{e}} \qquad \frac{T \rightsquigarrow T' \quad T' \ni t}{T \ni t}$$

$$\Gamma \vdash \;\rightarrow\; \boxed{\Gamma \vdash e \in S} \;\rightarrow\; \Gamma \vdash * \ni S$$

$$\frac{\dashv x : S}{x \in S} \qquad \frac{* \ni T \quad T \ni t}{t : T \in T} \qquad \frac{e \in S \quad S \rightsquigarrow S'}{e \in S'}$$

$$\boxed{s \rightsquigarrow t} \qquad \underline{t : T} \rightsquigarrow_0 t$$

things (I): functions

$$\frac{* \ni S \quad x : S \vdash * \ni T(x)}{* \ni \Pi x : S.\, T(x)} \qquad \frac{x : S \vdash T(x) \ni t(x)}{\Pi x : S.\, T(x) \ni \lambda x.\, t(x)}$$

$$\frac{f \in \Pi x : S.\, T(x) \quad S \ni s}{f\, s \in T(s : S)}$$

$$(\lambda x.\, t(x) : \Pi x : S.\, T(x))\; s \rightsquigarrow_\beta t(s : S) : T(s : S)$$

watch out for the types at which computation is active

$$\frac{x : S \vdash T(x) \ni (t_0 : \sim)\, x = (t_1 : \sim)\, x}{\underset{\sim\sim\sim}{\Pi x : S.\, T(x)} \ni t_0 = t_1}$$

things (II): tuples

$$\frac{* \ni S \quad x : S \vdash * \ni T(x)}{* \ni \Sigma x : S.\, T(x)} \qquad \frac{}{* \ni 1} \qquad \text{form}$$

$$\frac{S \ni s \quad T(s : S) \ni t}{\Sigma x : S.\, T(x) \ni s, t} \qquad \frac{}{1 \ni \star} \qquad \text{intro}$$

$$\frac{e \in \Sigma x : S.\, T(x)}{e\ \mathsf{car} \in S} \quad \frac{e \in \Sigma x : S.\, T(x)}{e\ \mathsf{cdr} \in T(e\ \mathsf{car})} \qquad \text{elim}$$

$$(s, t : \Sigma x : S.\, T(x)) \begin{array}{l} \mathsf{car} \\ \mathsf{cdr} \end{array} \begin{array}{l} \leadsto_\beta \\ \leadsto_\beta \end{array} \begin{array}{l} s : S \\ t : T(s : S) \end{array} \qquad \beta$$

$$\frac{S \ni \underbrace{(t_0 : \sim)\mathsf{car}} = (t_1 : \sim) \quad T(\sim) \ni (t_0 : \sim)\mathsf{cdr} = (t_1 : \sim)\mathsf{cdr}}{\underbrace{\Sigma x : S.\, T(x)} \ni t_0 = t_1} \qquad \frac{}{1 \ni t_0 = t_1} \qquad \eta$$

points $\qquad \Gamma \vdash \qquad \boxed{\Gamma \vdash \mathsf{PT}\ p}$

dimensions

$$\frac{\Gamma \vdash}{\Gamma, i \vdash} \qquad \frac{\dashv i}{\mathsf{PT}\ i}$$

endpoints

$$\frac{}{\mathsf{PT}\ 0} \qquad \frac{}{\mathsf{PT}\ 1}$$

affine rescaling ('mux')

$$\frac{\mathsf{PT}\ p \quad \nabla_i \vdash \mathsf{PT}\ p_i}{\mathsf{PT}\ \ p(p_0 - p_1)}$$

