Small Induction Recursion, Indexed Containers and Dependent Polynomials are equivalent*

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Abstract

There are several different approaches to the theory of data types. At the simplest level, polynomials and containers give a theory of data types as free standing entities. At a second level of complexity, dependent polynomials and indexed containers handle more sophisticated data types in which the data have an associated indices which can be used to store important computational information. The crucial and salient feature of dependent polynomials and indexed containers is that the index types are defined in advance of the data. At the most sophisticated level, induction-recursion allows us to define the data and the indices simultaneously.

The aim of this work is to investigate the relationship between the theory of *small* inductive recursive definitions and the theory of dependent polynomials and indexed containers. Our central result is that the expressiveness of small inductive recursive definitions is exactly the same as that of dependent polynomials and indexed containers. Formally, this result applies not just to the data types definable in these theories, but also to the morphisms between such data types. Indeed, we introduce the category of small inductive-recursive definitions and prove the equivalence of this category with the category of dependent polynomials/indexed containers.

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1 Introduction

One of the most important concepts in computer science is the notion of an inductive definition. It is difficult to trace back its origin since this concept permeates the history of proof theory and a large part of theoretical computer science[1]. In recent years, the desire to explore, understand, and extend the concept of an inductive definition has led different researchers to different but (extensionally) equivalent notions. The theory of containers [1], and polynomial functors [20, 14] are some of the outcomes of this research[2]. These theories

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[1] A gentle introductory survey of the history of this concept is given by Coquand and Dybjer in [7].
[2] The interested reader can consult the section on related works in [15] for further pointers to the literature.


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give a comprehensive account of those data types such as `Nat` (the natural numbers), `List a` (lists containing data of a given type `a`), and `Tree a` (trees containing, once more, data of a given type `a`) which are free-standing in that their definition does not require the definition of other inter-related data types.

These theories are too simple to capture more sophisticated data types possessing features such as: (*i*) variable binding as in the untyped and typed lambda calculus; (*ii*) constraints as in red black trees; and (*iii*) extra information about data having such types - the classic example of this are vectors which are lists indexed by their lengths. For this reason containers and polynomials have been generalised to indexed containers [2, 3] and dependent polynomials [14, 15] so as to capture not only free standing data types such as those mentioned above, but also data types where the data are indexed by computationally relevant information. Containers and (non-dependent) polynomials arise as specific instances of these generalised notions where the type of indices is chosen to be a singleton type.

However, even dependent polynomials and indexed containers fail to cover all the data types that we need to compute with because they require the indices to be defined before the data. The theory of induction-recursion (IR), developed in the seminal works of Peter Dybjer and Anton Setzer [11, 12, 13], remedies this deficiency. The key feature of an inductive-recursive definition is the simultaneous inductive definition of a type X of indices together with the recursive definition of a function $T : X \to D$ from X into a type D which assigns to every index the type of data with that index. Since X and T can be defined at the same time, the indices need not be defined in advance of the data. As we shall see later, universes (introduced by Martin-Löf in the early 70's [19]) are paradigm examples of inductive recursive definitions.

It is natural to ask what is the relationship between dependent polynomials and induction recursion. Can we characterise those inductive-recursive definitions which correspond to dependent polynomials? The aim of this paper is to address precisely this question. As we will show dependent polynomials and indexed containers correspond exactly to small inductive-recursive definitions, where the "smallness" refers to the size of the target-type D of the recursively defined function $T : X \to D$. More precisely, we will prove an equivalence between the category of dependent polynomials and the category of small inductive-recursive definitions. This result is not merely of theoretical importance - it also opens the way to programmers to convert definitions between the two forms, according to which works better for their own applications. To achieve this, as well as to make the paper more accessible, and to type check our translations, we have implemented our translations in Agda.

The paper is organised as follows: in Section 2 we set our notation, while Section 3 recalls indexed containers, dependent polynomials and induction recursion. In Section 4, we show an equivalence between data types definable by small IR and those data types definable using dependent polynomials and/or indexed containers. In Section 5 we introduce the category of small inductive-recursive definitions and show it equivalent to the category of dependent polynomials/indexed containers. In Section 6 we briefly recall the theory of indexed inductive-recursive definitions, and extend the previous equivalence to the case of indexed small induction recursion. We conclude in Section 7 with thoughts for future work.

The sources and additional materials for this paper are available from `http://personal.cis.strath.ac.uk/~conor/pub/SmallIR`.

2 Preliminaries and internal languages

We follow the standard approach of using extensional Martin-Löf type theory as the internal language to formalise reasoning with the locally cartesian closed structure of the category of sets — see [21, 16] for details [3]. Our notation follows Agda — indeed, this paper is a literate Agda development. We write identity types as $x \equiv y$ and assume uniqueness of identity proofs. We write ΣT or $(s : S) \times T\ s$ and ΠT or $(s : S) \to T\ s$ for the dependent sum and dependent product in Martin-Löf type theory of $T : S \to \mathsf{Set}$. The elements of $(s : S) \times T\ s$ are pairs (s, t) where $s : S$ and $t : T\ s$ may be projected by π_0 and π_1. The elements of $(s : S) \to T\ s$ are functions $\lambda\ x \to t\ x$ mapping each element $s : S$ to an element $t\ s$ of $T\ s$.

Categorically, we think of an I-indexed type as a morphism $f : X \to I$ with codomain I. These are objects of the slice category Set/I. Morphisms in Set/I from object $f : X \to I$ to object $f' : X' \to I$ are given by functions $h : X \to X'$ such that $f = f' \circ h$. Type theoretically, we can represent matters in more or less the same way – that is, an object in a slice Set/I is a pair (X, f) of a set X (the domain), and a function $f : X \to I$. However, another possibility is to model an I-indexed type by a function $F : I \to \mathsf{Set}$ where $F\ i$ represents the fibre of f above i, i.e. as $(X, f)^{-1}\ i$, defined as follows.

$$\begin{aligned} &\cdot^{-1} : \mathsf{Set}/I \to (I \to \mathsf{Set}) \\ &(X, f)^{-1}\ i = (x : X) \times f\ x \equiv i \end{aligned} \qquad \begin{aligned} &\exists. : (I \to \mathsf{Set}) \to \mathsf{Set}/I \\ &\exists.F = (\Sigma F, \pi_0) \end{aligned}$$

We write $\exists.F$ for the inverse of this operator: that these are inverse (given uniqueness of identity proofs) is at the heart of the well known equivalence between the categories Set/I and $I \to \mathsf{Set}$ which, in a sense, underlies the equivalences we describe in this paper.

Given a function $k : I \to J$, we can form three very important functors. The pullback along k of an object $f : X \to J$ of Set/J defines a *reindexing* functor $\Delta_k : \mathsf{Set}/J \to \mathsf{Set}/I$. Δ_k has both a left adjoint and a right adjoint, respectively $\Sigma_k, \Pi_k : \mathsf{Set}/I \to \mathsf{Set}/J$. In the internal language, we define these for $\cdot \to \mathsf{Set}$, as follows:

$$\begin{aligned} &\Delta_k : (J \to \mathsf{Set}) \to (I \to \mathsf{Set}) \\ &\Delta_k\ F\ i = F\ (k\ i) \end{aligned} \qquad \begin{aligned} &\Sigma_k : (I \to \mathsf{Set}) \to (J \to \mathsf{Set}) \\ &\Sigma_k\ F\ j = (i : I) \times k\ i \equiv j \times F\ i \\ &\Pi_k : (I \to \mathsf{Set}) \to (J \to \mathsf{Set}) \\ &\Pi_k\ F\ j = (i : I) \to k\ i \equiv j \to F\ i \end{aligned}$$

3 Three theories of data types

The foundation of our understanding of data types is initial algebra semantics. Thus, formally our theories of data types are in fact theories of functors which have initial algebras. In this section we recall the notions of dependent polynomials, indexed containers and induction recursion, each of which define certain classes of functors and hence data types.

▶ **Definition 1.** The collection of *dependent polynomials* with input indices I and output indices O is written $\mathsf{Poly}\ I\ O$ and consists of triples of morphisms (r, t, q) where

$$I \xleftarrow{r} P \xrightarrow{t} S \xrightarrow{q} O\ .$$

[3] The correspondence between lcccs and Martin Löf type theories is affected by coherence problems related to the interpretation of substitution. We refer to [8], [16] and more recently [6] for different solutions to these problems.

A *dependent polynomial functor* is any functor isomorphic to some $[\![(r, t, q)]\!]_{\mathsf{Poly}} = \Sigma_q \circ \Pi_t \circ \Delta_r : \mathsf{Set}/I \to \mathsf{Set}/O$, illustrated as follows:

$$\mathsf{Set}/I \xrightarrow{\Delta_r} \mathsf{Set}/P \xrightarrow{\Pi_t} \mathsf{Set}/S \xrightarrow{\Sigma_q} \mathsf{Set}/O\,.$$

While the definition above is concise, some readers may prefer a more concrete presentation. So we turn to the representation of dependent polynomials in the internal language. This leads us to the notion of an *indexed container*.

