Hand in solutions to nos. 1, 2 and 3 by 2PM on 22 March. Follow instructions for each.

1. Input your nine digit registration number at a website found from the course web page, to get seventeen numbers $n_1, n_2, \dots, n_{17}$. If you randomly pick three of these numbers, what is the probability that at least two of them are smaller than the average of all seventeen?

   Explain your reasoning carefully, and also what exact calculations you did. You have to do this "from scratch", that is, you cannot use formulas from others (unless you first prove them), only things we have gone through in the course so far.

   Show solution at beginning of problem

   Display, *at the beginning of your solution*, your registration number, the sum of your 17 numbers and their average, as well as your answer. Give your answer as an exact fraction of integers and as an approximation correctly rounded to 5 decimal places. Sample answer:

   201612345   193   11.35394 $\frac{378}{874} \approx 0.43249$

2. Suppose you play the following game: You flip 6 coins. If you get at least 4 heads, you win £300, otherwise you lose £200.

   (a) What should you expect your net outcome to be after a hundred games?

   (b) Suppose you flip 7 coins and need to get at least 5 heads, but you get to repeat the process n times, and you now win £100 if you get at least 5 heads in at least one of the n rounds, but lose £200 if you don't. How many rounds (how big an n) must you demand at least in order not to lose money (on average)?

   You have to do this "from scratch"; you cannot use formulas from others (unless you first prove them), only things we have gone through in the course so far.

   Follow instructions

   For both parts, show calculations and explain your reasoning carefully. For part (a), give an exact answer and a correctly rounded approximation to 3 decimal places. *At the beginning of your solution* of this problem, list each of your answers, one per line.

3. Two people play the following game, starting with a pile of n matches, where n is the number formed by the last five digits of your registration number: On each turn, a player removes 1 or 3 or 4 matches. The last person to remove a match wins. Who wins? Write a clear and convincing solution, where you describe the losing and winning positions and explain why they satisfy conditions 2 and 3 overleaf.

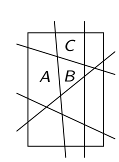
   Important instructions

   Begin your solution by describing all final, losing and winning positions for this game. Give a clear and concise argument for why your description is correct, that is, why these positions satisfy the conditions overleaf.

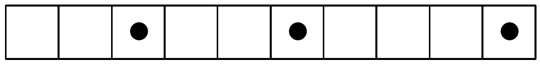
THREE BONUS PROBLEMS

4. **(No collaboration allowed, no use of external resources)** Two players take turns placing bishops on a chessboard so that none of them can take another. The player who places the last one wins. Who wins?

5. **(No collaboration, no external resources)** Suppose I draw a number of straight lines on a piece of paper, each stretching to the edges. This splits the paper into regions. How many colors do you need to color the regions so that no two regions sharing an edge get the same color? (For example, A and B share an edge, as do B and C, but A and C don't share an edge, just a point.)



6. **(No collaboration, no use of external resources)** On a strip of squares we have three coins, on squares 3, 6 and 10 from the left (see below). In a move, a player can move any one of the coins any number of squares to the left, but no farther than up to the next coin (and not off the left end). So, for example, the middle coin can be moved only 1 or 2 squares to the left. The last player to make a move wins. Who wins, and how? Your solution should work for any placement of three coins.



ANALYSING SYMMETRIC COMBINATORIAL GAMES

A *finite symmetric combinatorial game* is a game, guaranteed to end in a finite number of moves, played by two players who take turns making moves, where in any position the allowed moves do not depend on whose turn it is (which rules out chess and Tic-tac-toe, for example).

To find a strategy for winning such a game, you need to do a few things:

• Identify a set of *final positions*, in which no more moves can be made.

• Describe a set of *losing positions* ($\mathcal{L}$), which must contain all the final positions, and a set of *winning positions* ($\mathcal{W}$). The sets $\mathcal{L}$ and $\mathcal{W}$ must have the following properties:

1. Every position in the game belongs to either $\mathcal{L}$ or $\mathcal{W}$, but not both.
2. From any non-final position in $\mathcal{L}$, *every* move leads to a position in $\mathcal{W}$.
3. From any position in $\mathcal{W}$, there is *some* move that leads to a position in $\mathcal{L}$.

Once you have described the sets $\mathcal{L}$ and $\mathcal{W}$, and shown that they have the properties in points 1-3 above, to complete a strategy for the game you need to explain how to find, given any position in $\mathcal{W}$, a move to a position in $\mathcal{L}$. This, however, is often apparent from the argument for point 3 above.

MORE PROBLEMS

7. The numbers $1, 2, \ldots, 20$ are written on the blackboard. Two players take turns erasing one of the numbers, until there are two left. If the sum of these two is divisible by 3, the person who made the last move wins. Who wins, and how? Give a *simple* description of the strategy, so that any ten year old can follow your instructions!

   Of course, your solution should be easy to generalize to an arbitrary set of numbers, and arbitrary modulus instead of 3.

8. Imagine n light bulbs with light switches numbered 1 through n, all turned off. Now toggle all switches that are multiples of 1, then all switches that are multiples of 2, then all that are multiples of 3, and so on, up to n. Which light bulbs are on now?

9. A two person game is played with a number of matches, where the players take turns as follows: In each move, a player removes k matches, where k is an arbitrary (positive) squarefree integer. The person who takes the last match wins. Describe a winning strategy. In particular, it should explain for which n the first person to move wins.

   An integer is squarefree if not divisible by p^2 for any prime p

10. Given n matches, two players take turns, removing at least 1 and at most half the remaining matches. What are the losing positions in this game?