'∇_i' means 'for $i = 0$ and $i = 1$'

normal forms are reduced ordered binary decision diagrams

no redundant tests

test each dimension in order from Γ

test only dimensions

so $\nabla_i \vdash i(p_0 - p_1) = p_i$ and $p'(p - p) = p$

(these are not Thierry's de Morgan algebras)

paths (one dimensional)

$$\frac{i \vdash * \ni T(i) \qquad \nabla_i \vdash T(i) \ni t_i}{* \ni [i.T(i)]\,t_0 - t_1}$$

$$\frac{i \vdash T(i) \ni t(i) \qquad \nabla_i \vdash T(i) \ni t_i = t(i)}{[i.T(i)]\,t_0 - t_1 \ni \langle i \rangle t(i)}$$

$$\begin{array}{ccccc} 0 & \cdots & i & \cdots & 1 \\ t_0 & & T(i) & & t_1 \\ \parallel & & \ni & & \parallel \\ t(0) & & t(i) & & t(1) \\ & & p & & \\ & & \downarrow & & \\ & & t(p) & & \end{array}$$

$$\frac{e \in [i.T(i)]\,t_0 - t_1 \qquad \text{PT}\ p}{e\,p \in T(p)}$$

$$(\langle i \rangle t(i) : [i.T(i)]\,t_0 - t_1)\ p \leadsto_\beta t(p) : T(p)$$

moreover, if $e \in [i.T(i)]\,t_0 - t_1$ then $\nabla_i \vdash e\ i \leadsto_\nabla t_i$

write, e.g., $\langle i\ p_0 - p_1 \rangle t(i)$ for $\langle i \rangle t(i(p_0 - p_1))$, i.e. select a sub-path digression (I): extensionality

given $\quad \nabla_i \vdash f_i \in \Pi x:S.\ T(x)$

$\qquad\qquad f \in \Pi x:S.\ [\_.\ T(x)]\, f_0\, x - f_1\, x$

find $\quad [\_.\ \Pi x:S.\ T(x)]\, f_0 - f_1$

$\ni \langle i \rangle\ \lambda x.\ f\, x\, i$

(because $\nabla_i \vdash \ldots \ni \lambda x.\ f\, x\, i = \lambda x.\ f_i\, x = f_i$)

digression (II): contractibility of singletons

given $\nabla_i \vdash x_i \in T$

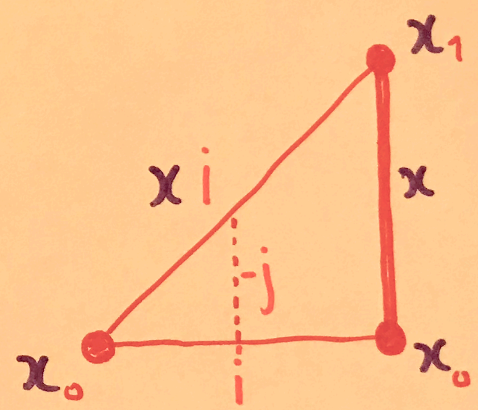
$x \in [\_ . T]\, x_0 - x_1$

oops

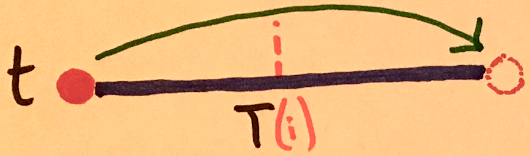
find $[\_ . \Sigma y : T . [\_ . T]\, x_0 - y]\, (x_0, \langle \_ \rangle\, x_0) - (x_1, x)$

oops

$\ni \langle i \rangle\, x\, i,\ \langle j : 0 - i \rangle\, x\, j$



transportation



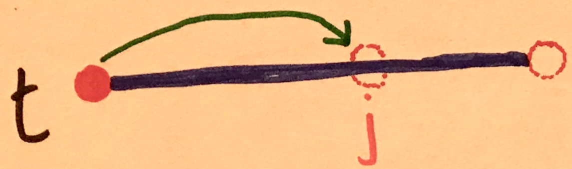
$$\frac{i \vdash * \ni T(i) \quad T(0) \ni t}{t\,[i.\,T(i)\rangle \in T(1)}$$

$$t\,[\_.\,T\rangle \rightsquigarrow t : T$$

"regularity"

note $\quad [j.\,T(j)]\,t - (t\,[i.\,T(i)\rangle)$

$\ni \langle j\rangle\, t\,[i\ 0{-}j.\,T(i)\rangle$



but ~~how~~ when does this compute ?