▶ **Definition 2.** The collection of *indexed containers* with input indices I and output indices O is written $\mathsf{IC}\,I\,O$ and consists of triples (S, P, n) where $S : O \to \mathsf{Set}$, $P : (o\!:\!O) \to S\ o \to \mathsf{Set}$ and $n : (o\!:\!O) \to (s\!:\!S\ o) \to P\ o\ s \to I$. Its extension is the functor

$$\begin{aligned}
&[\![\cdot]\!]_{\mathsf{IC}} : \mathsf{IC}\ I\ O \to (I \to \mathsf{Set}) \to (O \to \mathsf{Set})\\
&[\![(S, P, n)]\!]_{\mathsf{IC}}\ X\ o = (s\!:\!S\ o) \times (p\!:\!P\ o\ s) \to X\ (n\ o\ s\ p)
\end{aligned}$$

Every dependent polynomial functor (r, t, q) gives rise to an indexed container $(\hat{S}, \hat{P}, n)$.

$$\begin{aligned}
\hat{S}\ o &= (S, q)^{-1}\ o\\
\hat{P}\ o\ (s, _) &= (P, t)^{-1}\ s\\
n\ o\ (s, _)\ (p, _) &= r\ p
\end{aligned}$$

We may readily check that

$$\begin{aligned}
[\![(\hat{S}, \hat{P}, n)]\!]_{\mathsf{IC}}\ F\ o = &\ (sq\!:\!((S, q)^{-1}\ o)) \times (pq\!:\!((P, t)^{-1}\ (\pi_0\ sq))) \to F\ (r\ (\pi_0\ pq))\\
\cong &\ (s\!:\!S) \times (q\ s \equiv o)\ \times\ (p\!:\!P) \to (t\ s \equiv p)\ \to\ F\ (r\ p)\\
= &\ (\Sigma_q\ \circ\ \Pi_t\ \circ\ \Delta_r)\ F\ o
\end{aligned}$$

confirming the equivalence between indexed containers and dependent polynomials.

Polynomials (resp. containers) arise as a special case of dependent polynomials (indexed containers) by choosing $I = O = 1$. Notice the salient feature of both dependent polynomials and indexed containers — that the input and output indices I and O are fixed and must be defined in advance. This restriction means that neither dependent polynomials nor indexed containers suffice to define all the data types in which we are interested. Paradigmatic undefinable data types are universes of types. These are pairs (U, T) consisting of a set U, thought as a set of names or codes, and of a function $T : U \to Set$, thought as a "decoding function" which assigns a set $T\,u$ to every element u of U. For example, consider a universe containing the type of natural numbers $\mathbb{N}$ and closed under Σ-types. Such a universe will be the least solution of the

$$\begin{aligned}
U &= 1 + (u\!:\!U) \times T\ u\ \to\ U\\
T\ (\mathsf{inl}\ \star) &= \mathbb{N}\\
T\ (\mathsf{inr}\ (u, f)) &= (x\!:\!T\ u) \times T\ (f\ x)
\end{aligned}$$

Note how, in this example, the set of codes U must be defined simultaneously with the decoding function T - something not possible with dependent polynomials or indexed containers which require that U be defined before T. Dybjer and Setzer developed the theory of induction recursion to cover exactly such inductive definitions where the indices and the data must be defined simultaneously. The first presentation of induction-recursion [10] was

as an external schema. In later presentations, the concept of an inductive recursive definition is internalised using of a special type of codes $\mathsf{IR}\ I\ O$. [4]

▶ **Definition 3.** Let I, O be types. The type of $\mathsf{IR}\ I\ O$-codes has the following constructors

$$
\begin{array}{l}
\textbf{data}\ \mathsf{IR}\ (I\ O : \mathsf{Set}) : \mathsf{Set1}\ \textbf{where} \\
\quad \iota : (o : O) \to \mathsf{IR}\ I\ O \\
\quad \sigma : (S : \mathsf{Set})\ (K : S \to \mathsf{IR}\ I\ O) \to \mathsf{IR}\ I\ O \\
\quad \delta : (P : \mathsf{Set})\ (K : (P \to I) \to \mathsf{IR}\ I\ O) \to \mathsf{IR}\ I\ O
\end{array}
$$

In general I and O may be large types such as Set or $\mathsf{Set} \to \mathsf{Set}$ etc. Above, we encode *small* induction recursion (small IR) we mean the cases where I and O are sets.

Dybjer and Setzer prove that every IR code defines a functor. In the case of *small* IR, this functorial semantics can be given in terms of slice categories. Before giving this semantics, we note that slice categories have set indexed coproducts. That is, given a set A, and an A-indexed collection of objects $f_a : X_a \to I$ of Set/I, the cotuple $[f_a]_{a:A} : \coprod_{a:A} X_a \to I$ is the coproduct of the objects f_a in Set/I. We use $\mathsf{in}_a : X_a \to \coprod_{a:A} X_a$ for the a-th injection. In the internal language, the coproduct of an A-indexed family $X_a : I \to \mathsf{Set}$ is the function mapping i to $(a : A) \times X_a\ i$. We use these coproducts to give a definition of the functor denoted by an IR code more compact than - but of course equivalent to - that originally provided by Dybjer and Setzer.

▶ **Definition 4.** Let I, O be sets, $\gamma : \mathsf{IR}\ I\ O$. The action of the functor $[\![\gamma]\!] : \mathsf{Set}/I \to \mathsf{Set}/O$ on an object $f : X \to I$ of Set/I is defined by recursion on γ as follows

- if $\gamma = \iota\ o$ for some $o : O$

$$[\![\iota\ o]\!]\,(f : X \to I) \quad = \quad (\lambda\_.o) : 1 \to O$$

- if $\gamma = \sigma\ S\ K$ for some $S : \mathsf{Set}$, $K : S \to \mathsf{IR}\ I\ O$

$$[\![\sigma\ S\ K]\!]\,(f : X \to I) \quad = \quad \coprod_{s:S} [\![K\ s]\!]\,f$$

- if $\gamma = \delta\ P\ K$ for some $P : \mathsf{Set}$, $K : (P \to I) \to \mathsf{IR}\ I\ O$

$$[\![\delta\ P\ K]\!]\,(f : X \to I) \quad = \quad \coprod_{x:A\to X} [\![K(f \circ x)]\!]\,f$$

An IR functor is any functor isomorphic to one of the form $[\![\gamma]\!]$ for some $\gamma : \mathsf{IR}\ I\ O$.

We can give the above construction in type theory, using the *direct* translation of slices, closed under dependent sum, yielding an interpretation in the style of Dybjer and Setzer:

$$
\begin{array}{l}
[\![\cdot]\!]_{\mathsf{DS}} : \mathsf{IR}\ I\ O \to \mathsf{Set}/I \to \mathsf{Set}/O \\
[\![\iota\ o]\!]_{\mathsf{DS}}\ \ (X, f) = (1, \lambda\_ \to o) \\
[\![\sigma\ S\ K]\!]_{\mathsf{DS}}\ (X, f) = (s : S) \times [\![K\ s]\!]_{\mathsf{DS}}\ (X, f) \\
[\![\delta\ P\ K]\!]_{\mathsf{DS}}\ (X, f) = (x : P \to X) \times [\![K\ (f \circ x)]\!]_{\mathsf{DS}}\ (X, f)
\end{array}
$$

For any $\gamma : \mathsf{IR}\ I\ I$, we can then follow their construction of an inductive datatype simultaneously with its recursive decoder as the initial algebra, $((\mu\ \gamma, \mathsf{decode}\ \gamma), \mathsf{in})$, of $[\![\gamma]\!]_{\mathsf{DS}}$.

$$
\begin{array}{ll}
\textbf{data}\ \mu\ (\gamma : \mathsf{IR}\ I\ I) : \mathsf{Set}\ \textbf{where} & \mathsf{decode} : (\gamma : \mathsf{IR}\ I\ I) \to \mu\ \gamma \to I \\
\quad \mathsf{in} : \mathsf{dom}\ ([\![\gamma]\!]_{\mathsf{DS}}\ (\mu\ \gamma, \mathsf{decode}\ \gamma)) \to \mu\ \gamma & \mathsf{decode}\ \gamma\ (\mathsf{in}\ t) = \mathsf{fun}\ ([\![\gamma]\!]_{\mathsf{DS}}\ (\mu\ \gamma, \mathsf{decode}\ \gamma))\ t
\end{array}
$$

As an example, we show that all containers can be defined by induction recursion:

[4] Dybjer and Setzer treated only the case where I and O are the same. Our mild generalization allows the construction of partial fixed points.

▶ **Example 5** (containers and W-types). Given a simple container (S, P), where $S : \mathsf{Set}$ and $P : S \to \mathsf{Set}$, we can represent it by an $\mathsf{IR}\ 1\ 1$ code as follows:

$$\begin{aligned}&\mathsf{cont} : (S : \mathsf{Set}) \to (P : S \to \mathsf{Set}) \to \mathsf{IR}\ 1\ 1\\&\mathsf{cont}\ S\ P = (\sigma\ S\ \lambda\ s \to \delta\ (P\ s)\ \lambda\ p \to \iota\ \star)\end{aligned}$$

We note that $\mathsf{dom}\ [\![\mathsf{cont}\ S\ P]\!]_{\mathsf{DS}}\ (X, \_) = (s{:}S) \times (P\ s \to X) \times 1$ and that $\mu\ (\mathsf{cont}\ S\ P)$ thus amounts to Martin-Löf's well-ordering type $\mathsf{W}\ S\ P$. As a corollary of our main result we shall see that $\mathsf{IR}\ 1\ 1$ codes describe exactly the category of containers and their morphisms.