computation and canonicity

x blah blah blah

free variable = excuse for not being Canonical

$e\ (i(p_0 - p_1))$

free dimension $\neq$ excuse for not being Canonical

for why is e not $\langle j \rangle\, t(j)$?

in fact, a free dimension increases demand for Canonicity

and transportation $\left\{ \begin{array}{l} \text{must deliver} \\ \text{may exploit} \end{array} \right\}$ this fact

transportation and Canonicity

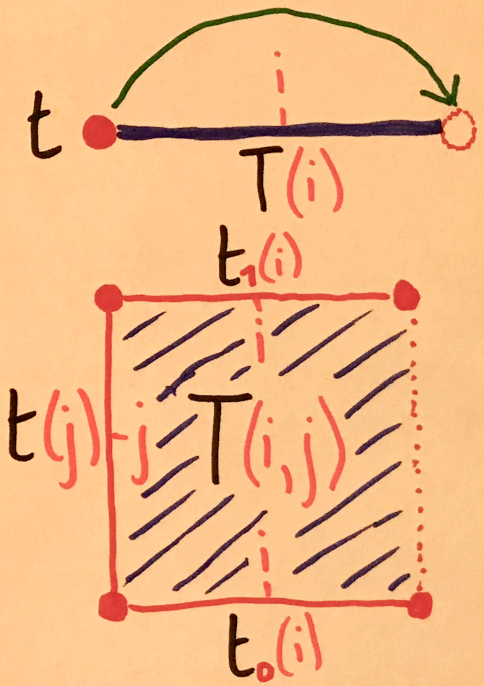
with only free dimensions...

$t\ [i.T(i)\rangle$

must be canonical, e.g.,

$(\langle j\rangle\, t(j))\ [i.[j.T(i,j)]\, t_0(i) - t_1(i)\rangle$

must be canonical, e.g., ...



we must understand transportation in arbitrary dimensions

∇, or how to think in n dimensions without really trying

$$\frac{\bar{\imath} \vdash * \ni T(\bar{\imath}) \qquad \nabla\bar{\imath} \vdash T(\bar{\imath}) \ni \bar{t}(\bar{\imath})}{* \ni [\bar{\imath}.\,T(\bar{\imath})]\,\bar{t}(\bar{\imath})}$$

"on the surface of $\bar{\imath}$"

a vector of n pairs of opposed faces, each with $n-1$ dimensions

an n-fold iterated path type,

$$[i_0.\,[i_1.\,\ldots[i_{n-1}.\,T(\bar{\imath})]\,t_0^{n-1}(i_0,..i_{n-2}) - t_1^{n-1}(i_0,..i_{n-2})\ldots$$
$$](\langle i_2,..i_{n-1}\rangle t_0^{1}(i_0,i_2,..i_{n-1})) - (\langle i_2,..i_{n-1}\rangle t_1^{1}(i_0,i_2,..i_{n-1}))$$
$$](\langle i_1,..i_{n-1}\rangle t_0^{0}(i_1,..i_{n-1})) - (\langle i_1,..i_{n-1}\rangle t_1^{0}(i_1,..i_{n-1}))$$

really trying

$$\frac{\bar{\imath} \vdash T(\bar{\imath}) \ni t(\bar{\imath}) \qquad \nabla\bar{\imath} \vdash T(\bar{\imath}) \ni \bar{t}(\bar{\imath}) \Leftarrow t(\bar{\imath})}{[\bar{\imath}.\,T(\bar{\imath})]\,\bar{t}(\bar{\imath}) \ni \langle\bar{\imath}\rangle\, t(\bar{\imath})}$$

"interior t meets surface $\bar{t}$"

∇-judgments $\qquad \bar{\imath} \vdash * \ni T(\bar{\imath}) \qquad \boxed{\nabla_{\bar{\imath}} \vdash T(\bar{\imath}) \ni \bar{t}(\bar{\imath})}$

faces — $\bar{\jmath} \vdash T(0,\bar{\jmath}) \ni t_0(\bar{\jmath}) \qquad \bar{\jmath} \vdash T(1,\bar{\jmath}) \ni t_1(\bar{\jmath})$

tube — $i \vdash \nabla_{\bar{\jmath}} \vdash T(i,\bar{\jmath}) \ni \bar{t}(i)(\bar{\jmath})$

meeting — $\nabla_{\bar{\jmath}} \vdash T(0,\bar{\jmath}) \ni \bar{t}(0)(\bar{\jmath}) \Leftarrow t_0(\bar{\jmath}) \qquad \nabla_{\bar{\jmath}} \vdash T(1,\bar{\jmath}) \ni \bar{t}(1)(\bar{\jmath}) \Leftarrow t_1(\bar{\jmath})$

$$\frac{}{\nabla \vdash T \ni} \qquad \frac{}{\nabla_{i,\bar{\jmath}} \vdash T(i,\bar{\jmath}) \ni t_0(\bar{\jmath}) - t_1(\bar{\jmath}), \bar{t}(i)(\bar{\jmath})}$$

oops

$\bar{\imath} \vdash * \ni T(\bar{\imath}) \qquad \nabla_{\bar{\imath}} \vdash T(\bar{\imath}) \ni \bar{t}(\bar{\imath}) \qquad \bar{\imath} \vdash T(\bar{\imath}) \ni t(\bar{\imath}) \qquad \boxed{\nabla_{\bar{\imath}} \vdash T(\bar{\imath}) \ni \bar{t}(\bar{\imath}) \Leftarrow t(\bar{\imath})}$

$$\frac{}{\nabla \vdash T \ni \Leftarrow t} \qquad \frac{\begin{array}{c}\bar{\jmath} \vdash T(0,\bar{\jmath}) \ni t_0(\bar{\jmath}) = t(0,\bar{\jmath}) \qquad \bar{\jmath} \vdash T(1,\bar{\jmath}) \ni t_1(\bar{\jmath}) = t(1,\bar{\jmath}) \\ i \vdash \nabla_{\bar{\jmath}} \vdash T(i,\bar{\jmath}) \ni \bar{t}(i)(\bar{\jmath}) \Leftarrow t(i,\bar{\jmath})\end{array}}{\nabla_{i,\bar{\jmath}} \vdash T(i,\bar{\jmath}) \ni t_0(\bar{\jmath}) - t_1(\bar{\jmath}), \bar{t}(i)(\bar{\jmath}) \Leftarrow t(i,\bar{\jmath})}$$

$(n+1)$-dimensional *transportation*

$\bar{\imath}, \bar{\jmath} \vdash * \ni T(\bar{\imath}, \bar{\jmath})$ ‘typespace’

$\bar{\jmath} \vdash T(0, \bar{\jmath}) \ni t_0(\bar{\jmath})$ ‘start’ or ‘base’

$\bar{\imath} \vdash \nabla_{\bar{\jmath}} \vdash T(\bar{\imath}, \bar{\jmath}) \ni \bar{t}(\bar{\imath})(\bar{\jmath})$ ‘tube’ or ‘sides’

$\nabla_{\bar{\jmath}} \vdash T(0, \bar{\jmath}) \ni \bar{t}(0)(\bar{\jmath}) \Leftarrow t_0(\bar{\jmath})$ start of tube encloses start

PT $\bar{p}$ ← coordinates on ‘end’ or ‘lid’

$\underbrace{t_0(\bar{\jmath}), \bar{t}(\bar{\imath})(\bar{\jmath})}_{\text{‘box’ } \overset{\sqsubset}{t}(\bar{\imath}, \bar{\jmath})} [\bar{\imath} . \bar{\jmath} . T(\bar{\imath}, \bar{\jmath})\rangle\ \bar{p} \in T(1, \bar{p})$ ← everything in sight is a T