▶ **Example 6** (A Language of Sums and Products). If $\mathsf{Fin} : \mathbb{N} \to \mathsf{Set}$ maps n to a set with n elements, we can implement finitary summation and product with the following types:

$$\mathsf{sum}\ \mathsf{prod} : (n{:}\mathbb{N}) \to (\mathsf{Fin}\ n \to \mathbb{N}) \to \mathbb{N}$$

Having done so, we may now encode a datatype of numerical expressions closed under constants, sums and products, where each expression decodes to its numerical value — we need to know these values to compute the correct domains for the sums and the products.

$$\begin{aligned}&\textbf{data}\ \mathsf{Tag} : \mathsf{Set}\ \textbf{where}\ \mathsf{fin}'\ \mathsf{sum}'\ \mathsf{prod}' : \mathsf{Tag}\\&\mathsf{lang} : \mathsf{IR}\ \mathbb{N}\ \mathbb{N}\\&\mathsf{lang} = \sigma\ \mathsf{Tag}\ \lambda\ \{\mathsf{fin}' \to \sigma\ \mathbb{N}\ \lambda\ n \to \iota\ n\\&\qquad\qquad ;\mathsf{sum}' \to \delta\ 1\ \lambda\ n \to \delta\ (\mathsf{Fin}\ (n\ \star))\ \lambda\ f \to \iota\ (\mathsf{sum}\ (n\ \star)\ f)\\&\qquad\qquad ;\mathsf{prod}' \to \delta\ 1\ \lambda\ n \to \delta\ (\mathsf{Fin}\ (n\ \star))\ \lambda\ f \to \iota\ (\mathsf{prod}\ (n\ \star)\ f)\}\\&\mathsf{example} : \mu\ \mathsf{lang}\\&\mathsf{example} = \mathsf{in}\ (\mathsf{sum}', (\lambda\ \_ \to \mathsf{in}\ (\mathsf{fin}', 5, \star)), (\lambda\ n \to \mathsf{in}\ (\mathsf{fin}', n, \star)), \star)\end{aligned}$$

The example expression denotes $\sum_{n<5} n$, and indeed, $\mathsf{decode}\ \mathsf{lang}\ \mathsf{example} = 10$.

Having introduced dependent polynomials, indexed containers and small induction recursion, we can now turn to the main focus of the paper, namely showing that they define the same class of functors and hence define the same class of data types. The key to the construction is observing that we may just as well interpret $\mathsf{IR}\ I\ O$ with our $I \to \mathsf{Set}$ presentation of slices.

$$\begin{aligned}&[\![\cdot]\!]_{\mathsf{IR}} : \mathsf{IR}\ I\ O \to (I \to \mathsf{Set}) \to (O \to \mathsf{Set})\\&[\![\iota\ o']\!]_{\mathsf{IR}}\ \ F\ o = o' \equiv o\\&[\![\sigma\ S\ K]\!]_{\mathsf{IR}}\ F\ o = (s{:}S) \times ([\![K\ s]\!]_{\mathsf{IR}}\ F\ o)\\&[\![\delta\ P\ K]\!]_{\mathsf{IR}}\ F\ o = (\mathit{if}{:}P \to \Sigma F) \times ([\![K\ (\pi_0 \circ \mathit{if})]\!]_{\mathsf{IR}}\ F\ o)\end{aligned}$$

The correspondence up to trivial isomorphism between $[\![\cdot]\!]_{\mathsf{IR}}$ and $[\![\cdot]\!]_{\mathsf{DS}}$ is readily observed by considering F here to be an arbitrary $(X, f)^{-1}$.

4 From Poly to small IR and back

We divide this section into two: (i) we first show how to translate dependent polynomials, and hence indexed containers, into IR codes; and (ii) we then show how every small IR code can be translated into a dependent polynomial. Crucially, we show that these translations preserve the functorial semantics of dependent polynomials and IR codes.

4.1 From Poly to small IR

We have already seen (example 5) that the extension of a container is an IR functor. We now extend this result to indexed containers and dependent polynomials.

▶ **Lemma 7.** *Every dependent polynomial functor is an* IR *functor.*

It is enough to show that, for every dependent polynomial $(r, t, q) : \mathsf{Poly}\ I\ O$, there is an $\mathsf{IR}\ I\ O$-code, whose interpretation is isomorphic to the dependent polynomial functor $[\![(r, t, q)]\!]_{\mathsf{Poly}}$. Our candidate for this IR-code is given and interpreted as follows

$$[\![\sigma\ S\ \lambda\ s\ \to\ \delta\ ((P, t)^{-1}\ s)\ \lambda\ i\ \to\ \sigma\ (i \equiv r\ \circ\ \pi_0)\ \lambda\ \_\ \to\ \iota\ (q\ s)]\!]_{\mathsf{IR}}\ F\ o\ =$$
$$(s : S) \times (\mathit{if} : ((P, t)^{-1}\ s\ \to\ \Sigma F)) \times (\pi_0\ \circ\ \mathit{if} \equiv r\ \circ\ \pi_0)\ \times\ (q\ s \equiv o)$$

which is readily seen to be isomorphic to $\Sigma_q\ \Pi_t\ \Delta_r\ F\ o$

$$(s : S) \times (q\ s \equiv o)\ \times\ (p : P) \to (t\ p \equiv s)\ \to\ F\ (r\ p)$$

as the former effectively constrains the function *if* to choose $r\ p$ as the index of its F, for each position $(p, \_) : (P, t)^{-1}\ s$.

4.2 From small IR to Poly

The essence of our embedding of $\mathsf{IR}\ I\ O$ into $\mathsf{Poly}\ I\ O$ consists of showing how three constructors for $\mathsf{IR}\ I\ O$-codes can be interpreted in $\mathsf{Poly}\ I\ O$.

▶ **Definition 8.** To each code $\gamma : \mathsf{IR}\ I\ O$ we associate a dependent polynomial

$$I \xleftarrow{\mathsf{t}\ \gamma} \mathsf{P}\ \gamma \xrightarrow{\mathsf{r}\ \gamma} \mathsf{S}\ \gamma \xrightarrow{\mathsf{q}\ \gamma} O$$

by structural recursion on γ.

- if γ is $\iota\ o$, then we define $\mathsf{S}\ \gamma = 1$, $\mathsf{P}\ \gamma = 0$, $\mathsf{r}\ \gamma = !_I$, $\mathsf{t}\ \gamma = !_1$, and $\mathsf{q}\ \gamma\ \star = o$. As a diagram, this is as follows.

$$I \xleftarrow{!_I} 0 \xrightarrow{!_1} 1 \xrightarrow{o} O$$

- if γ is $\sigma\ S\ K$ then the diagram is as follows.

$$I \xleftarrow{[\mathsf{r}\ (K\ s)]_{s:S}} \coprod_{s:S} \mathsf{P}\ (K\ s) \xrightarrow{\coprod_{s:S} \mathsf{t}\ (K\ s)} \coprod_{s:S} \mathsf{S}\ (K\ s) \xrightarrow{[\mathsf{q}\ (K\ s)]_{s:S}} O$$

  Here (and in the next clause) we use $\coprod_{s:S} m\ s$ to abbreviate the cotuple $[\mathsf{in}_s \circ m\ s]_{s:S}$.
- if γ is $\delta\ P\ K$, the diagram is as follows.

$$\begin{array}{ccc} \coprod_{i:P\to I} (P \times \mathsf{S}\ (K\ i)) + \mathsf{P}\ (K\ i) & \xrightarrow{\coprod_{i:P\to I} [\pi_0, \mathsf{t}\ (K\ i)]} & \coprod_{i:P\to I} \mathsf{S}\ (K\ i) \\ \downarrow{\scriptstyle [[i \circ \pi_0, \mathsf{r}\ (K\ i)]]_{i:P\to I}} & & \downarrow{\scriptstyle [\mathsf{q}\ (K\ i)]_{i:P\to I}} \\ I & & O \end{array}$$

  Note that in the last clause, it is crucial that we are dealing with small IR so that I is a set, hence $P\ \to\ I$ is a set and hence the coproducts used are also small.

  We can now state the result concerning the second half of our isomorphism.

▶ **Lemma 9.** *Every small* IR *functor is a dependent polynomial functor.*

To prove the lemma we define a function $\phi\ :\ \mathsf{IR}\ I\ O\ \to\ \mathsf{Poly}\ I\ O$ by recursion on the structure of IR codes and then we prove by induction that the functorial semantics is preserved. Details of the proof can be found in the online Appendix.