for such a box, define its *gathering*

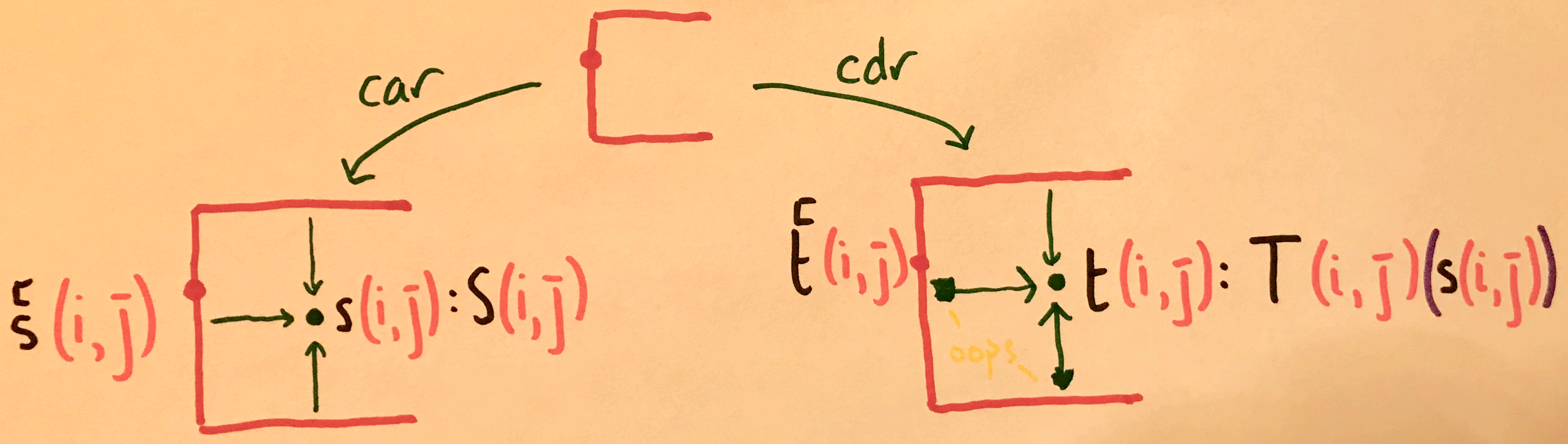
$$\overset{\blacksquare}{t}(\bar{\imath}', \bar{\jmath}') = \overset{\sqsubset}{t}(\bar{\imath}, \bar{\jmath}) [\bar{\imath}\ 0 - \bar{\imath}' . \bar{\jmath} . T(\bar{\imath}, \bar{\jmath})\rangle\ \bar{\jmath}' \in T(\bar{\imath}', \bar{\jmath}')$$

in covariant positions, gather

$$\left(\overset{\sqsubset}{s}(i,\bar{j}),\ \overset{\sqsubset}{t}(i,\bar{j})\right)\ [i.\bar{j}.\ \textstyle\sum x : S(i,\bar{j}).\ T(i,\bar{j})(x)\rangle\ \bar{j}$$

$$= s(1,\bar{j}), t(1,\bar{j})\ :\ \textstyle\sum x : S(1,\bar{j}).\ T(1,\bar{j})(x)$$

where $\overset{\sqsubset}{t}(i,\bar{j})$ is a box for $T(i,\bar{j})(s(i,\bar{j}))$



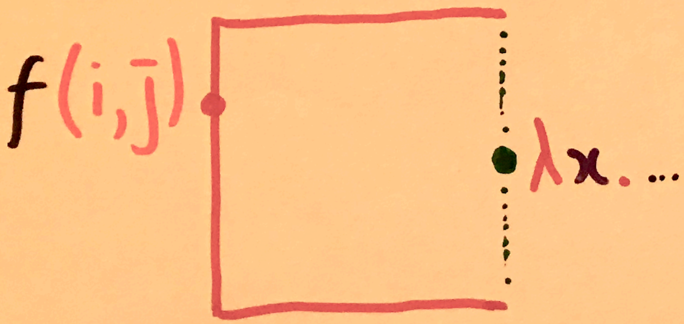
in contravariant positions, scatter

$$(\lambda x.\ \overset{E}{t}(i,\bar{j})(x))\ [i.\bar{j}.\ \Pi x : S(i,\bar{j}).\ T(i,\bar{j})(x)\rangle\ \bar{j}$$