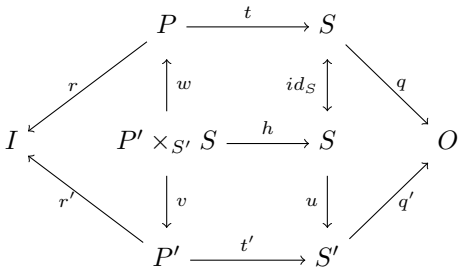
5 Equivalence between small IR and Poly

In the previous section we saw that every small IR functor gives rise to an isomorphic dependent polynomial functor and vice versa. What can we say about natural transformations between these functors? Before trying to answer this question we need to turn Poly $I\ O$ and IR $I\ O$ into categories. This section is therefore organised as follows: (i) we first recall the notion of morphism between dependent polynomials/indexed containers; (ii) then we introduce morphisms of IR codes, showing that the interpretation function, $[\![\_]\!]_{\mathsf{IR}} : \mathsf{IR}\ I\ O \to [\mathsf{Set}/I, \mathsf{Set}/O]$ can be extended to a functor which is full and faithful; and (iii) finally we prove the equivalence between the two categories IR $I\ O$ and Poly $I\ O$.

5.1 The categories Poly $I\ O$ and IC $I\ O$

Dependent polynomials/indexed containers with fixed input and output index sets, I and O, form a category. In this section we recall the definition of the morphisms between dependent polynomials and their interpretation as natural transformations. We conclude by stating some properties of the categories of dependent polynomials/indexed containers which allows us to recast in elementary terms the dependent polynomials introduced in definition 8.

▶ **Definition 10.** A morphism between dependent polynomials (r, t, q) and (r', t', q') is given by a diagram of the form



where the bottom square is a pullback of u and t'.

From now on, Poly $I\ O$ will indicate the category of dependent polynomials with fixed input and output index sets I, O and their morphisms. In a similar manner we can define morphism between indexed containers.

▶ **Definition 11.** A morphism between (S, P, n) and (S', P', n') consists of
- a function $u : (o : O) \to S\ o \to S'\ o$;
- a function $f : (o : O) \to S\ o \to P'\ o\ (u\ o\ s) \to P\ o\ s$;

such that for every $o : O$, $s : S\ o$ and $p' : P\ o\ (u\ o\ s)$ we have $n\ o\ s\ (f\ o\ s\ p') = n'\ o\ (u\ o\ s)\ p'$.

We will indicate with IC $I\ O$ the category of indexed containers and their morphisms. The main result concerning these morphisms is the following (Theorem 2.12 in [15]). We state the result for dependent polynomials but clearly an analogue result holds also for indexed containers.

▶ **Theorem 12** ([15] Theorem 2.12)**.** *Given dependent polynomials (r, t, q) and (r', t', q'), every natural transformation $[\![(r, t, q)]\!] \to [\![(r', t', q')]\!]$ is represented in an essentially unique way by a commuting diagram as in definition 10.*

This theorem ensures that the assignment to each dependent polynomial of its extension is a functor, and moreover this functor is full and faithful. In the following we indicate with $\mathit{PolyFun}\ I\ O$ the full subcategory of $[\mathsf{Set}/I, \mathsf{Set}/O]$ whose objects are dependent polynomial functors and whose morphisms are natural transformation between them[5].

▶ **Corollary 13** (Representation). *For any pair of sets I, O the functor*

$$[\![\ ]\!] : \mathsf{Poly}\ I\ O \to \mathit{PolyFun}(I, O)$$

is an equivalence of categories.

Dependent polynomials and indexed containers have several interesting closure properties. Here we only need closure under set-indexed coproducts and binary product. Note that we had to define morphisms before introducing these closure properties to ensure that they have the required categorical universal properties. The sum of a K-indexed family of dependent polynomials $\{Q_k = (r_k, t_k, q_k) \mid k : K\}$, for an arbitrary set K, is the dependent polynomial $\coprod_{k:K} Q_k$ given by the following diagram

$$I \xleftarrow{[r_k]_{k:K}} \coprod_{k:K} P_k \xrightarrow{\coprod_{k:K} t_k} \coprod_{k:K} S_k \xrightarrow{[q_k]_{k:K}} O$$

where $\coprod_{k:K} t_k = [\mathsf{in}_k \circ t_k]_{k:K}$. Note that the dependent polynomial associated to $\sigma\ S\ K : \mathsf{IR}\ I\ O$ is of exactly this form. The product of two dependent polynomial (r, t, q) and (r', t', q') is the evident dependent polynomial

$$I \longleftarrow (P' \times_O S) + (P \times_O S') \longrightarrow S \times_O S' \longrightarrow O.$$

We can now describe the dependent polynomial associated to a code $\delta\ P\ K : \mathsf{IR}\ I\ O$ as the sum of products of a family of dependent polynomials. We start with a family of dependent polynomials $\{(\mathsf{r}\ (K\ i), \mathsf{t}\ (K\ i), \mathsf{q}\ (K\ i) \mid i : P \to I\}$. For each element of this family we take the product of it with the dependent polynomial

$$I \xleftarrow{i \circ \pi_0} P \times O \xrightarrow{\pi_1} O \xrightarrow{id_O} O$$

and then we take the sum of these products over the set $P \to I$.

5.2 The category of small IR codes

We know how to define small IR codes and interpret them as functors between slices of Set. In this section we introduce morphisms between small $\mathsf{IR}\ I\ O$-codes. Our definition will ensure that every such morphism gives rise to a natural transformation between the corresponding IR functors – and *vice versa*. We start this section developing the appropriate categorical description of the semantics of IR constructors. The constructor ι simply represents constant functors while the constructor σ takes coproducts of functors. The following lemma tells us more about the semantics of δ.

[5] The original result for polynomial functors (Theorem 2.12 in [15]) is stated in terms of strong natural transformations. We can avoid mention of strength since natural transformations between functors on slices of Set are automatically strong.

▶ **Lemma 14.** *Given an object $k : X \to I$, there is a natural isomorphism*

$$[\![\delta\ P\ K]\!]\, k \quad \cong \quad \coprod_{i:P\to I} Hom_{\mathsf{Set}/I}(i,k) \otimes [\![K\ i]\!]_{\mathsf{IR}}\, k$$

Here $\otimes$ indicates the tensor product. Given a set X and an object $i : Y \to I$ of Set/I the object $X \otimes i$ is nothing but the copower $\coprod_{x:X} i$, i.e the X-fold coproduct of the object i.

Proof. We have a natural isomorphism

$$\begin{array}{rcl} [\![\delta\ P\ K]\!]\, k & = & \coprod_{x:P\to X} [\![K(k\circ x)]\!]_{\mathsf{IR}}\, k \\ & \cong & \coprod_{i:P\to I} \coprod_{x:P\to X} (i \equiv k\circ x) \otimes [\![K\ i]\!]_{\mathsf{IR}}\, k. \end{array}$$

Then observe that $\coprod_{x:P\to X}(i \equiv k\circ x) \cong Hom_{\mathsf{Set}/I}(i,k)$.

◀

Thanks to this lemma, we are able to characterise the semantics of δ-codes through a well-known universal construction in category theory: the left Kan extension.

If $i : X \to I$ is an object in Set/I we use $(+\ i)$, in the following lemma, to indicate the functor

$$\begin{array}{rccc} (+i): & \mathsf{Set}/I & \longrightarrow & \mathsf{Set}/I \\ & k & \longmapsto & [i,k]. \end{array}$$

▶ **Theorem 15.** *There is a natural isomorphism*

$$[\![\delta\, P\, F]\!] \quad \cong \quad \coprod_{i:P\to I} Lan_{(+i)}[\![F\ i]\!]$$

Our definition of $\mathsf{IR}\ I\ O$-morphisms is based on this isomorphism. First, we recall the universal property characterising the left Kan extension $Lan_G F : \mathbb{B} \to \mathbb{C}$ of a functor $F : \mathbb{A} \to \mathbb{C}$ along $G : \mathbb{A} \to \mathbb{B}$; for every functor $H : \mathbb{B} \to \mathbb{C}$ there is a bijection

$$Nat(Lan_G F, H) \cong Nat(F, H \circ G)$$

natural in H. We also need to check that $\mathsf{IR}\ I\ O$-functors are closed by precomposition with functors of the form $(+i)$. Fortunately, this can be easily checked by structural induction on codes. We just state the result.

▶ **Lemma 16.** *Given $\gamma : \mathsf{IR}\ I\ O$, and a function $i : P \to I$ there exists $\gamma^i : \mathsf{IR}\ I\ O$-code such that*

$$[\![\gamma]\!]_{\mathsf{IR}} \circ (+i) = [\![\gamma^i]\!]_{\mathsf{IR}}$$

We can now define IR morphisms by structural induction on codes as follows.

▶ **Definition 17.** Let $\gamma, \gamma' : \mathsf{IR}\ I\ O$ we define the homset $\mathsf{IR}(\gamma,\gamma')$ as follows.

Morphisms from ι-codes:

1A. $IR(\iota\ o, \iota\ o') = o \equiv o'$
1B. $IR(\iota\ o, \sigma\ S\ K) = \coprod_{s:S} IR(\iota\ o, K\ s)$
1C. $IR(\iota\ o, \delta\ P\ K) = \coprod_{e:P\to\emptyset} IR(\iota\ o, K\ (!\circ g))$

Morphisms from σ-codes:

2. $IR(\sigma\ S\ K, \gamma) = \prod_{s:S} IR(K\ s, \gamma)$

Morphisms from δ-codes:

3. $IR(\delta\ P\ K, \gamma) = \prod_{i:P\to I} IR(K\ i, \gamma^i)$

The following theorem shows we have the right notion of morphism for IR codes.