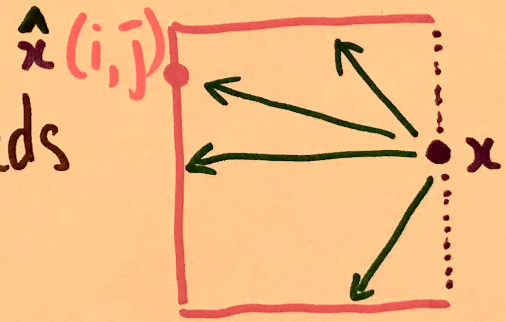
$$= \lambda x.\ \overset{E}{t}(i,\bar{j})(\hat{x}(i,\bar{j}))\ [i.\bar{j}.\ T(i,\bar{j})(\hat{x}(i,\bar{j}))\rangle\ \bar{j}$$

$$: \Pi x : S(1,\bar{j}).\ T(1,\bar{j})(x)$$

where $\hat{x}(i',\bar{j}') = x\,[k.\ S(k(1,\bar{j}-i',\bar{j}'))\rangle$



needs



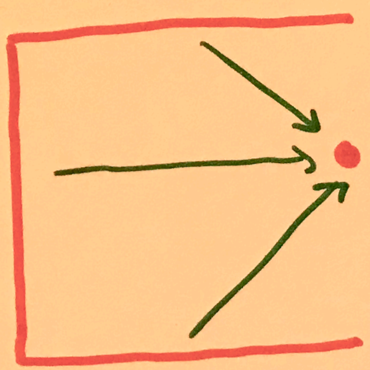
this is pretty much Observational Type Theory

BUT without destroying higher structure

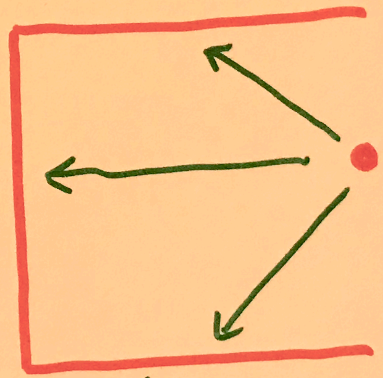
$\bullet \xrightarrow{\text{coerce}} \bullet$ $\qquad$ $\bullet \xleftarrow{\text{coerce}^{-1}} \bullet$