▶ **Theorem 18.** *The interpretation* $[\![\_]\!]_{\mathsf{IR}}$ *of* $\mathsf{IR}\ I\ O$*-codes can be extended to morphisms: we can associate to each* $\mathsf{IR}\ I\ O$*-morphism* $f : \gamma \to \gamma'$ *a natural transformation* $[\![f]\!]_{\mathsf{IR}} : [\![\gamma]\!]_{\mathsf{IR}} \to [\![\gamma']\!]_{\mathsf{IR}}$*. Moreover The assignment*

$$[\![\_]\!]_{\mathsf{IR}} : \mathsf{IR}\ I\ O\ \to\ [\mathsf{Set}/I, \mathsf{Set}/O]$$

is full and faithful.

The theorem is proved by induction on the structure of IR morphisms. As corollary we have the following important result.

▶ **Corollary 19.** $\mathsf{IR}\ I\ O$*-codes and their morphisms define a category. The interpretation*

$$[\![\_]\!]_{\mathsf{IR}} : \mathsf{IR}\ I\ O\ \to\ [\mathsf{Set}/I, \mathsf{Set}/O]$$

is full and faithful.

5.3 An equivalence

In the previous sections we have seen how to represent $\mathsf{IR}\ I\ O$-codes as dependent polynomials in $\mathsf{Poly}\ I\ O$ and vice versa. To sum up:

- In subsection 4.1, we saw how to translate a dependent polynomial (r, t, q) into an $\mathsf{IR}\ I\ O$-code, $\gamma^{(r,t,q)}$. Therefore we can define a function

$$\begin{aligned} \psi :\ & \mathsf{Poly}\ I\ O \to \mathsf{IR}\ I\ O \\ & (r, t, q) \mapsto (\gamma^{(r,t,q)}) \end{aligned}$$

  such that $[\![\_]\!]_{\mathsf{IC}} \cong [\![\_]\!]_{\mathsf{IR}} \circ \psi$
- in subsection 4.2 we showed how to define a function,

$$\begin{aligned} \phi : \mathsf{IR}\ I\ O &\to \mathsf{Poly}\ I\ O \\ \gamma &\mapsto (\mathsf{r}\ \gamma, \mathsf{t}\ \gamma, \mathsf{q}\ \gamma) \end{aligned}$$

  such that $[\![\_]\!]_{\mathsf{IR}} \cong [\![\_]\!]_{\mathsf{IC}} \circ \phi$.

We sum up these results in the following corollary.

▶ **Corollary 20.** *For every* $\gamma : \mathsf{IR}\ I\ O$ *and, for every* $(r, t, q) : \mathsf{Poly}\ I\ O$

1) $[\![\psi \circ \phi\ (\gamma)]\!]_{\mathsf{IR}} \cong [\![\gamma]\!]_{\mathsf{IR}}$,
2) $[\![\phi \circ \psi\ (r, t, q)]\!]_{\mathsf{Poly}} \cong [\![(r, t, q)]\!]_{\mathsf{Poly}}$

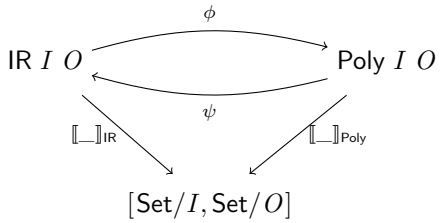
These isomorphisms deal just with objects of the two categories $\mathsf{IR}\ I\ O$ and $\mathsf{Poly}\ I\ O$. But what can we say about morphisms? As we show in the next theorem the equivalence of these two categories, is an immediate consequence of the previous results combined with full and faithfulness of the respective interpretation functions:

▶ **Theorem 21.** *The two categories* $\mathsf{IR}\ I\ O$ *and* $\mathsf{Poly}\ I\ O$ *are equivalent.*

It is immediate to show full and faithfulness of ϕ (or, equivalently of ψ):

$$\begin{aligned} \mathsf{IR}\ I\ O(\gamma, \gamma') &\cong Nat([\![\gamma]\!]_{\mathsf{IR}}, [\![\gamma']\!]_{\mathsf{IR}}) && \text{(corollary 19)} \\ &\cong Nat([\![\phi(\gamma)]\!]_{\mathsf{Poly}}, [\![\phi(\gamma')]\!]_{\mathsf{Poly}}) && \text{(lemma 9)} \\ &\cong \mathsf{Poly}\ I\ O(\phi(\gamma), \phi(\gamma')) && \text{(corollary 13)} \end{aligned}$$

Now, since we have already showed that each dependent polynomial, (r, t, q) is isomorphic to $\phi(\gamma)$ for some $\gamma : \mathsf{IR}\ I\ O$ (namely $\gamma = \psi(r, t, q)$), this is enough to conclude the stated equivalence (see theorem 1, par. 4, ch. IV in [18]). Here is a commutative diagram which represents the statement of theorem 21:



6 Small indexed Induction Recursion

The theory of induction recursion has been extended by Dybjer and Setzer in [13] in order to capture more sophisticated inductive-recursive definitions. As indexed container and dependent polynomials generalise polynomials and containers respectively, the theory of indexed induction-recursion (IIR) generalises the theory inductive-recursive definitions in order to capture, not only ordinary inductive-recursive definition, but also families of inductive-recursive definitions which admit extra indexing. IR then appears as the fragment of IIR given by those definitions indexed over a singleton.

We will briefly recall the axiomatic presentation of IIR which closely follows that of IR. We then show how the theory of small indexed inductive-recursive definitions (*small* IIR) can be reduced to small IR. This simple fact will automatically transfer the results of the previous sections to small IIR, allowing to conclude a generalisation of the equivalence stated in theorem 21. We now give the coding for small IIR.

$$
\begin{array}{l}
\textbf{data}\ \mathsf{IIR}\ (D : I \to \mathsf{Set})\ (E : J \to \mathsf{Set}) : \mathsf{Set1}\ \textbf{where} \\
\quad \iota : (je : \Sigma E) \to \mathsf{IIR}\ D\ E \\
\quad \sigma : (S : \mathsf{Set})\ (K : S \to \mathsf{IIR}\ D\ E) \to \mathsf{IIR}\ D\ E \\
\quad \delta : (P : \mathsf{Set})\ (i : P \to I)\ (K : ((p : P) \to D\ (i\ p)) \to \mathsf{IIR}\ D\ E) \to \mathsf{IIR}\ D\ E
\end{array}
$$

Note that δ carries an extra argument i, selecting the index for each position in P. One way to interpret these codes is by translation to the codes for $\mathsf{IR}\ \Sigma D\ \Sigma E$, as follows:

$$
\begin{array}{l}
\lfloor \cdot \rfloor : \mathsf{IIR}\ D\ E \to \mathsf{IR}\ \Sigma D\ \Sigma E \\
\lfloor \iota\ je \rfloor = \iota\ je \\
\lfloor \sigma\ S\ K \rfloor = \sigma\ S\ \lambda\ s \to \lfloor K\ s \rfloor \\
\lfloor \delta\ P\ i\ K \rfloor = \delta\ P\ \lambda\ iD \to \sigma\ (i \equiv (\pi_0 \circ iD))\ \lambda\ q \to \lfloor K\ (\pi_1 \circ iD) \rfloor
\end{array}
$$

In the δ case, the generated IR code yields a ΣD for each position in P, so we constrain its first component to coincide with the index required by the i in the IIR code. Given this embedding, we can endow small IIR with the categorical machinery developed for small IR in Section 5.2. We therefore can straightforwardly define a category of $\mathsf{IIR}\ D\ E$ -codes and their morphisms. Theorem 21 in Section 5 immediately give us the following corollary.

▶ **Corollary 22.** *The category* $\mathsf{IIR}\ D\ E$ *and the category* $\mathsf{Poly}\ \Sigma D\ \Sigma E$ *are equivalent.*

We can also follow Dybjer and Setzer by giving a direct interpretation of an IIR code as a functor between families of slice categories.

$$
\begin{array}{l}
[\![\cdot]\!]_{\mathsf{IIR}} : \mathsf{IIR}\ D\ E \to ((i : I) \to \mathsf{Set}/(D\ i)) \to ((j : J) \to \mathsf{Set}/(E\ j)) \\
[\![\iota\ (j', e)]\!]_{\mathsf{IIR}}\ \ G\ j = ((j' \equiv j), \lambda\ q \to \cdot\ q\ e) \\
[\![\sigma\ S\ K]\!]_{\mathsf{IIR}}\ \ G\ j = (s : S) \times ([\![K\ s]\!]_{\mathsf{IIR}}\ G\ j) \\
[\![\delta\ P\ i\ K]\!]_{\mathsf{IIR}}\ G\ j = (ig : (p : P) \to \mathsf{dom}\ (G\ (i\ p))) \times ([\![K\ (\lambda\ p \to \mathsf{fun}\ (G\ (i\ p))\ (ig\ p))]\!]_{\mathsf{IIR}}\ G\ j)
\end{array}
$$

We note that keeping I and D small ensures the following:

$$(i : I) \to \mathsf{Set}/(D\ i) \ \cong\ (i : I) \to D\ i \to \mathsf{Set} \ \cong\ \Sigma D \to \mathsf{Set} \ \cong\ \mathsf{Set}/\Sigma D$$

Consider $G\ i = (\exists.(F \circ (i,)))$ for some $F : \Sigma D \to \mathsf{Set}$ to see that $[\![\gamma]\!]_{\mathsf{IIR}}\ G$ corresponds to $[\![\lfloor\gamma\rfloor]\!]_{\mathsf{IR}}\ F$, up to bureaucratic isomorphism.

Once again, we construct simultaneously an indexed family of data types $\mu\ \gamma\ i$ and their decoders $\mathsf{decode}\ i$ as the initial algebra for $[\![\gamma]\!]_{\mathsf{IIR}}$.

```
μd : (γ : IIR D D) → (i : I) → Set/(D i)
μd γ i = (μ γ i, decode γ i)
data μ (γ : IIR D D) (i : I) : Set where
  in : dom ([[γ]]_IIR (μd γ) i) → μ γ i
decode : (γ : IIR D D) → (i : I) → μ γ i → D i
decode γ i (in t) = fun ([[γ]]_IIR (μd γ) i) t
```

The corresponding fixpoint of $[\![\lfloor\gamma\rfloor]\!]_{\mathsf{IR}}$ gives the inductive family indexed by pairs in ΣD.

▶ **Example 23.** The Bove-Capretta method, applied to call-by-value computation Bove and Capretta [5] make use of indexed induction-recursion to model the *domains* of partial function. A partial function $d : (i : I) \rightharpoonup D\ i$ has a domain given a code $\gamma : \mathsf{IIR}\ D\ D$. If $h : \mu\ \gamma\ i$ gives evidence that the domain is inhabited at argument i, then $\mathsf{decode}\ \gamma\ i\ h$ is sure to compute the result.

Let us take a concrete example. One might define a type of λ-terms and seek to give a call-by-value evaluator for them, as follows.

```
data Tm : Set where
  var : ℕ → Tm
  app : Tm → Tm → Tm
  lam : Tm → Tm

cbv : Tm ⇀ Tm
cbv (var x) = var x
cbv (lam t) = lam t
cbv (app f s) with cbv f
...           | lam t = cbv (subst0 (cbv s) t)
...           | f'    = app f (cbv s)
```

where, say, we adopt a de Bruijn indexing convention and define $\mathsf{subst0}\ s\ t$ to substitute s for variable *0* in t. Of course, cbv is not everywhere defined. Can we say when it is defined? It is hard to define the domain *inductively*, because the $\mathsf{app}\ f\ s$ case will require that $\mathsf{subst0}\ (\mathsf{cbv}\ s)\ t$ is in the domain whenever f is in the domain *and evaluates to* $\mathsf{lam}\ t$. We need to define the domain simultaneously with evaluation — a job for induction-recursion.

It will prove convenient to define the special case of δ when $P = 1$.

$$
\begin{array}{l}
\delta_1 : (i : I) \to (K : D\ i \to \mathsf{IIR}\ D\ E) \to \mathsf{IIR}\ D\ E \\
\delta_1\ i\ K = \delta\ 1\ (\lambda\ \_ \to i)\ \lambda\ d \to K\ (d\ \star)
\end{array}
$$

In the code for a domain predicate, a recursive call at i gives rise to a $\delta_1\ i\ K$ code, where K explains how to carry on if the call returns. Let us give the domain of cbv.

```
cbvD : IIR {Tm} {Tm} (λ _ → Tm) (λ _ → Tm)
cbvD = σ Tm λ
  {(var x)    → ι (var x, var x)
  ;(lam t)    → ι (lam t, lam t)
  ;(app f s) → δ₁ f λ {(lam t) → δ₁ s λ s' → δ₁ (subst0 s' t) λ t' → ι (app f s, t')
                      ;f'      → δ₁ s λ s'                        → ι (app f s, app f' s')
  }                   }
```

Note the way the application case makes key use of the delivered values in subsequent recursive calls, and in every case, the final ι delivers an input-output pair. The type μ cbvD t thus contains the evidence that cbv t terminates without presupposing a particular value — decoding that evidence will yield t's value. The equivalence we have demonstrated in this paper ensures that the corresponding inductive *family* indexed over Tm × Tm is exactly the big-step evaluation relation for cbv.

7 Conclusion and further work

The pioneering work of Dybjer and Setzer on induction recursion has opened up new realms of data types. They have given us tools to program with universes closed under dependently typed operators. Despite its evident potential the theory of induction recursion has not become as widely understood and used as it should be. In this paper we seek to broaden appreciation of their work by comparing it with better-known theories of data types based on dependent polynomials, and more practically with indexed containers. In the case of *small* IR, these three analyses coincide. We can now pick up the fruits of our central result (theorem 21).

Initial algebras. When interpreting codes in IR I I we get endofunctors on Set/I. Theorem 21 ensures that initial algebras for these functors always exist, since they are nothing but Peterson-Synek's general tree types, i.e. initial algebras for dependent polynomial endofunctors. Altenkirch and Morris have generalized in [3] the general tree types to account for parametrized initial algebras of indexed containers of type IC $(I + O)$ O : in fact, the same construction carries over into IR $(I + O)$ O functors. We also have a clear understanding of the dual of initial algebras of IR functors: final coalgebras for small IR functors, recently investigated by Capretta with the name of wander. These are nothing but coalgebras for dependent polynomial functors/indexed containers.

Closure properties. The axiomatization of small IR and its semantics provides a new (but equivalent) grammar to work with the categories Poly and IC. It is known that these categories have very rich closure properties such as sums, products, composition, as well as linear and differential structure. Clearly we can transport these properties along the equivalence of theorem 21.

Compositions. A difficult open question in the theory of induction-recursion is whether the Dybjer-Setzer functors are closed under composition: given codes γ : IR I J and γ' : IR J O is it always possible to find a code ξ in IR I O such that $[\![\gamma']\!] \circ [\![\gamma]\!] \cong [\![\xi]\!]$? Theorem 21 ensures that we can transport composition in Poly or IC to obtain closure under composition of small IR functors.

Further work. We have proved that Poly I O, IC I O and IR I O are equivalent categories which define the same class of functors. It is easy to generalize this result to a biequivalence of bicategories. Since it is possible to define reindexing of IR codes and IR functors, in future work we would like to explore this extra-structure of small IR and compare

it with the double category of Poly. Abstracting from the category of sets we also aim to investigate to which extent this result applies to arbitrary locally cartesian closed categories.

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A Proofs of section 4

```
thecode : {O S P I : Set} →
  (q : S → O) → (t : P → S) → (r : P → I) → IR I O
thecode {O} {S} {P} {I} q t r = σ S (λ s →
                                  let Ps = (P , t) ⁻¹ s in
                                  δ Ps (λ i →
                                  σ ((a : Ps) → i a ≡ r (π₀ a)) (λ _ →
                                  ι (q s))))

iso : (I : Set) → (O : Set) → Set
iso I O = (f : I → O) × (g : O → I) ×
          ((i : I) → g (f i) ≡ i) × ((o : O) → f (g o) ≡ o)

transport : {I : Set} → {F : I → Set} → {i i' : I} → F i → i ≡ i' → F i'
transport fi refl = fi

postulate ext : forall {S : Set} {T : S → Set} (f g : (x : S) → T x) →
          ((x : S) → f x ≡ g x) → f ≡ g

dpright : forall {S : Set} {T : S → Set} {s : S} {t t' : T s} → t ≡ t' →
  _==_ {(· : S) × ·} (s , t) (s , t')
dpright refl = refl

PIrr : Set → Set
PIrr X = forall (x y : X) → x ≡ y

uip : forall {X} {x y : X} → PIrr (x ≡ y)
uip refl refl = refl

SgPIrr : forall {S T} → PIrr S → ((s : S) → PIrr (T s)) → PIrr (· : S) × ·
SgPIrr si ti (s , t) (s' , t') with si s s'
SgPIrr si ti (s , t) (.s , t') | refl with ti s t t'
SgPIrr si ti (s , t) (.s , .t) | refl | refl = refl

PiPIrr : forall {S T} → ((s : S) → PIrr (T s)) → PIrr (· : S) → ·
PiPIrr ti f g = ext f g (λ x → ti x (f x) (g x))

pi1Irr : forall {S T} → ((s : S) → PIrr (T s)) → {s s' : S} → (s ≡ s') →
  {t : T s} {t' : T s'} → _==_ {(· : S) × ·} (s , t) (s' , t')
pi1Irr ti refl = dpright (ti _ _ _)

transl : forall {S T s} {x : (· : ·) × S T} (q : π₀ x ≡ s) →
         (s , transport {F = T} (π₁ x) q) ≡ x
transl refl = refl

theclaim : {O S P I : Set} →
  (q : S → O) → (t : P → S) → (r : P → I) →
  (F : I → Set) → (o : O) →
  let lhs = ⟦thecode q t r⟧IR F o
      rhs = Σ. q Π_t Δ_r F o
  in iso lhs rhs
theclaim {O} {S} {P} {I} q t r
  F o = let l2r : ⟦thecode q t r⟧IR F o → Σ. q Π_t Δ_r F o
            l2r x = let s : S
                        s = π₀ x
                        Ps : Set
```

```
                Ps = (P, t)^-1 s
                thing = π1 x
                fun : Ps → (i:I) × F i
                fun = π0 thing
                fun0 : Ps → I
                fun0 x = π0 (fun x)
                fun1 : (pe:Ps) → F (fun0 pe)
                fun1 x = π1 (fun x)
                otherthing : ((pe : Ps) → fun0 pe ≡ r (π0 pe)) × (q s ≡ o)
                otherthing = π1 thing
                ue : (pe : Ps) → fun0 pe ≡ r (π0 pe)
                ue = π0 otherthing
                secondbit : q s ≡ o
                secondbit = π1 otherthing
                thirdbit : (p : P) → t p ≡ s → F (r p)
                thirdbit p etps = let pe = (p, etps)
                                  in transport {I} {F} {fun0 pe} {r p} (fun1 pe) (ue pe)
            in (s, (secondbit, thirdbit))
        r2l : Σ. q Π_t Δ_r F o → [[thecode q t r]]_IR F o
        r2l y = let s : S
                s = π0 y
                thing : (q s ≡ o) × ((p : P) → t p ≡ s → F (r p))
                thing = π1 y
                eqso : q s ≡ o
                eqso = π0 thing
                tofrp : (p : P) → t p ≡ s → F (r p)
                tofrp = π1 thing
                Ps = (P, t)^-1 s
                secondbit : Ps → ((i:I) × F i)
                secondbit petps = (r (π0 petps), tofrp (π0 petps) (π1 petps))
                thirdbit : (pe : Ps) → π0 (secondbit pe) ≡ r (π0 pe)
                thirdbit _ = refl   -- plain weird
            in (s, (secondbit, (thirdbit, eqso)))
    in l2r, (r2l, ((λ {(s, (pif, (paf, qs))) →
      dpright (pi1Irr (λ if → SgPIrr (PiPIrr (λ _ → uip)) (λ _ → uip))
        (ext _ _ (λ x → transl (paf x))))}),
          (λ _ → refl)))   -- not much hope of this – au contraire
```