becomes

covariant



gather



contravariant

scatter

but Canonical transport is as before

at the crossroads

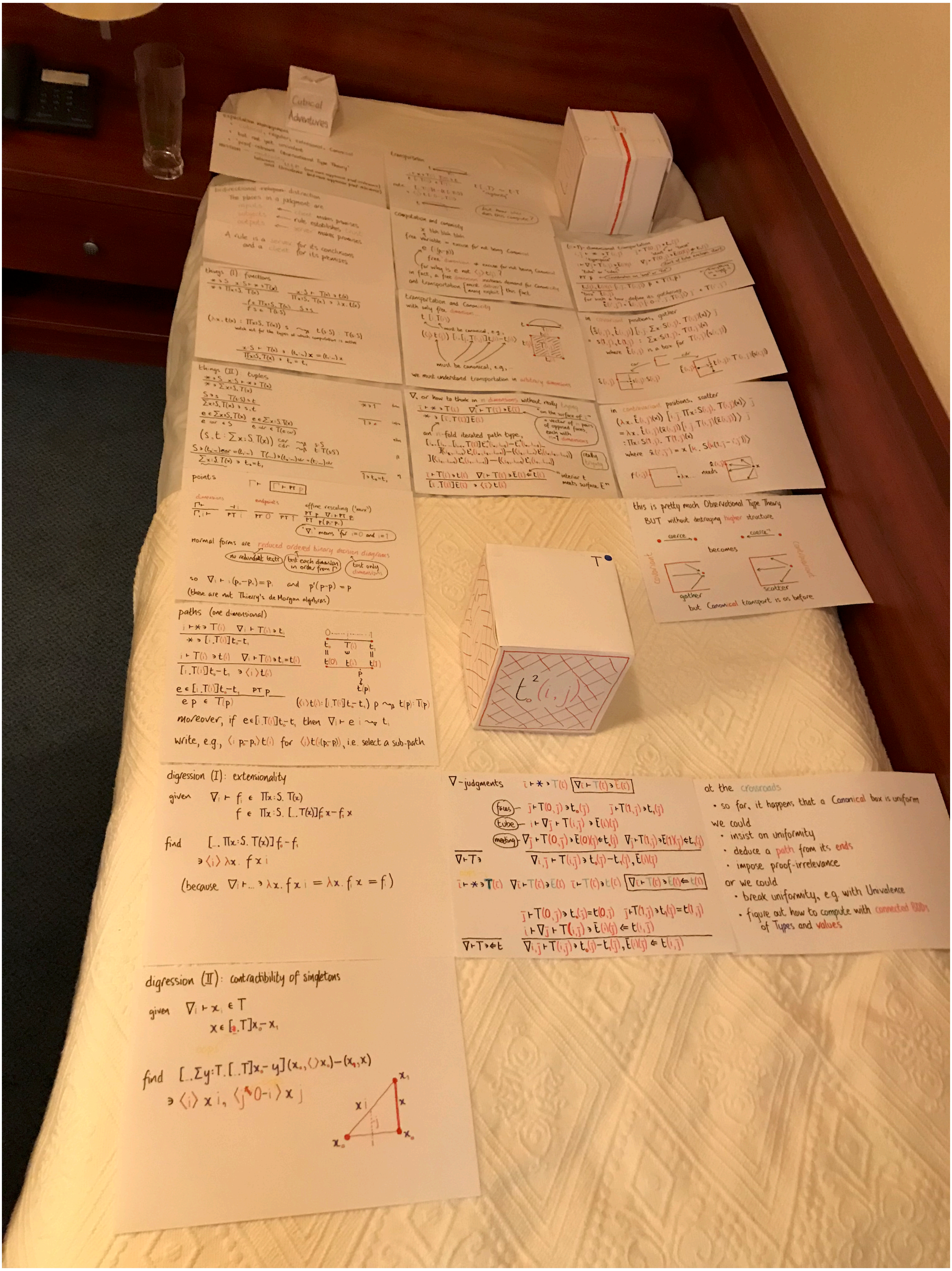
- so far, it happens that a Canonical box is uniform

we could

- insist on uniformity
- deduce a path from its ends
- impose proof-irrelevance

or we could

- break uniformity, e.g. with Univalence
- figure out how to compute with connected BDDs of Types and values



Cubical Adventures
this is pretty much Observational Type Theory
BUT without destroying higher structure
coerce
becomes
gather
scatter
but Canonical transport is as before
paths (one dimensional)
moreover, if e∈[i.T(i)]t₀−t₁ then ∇i ⊢ e i ⇝ tᵢ
write, e.g., ⟨i p₀−p₁⟩t(i) for ⟨i⟩t(i(p₀−p₁)), i.e. select a sub-path
normal forms are reduced ordered binary decision diagrams
(these are not Thierry's de Morgan algebras)
digression (I): extensionality
given ∇i ⊢ fᵢ ∈ Πx:S. T(x)
f ∈ Πx:S. [..T(x)]f₀x−f₁x
find [..Πx:S. T(x)]f₀−f₁
∋ ⟨i⟩ λx. f x i
(because ∇i ⊢ .. ∋ λx. f x i = λx. fᵢ x = fᵢ)
∇-judgments
at the crossroads
• so far, it happens that a Canonical box is uniform
we could
• insist on uniformity
• deduce a path from its ends
• impose proof-irrelevance
or we could
• break uniformity, e.g. with Univalence
• figure out how to compute with connected BDDs of Types and values
digression (II): contractibility of singletons
given ∇i ⊢ xᵢ ∈ T
x ∈ [i.T]x₀−x₁
find [..Σy:T.[..T]x₀−y](x₀,⟨⟩x₀)−(x₁,x)
∋ ⟨i⟩ x i, ⟨j 0−i⟩ x j