Proof of Lemma 7. We will show that a dependent polynomial (r, t, q): Poly I O, has the same functorial semantics as the IR-code

```
σ S λ s →
  let Ps = (P, t)^-1 s in
  δ Ps λ i →
  σ ((a : Ps) → i a ≡ r (π0 a)) λ _ →
  ι (q s)
```

Some definitions:

$$\begin{aligned}&\hat{S} \;:\; O \to \mathsf{Set}\\ &\hat{S} \;=\; (S,q)^{-1}\end{aligned}$$

Then

$$\begin{aligned}(o\!:\!O) \times \hat{S}\ o &= (o\!:\!O) \times (s\!:\!S) \times q\ s \equiv o\\ &\cong (s\!:\!S) \times (o\!:\!O) \times q\ s \equiv o\\ &\cong S\end{aligned}$$

$$\begin{aligned}&\hat{P} \;:\; (o \;:\; O) \to \hat{S}\ o \to \mathsf{Set}\\ &\hat{P}\ o\ s \;=\; (P,t)^{-1}\ (o,s)\end{aligned}$$

where we have converted the dependent polynomial (r,t,q) into its representation as an indexed container $(\hat{S},\hat{P},n)$ in the internal language. This means we have $\hat{S} : O \to Set$, $\hat{P} : (o : O) \to \hat{S}\,o \to Set$ and $n : (o\ O) \to (s : \hat{S}\,o) \to \hat{P}\,o\,s \to I$.

We can now prove that the interpretation of this code corresponds to the extension of the given dependent polynomial. We do this using the internal language. So, given $X : I \to Set$

$$\begin{aligned}&[\![\sigma(o : O).\ \sigma(s : \hat{S}\,o).\ \delta(f : \hat{P}\,o\,s \to I).\ \sigma(\_ : f \equiv n\,o\,s).\ \iota\,o]\!]_{IR}\,X\,o'\\ &= \sum o : O.\ \sum s : \hat{S}\,o.\ [\![\delta(f : \hat{P}\,o\,s \to I).\ \sigma(\_ : f \equiv n\,o\,s).\ \iota\,o]\!]_{IR}\,X\,o'\\ &= \sum o : O.\ \sum s : \hat{S}\,o.\ \sum g : \hat{P}\,o\,s \to I.\ (\prod p : \hat{P}\,o\,s.\ X\,(g\,p)) \times [\![\sigma(\_ : g \equiv n\,o\,s).\ \iota\,o]\!]_{IR}\,X\,o'\\ &= \sum o : O.\ \sum s : \hat{S}\,o.\ \sum g : \hat{P}\,o\,s \to I.\ (\prod p : \hat{P}\,o\,s.\ X\,(g\,p)) \times (g \equiv n\,o\,s) \times o \equiv o'\\ &= \sum s : \hat{S}\,o'.\ \sum g : \hat{P}\,o'\,s \to I.\ (\prod p : \hat{P}\,o'\,s.\ X\,(g\,p)) \times (g \equiv n\,o'\,s)\\ &= \sum s : \hat{S}\,o'.\ \prod p : \hat{P}\,o'\,s.\ X\,(n\,o'\,s\,p)\end{aligned}$$

The last line is exactly $[\![\hat{S},\hat{P},n]\!]_{\mathsf{IC}}X\,o'$. Thus the dependent polynomial (r,t,q) and the IR code given above have the same functorial semantics in the internal language. They thus have the same functorial semantics. ◀

Proof of Lemma 9. We prove the lemma by constructing a function $\phi : \mathsf{IR}\ I\ O \to \mathsf{Poly}\ I\ O$ defined structural recursion on its argument. We will then prove that for every $\gamma : \mathsf{IR}\ I\ O$

$$[\![\gamma]\!]_{IR} = [\![\phi(\gamma)]\!]_{\mathsf{Poly}}. \tag{1}$$

Let $\gamma : \mathsf{IR}\ I\ O$ we define $\phi(\gamma)$ as follow:

1. if γ is $\iota\ o$ for some $o : O$ then
$$\phi(\gamma) = (r^{\iota\,o}, t^{\iota\,o}, q^{\iota\,o})$$
2. if γ is $\sigma\,A\,f$ for some $A : Set$, $f : A \to \mathsf{IR}\ I\ O$ then
$$\phi(\gamma) = (r^{\sigma\,A\,h}, t^{\sigma\,A\,h}, q^{\sigma\,A\,h})$$
where $h = \phi \circ f : A \to \mathsf{Poly}\ I\ O$
3. if γ is $\delta\,A\,F$ for some $A : Set$, $F : (A \to I) \to \mathsf{IR}\ I\ O$ then
$$\phi(\gamma) = (r^{\delta\,A\,H}, t^{\delta\,A\,H}, q^{\delta\,A\,H})$$
where $H = \phi \circ F : I^A \to \mathsf{Poly}\ I\ O$.

We can now prove that (1) holds. As before we will use the internal language and we will associate to $k : X \to I$ in Set/I the Set-valued function $X : I \to Set$. Moreover, for every dependent polynomial $(r^\gamma, t^\gamma, q^\gamma)$ representing an IR code we define by recursion on the structure of the code its representation as an indexed container $(\hat{S}^\gamma, \hat{P}^\gamma, n^\gamma)$.

1. if γ is $\iota\ o\ :\ \mathsf{IR}\ I\ O$ we can compute the fibre at $o' : O$ of $[\![\phi(\iota\, o)]\!]_{\mathsf{Poly}}\, k$ as follows

$$
\begin{aligned}
&[\![\phi(\iota\, o)]\!]_{\mathsf{IC}}\, X\, o' \\
&\qquad = [\![(\hat{S}^{\iota\, o}, \hat{P}^{\iota\, o}, n^{\iota\, o})]\!]_{\mathsf{IC}}\, X\, o' \\
&\qquad = \sum s : \hat{S}^{\iota\, o}\, o'.\ \prod p : \hat{P}^{\iota\, o}\, s.\, X\, (n^{\iota\, o} o'\, s\, p) \\
&\qquad = \sum m : (o' \equiv o).\ \prod p : \emptyset.\, X\, (n^{\iota\, o} o'\, s\, p) \\
&\qquad \cong (o' \equiv o) \\
&\qquad = [\![\iota\, o]\!]_{IR}\, X\, o'
\end{aligned}
$$

2. if γ is $\sigma\ A\ f : \mathsf{IR}\ I\ O$ we can compute the fibre at $o : O$ of $[\![\phi(\sigma\, A\, f)]\!]_{\mathsf{Poly}}\, k$ as follows

$$
\begin{aligned}
&[\![\phi(\sigma\, A\, f)]\!]_{\mathsf{IC}}\, X\, o \\
&\qquad = [\![(\hat{S}^{\sigma A h}, \hat{P}^{\sigma A h}, n^{\sigma A h})]\!]_{\mathsf{IC}}\, X\, o \\
&\qquad = \sum s : \hat{S}^{\sigma A h}\, o.\ \prod p : \hat{P}^{\sigma A h}\, s.\, X\, (n^{\sigma A h} o\, s\, p) \\
&\qquad = \sum a : A.\ \sum s : \hat{S}^{h(a)}\, o.\ \prod p : \hat{P}^{h(a)}\, s.\, X\, (n^{h(a)} o\, s\, p) \\
&\qquad \cong \sum a : A.\, [\![(\hat{S}^{h(a)}, \hat{P}^{h(a)}, n^{h(a)})]\!]_{\mathsf{IC}}\, X\, o \\
&\qquad = \sum a : A.\, [\![\phi(f\, a)]\!]_{\mathsf{IC}}\, X\, o \qquad \text{(inductive hypothesis)} \\
&\qquad = \sum a : A.[\![f\, a]\!]_{IR}\, X\, o \\
&\qquad = [\![\sigma\, A\, f]\!]_{IR}\, X\, o
\end{aligned}
$$

3. if γ is $\delta\, A\, F : \mathsf{IR}\ I\ O$ we can compute the fibre at $o : O$ of $[\![\phi(\delta\, A\, F)]\!]_{\mathsf{Poly}}\, k$ as follows

$$
\begin{aligned}
&[\![\phi(\delta\, A\, F)]\!]_{\mathsf{IC}}\, X,\, o \\
&= [\![(\hat{S}^{\delta\, A\, H}, \hat{P}^{\delta\, A\, H}, n^{\delta\, A\, H})]\!]_{\mathsf{IC}}\, X,\, o \\
&= \sum s : \hat{S}^{\delta\, A\, H} o.\ \prod p : \hat{P}^{\delta\, A\, H}\, s.\, X\, (n^{\delta\, A\, H} o\, s\, p) \\
&= \sum g : A \to I.\ \sum s : \hat{S}^{H\, g}\, o.\ \prod p : A + \hat{P}^{H\, g}\, s.\, X[g, (n^{H\, g}\, o\, s)](p) \\
&\qquad\qquad \text{(universal property of coproduct)} \\
&\cong \sum g : A \to I.\ \sum s : \hat{S}^{H\, g}\, o.\, (\prod a : A.\, X\, (g\, a)) \times (\prod p : \hat{P}^{H\, g}\, s.\, X\, n^{H\, g}\, o\, s\, p)) \\
&\cong \sum g : A \to I.\, (\prod a : A.\, X\, (g\, a)) \times (\sum s : \hat{S}^{H\, g}\, o.\ \prod p : \hat{P}^{H\, g}\, s.\, X\, (n^{H\, g}\, o\, s\, p)) \\
&= \sum g : A \to I.\, (\prod a : A.\, X\, (g\, a)) \times [\![(\hat{S}^{H\, g}, \hat{P}^{H\, g}, n^{H\, g})]\!]_{\mathsf{IC}}\, X\, o \qquad \text{(definiton of } H) \\
&= \sum g : A \to I.(\prod a : A.\, X\, (g\, a)) \times [\![\phi(F\, g)]\!]_{\mathsf{IC}}\, X\, o \qquad \text{(inductive hypothesis)} \\
&= \sum g : A \to I.\, (\prod a : A.\, X\, (g\, a)) \times [\![F\, g]\!]_{IR}\, X\, o \\
&= [\![\delta\, A\, F]\!]_{IR}\, X\, o
\end{aligned}
$$

◀

B Proofs of section 5.2

Proof of Theorem 15.

$[\![\delta\, A\, F]\!]_{IR}\,(k : X \to I)$

(lemma 14)

$\cong \coprod_{g:A\to I} [\![F\, g]\!]\, k \otimes Hom_{Set/I}(g,k)$

(every functor is its own left Kan extension along the identity functor)

$\cong \coprod_{g:A\to I} \left(Lan_{Id_{Set/I}} [\![F\, g]\!]\right) \otimes Hom_{Set/I}(g,k)$

(compute Lan with coend and tensor product)

$\cong \coprod_{g:A\to I} \left(\int^{l:Set/I} [\![F\, g]\!]\, l \otimes Hom_{Set/I}(l,k)\right) \otimes Hom_{Set/I}(g,k)$

(tensor distributes over colimits)

$\cong \coprod_{g:A\to I} \int^{l:Set/I} \left([\![F\, g]\!]\, l \otimes Hom_{Set/I}(l,k) \otimes Hom_{Set/I}(g,k)\right)$

(tensor between sets is product)

$\cong \coprod_{g:A\to I} \int^{l:Set/I} \left([\![F\, g]\!]\, l \otimes Hom_{Set/I}(l,k) \times Hom_{Set/I}(g,k)\right)$

(universal property of coproduct)

$\cong \coprod_{g:A\to I} \int^{l:Set/I} \left([\![F\, g]\!]\, l \otimes Hom_{Set/I}(g+l,k)\right)$

(definition of $(+g)$)

$\cong \coprod_{g:A\to I} \int^{l:Set/I} \left([\![F\, g]\!]\, l \otimes Hom_{Set/I}((+g)\, l,k)\right)$

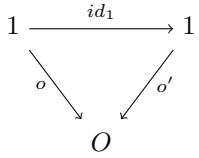
definition of Lan with coend and tensor product

$\cong \coprod_{g:A\to I} \left(Lan_{(+g)} [\![F\, g]\!]\right) k$

◀

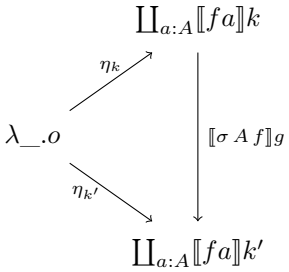
Proof of Theorem 18. We first prove by induction on the structure of IR morphisms that the assignment of natural transformations to IR morphisms is injective and surjective. This enable us to define a category of IR $I\, O$-codes.

1.A Since $[\![\iota\, o]\!]$ is the constant functor with value $\lambda\_.o : 1 \to O$, a natural transformation $\eta : [\![\iota\, o]\!] \dot{\to} [\![\iota\, o']\!]$ consists of a morphism in Set/O



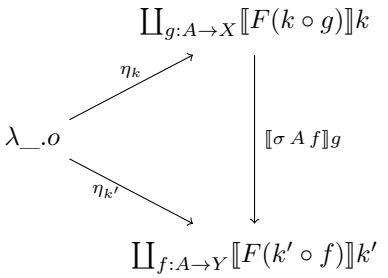
Such a morphism exists if and only if $o \equiv o'$

1.B Given $g : k \to k'$ in Set/I, a naturality square for $\eta : [\![\iota\, o]\!] \dot{\to} [\![\sigma\, A\, f]\!]$ is a commuting diagram in Set/O of the form



Now observe that every component η_k is uniquely determined by $\eta_k = [\![\sigma\, A\, f]\!]!_k \circ \eta_!$ where $! : \emptyset \to I$ is the initial object in Set/I and $!_k : ! \to k$ is the unique morphism from the initial object into k. Therefore $\eta_!$ define for some $a : A$, a morphism $\eta^a_! : [\![\iota\, o]\!]! \to [\![fa]\!]!$. This morphism gives, for all $k : \mathsf{Set}/I$, a morphism $\eta^a_k : [\![\iota\, o]\!]k \to [\![fa]\!]k$, natural in k, defined by $\eta^a_k = [\![fa]\!]!_k \circ \eta^a_!$, and therefore a natural transformation $\eta^a : [\![\iota\, o]\!] \dot{\to} [\![fa]\!]$.

1.C Given two objects $k : X \to I$, $k' : Y \to I$ in Set/I and a morphism $g : k \to k'$ $(k = k' \circ g)$, a naturality square for $\eta : [\![\iota\, o]\!] \dot{\to} [\![\delta\, A\, F]\!]$ is a commuting diagram in Set/O of the form



Reasoning as above we have, for every k in Set/I, $\eta_k = [\![\delta\, A\, F]\!]!_k \circ \eta_!$. Therefore $\eta_!$ define a morphism $\eta^g_! : [\![\iota\, o]\!]! \to [\![F(! \circ g)]\!]!$ for some $g : A \to \emptyset$. For every k in Set/I we define the component at k of $\eta^g : [\![\iota\, o]\!] \dot{\to} [\![F(! \circ g)]\!]$ to be $[\![F(! \circ g)]\!]!_k \circ \eta^g_!$.

2.

$$
\begin{aligned}
&Nat([\![\sigma\, A\, f]\!][\![\gamma]\!]) \\
&\qquad = Nat(\coprod_{a:A}[\![f(a)]\!], [\![\gamma]\!]) \\
&\qquad \cong \prod_{a:A} Nat([\![f(a)]\!], [\![\gamma]\!]) && \text{(inductive hypothesis)} \\
&\qquad \cong \prod_{a:A} IR(f(a), \gamma) \\
&\qquad \cong IR(\sigma\, A\, f, \gamma)
\end{aligned}
$$

3.

$$
\begin{aligned}
&Nat([\![\delta\, A\, F]\!][\![\gamma]\!]) \\
&\qquad \cong Nat(\coprod_{g:A\to I} Lan_{+g}[\![F\, g]\!], [\![\gamma]\!]) \\
&\qquad \cong \prod_{g:A\to I} Nat([\![F\, g]\!], [\![\gamma]\!] \circ (+g)) \\
&\qquad \cong \prod_{g:A\to I} Nat([\![F\, g]\!], [\![\gamma^g]\!]) && \text{(inductive hypothesis)} \\
&\qquad \cong \prod_{g:A\to I} IR(F\, g, \gamma^g) \\
&\qquad = IR(\delta\, A\, F, \gamma)
\end{aligned}
$$

◀

Proof of Corollary 19. Theorem 18 already established a full and faithful assignment of natural transformation to IR morphisms. This guarantees that we can define identity and composition, making $\mathsf{IR}\, I\, O$ a category. ◀