Introduction to Homotopy Type Theory

Lecture 1: Type theory from a homotopy theory perspective

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University of Strathclyde, Glasgow
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Course plan

- **Today**: Type theory from a homotopy theory perspective
- **Tomorrow**: Equivalences, the Univalence Axiom
- **Saturday**: Propositional truncation, Univalent logic
- **Sunday**: Higher inductive types, synthethic homotopy theory
Main source material

- Homotopy Type Theory blog
- Homotopy Type Theory Google group
- Slides and exercises: https://tinyurl.com/hott-ohrid
Homotopy Type Theory
Homotopy (Type Theory)
(Homotopy Type) Theory
Univalent Foundations and Homotopy Type Theory

Two separate origins:

- **UF**: Voevodsky [2010–].
Univalent Foundations

A higher-dimensional foundation of mathematics: basic objects are not discrete sets, but $\infty$-groupoids.
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Main motivation: everything we write down should be invariant under equivalence.
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Miracle: Martin-Löf Type Theory essentially already is such a foundation.
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Main motivation: everything we write down should be invariant under equivalence.

Miracle: Martin-Löf Type Theory essentially already is such a foundation.

Side remark: “univalent” derives from Russian word for “faithful” [Voevodsky IHP talk 2014].
Homotopy Type Theory

Started as investigations into models of Martin-Löf Type Theory into abstract homotopy theory.
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Consequence: can prove results in homotopy theory synthetically using Type Theory, extended by axioms suggested by the models.
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Miracle: these axioms imply most of the “missing features” of plain Type Theory, such as function extensionality, and quotient types.
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Miracle: these axioms imply most of the “missing features” of plain Type Theory, such as function extensionality, and quotient types.

We now also know that these axioms are computationally well-behaved thanks to Cubical Type Theory [Cohen, Coquand, Huber, Mörtberg 2017].
## Intuition

<table>
<thead>
<tr>
<th>Type Theory</th>
<th>Interpretation</th>
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<tr>
<td>$A$ type</td>
<td>space $A$</td>
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<tr>
<td>$a : A$</td>
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</tr>
<tr>
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<tr>
<td>universe $\mathcal{U}$</td>
<td>space of small spaces</td>
</tr>
<tr>
<td>$a =_A a'$</td>
<td>space of paths connecting $a$ and $a'$ in $A$</td>
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A fibration is a “parameterised space with a homotopy lifting property” — the notion needed if identity is weakened to paths.

The total space of a fibration is the disjoint union of all the fibres.

A section is in particular a \textit{continuous} function — worth keeping in mind when translating concepts.
Thinking of identities as paths
Thinking of identities as paths

Inverse paths $p^{-1}$ (symmetry) ➔ Path concatenation $p \cdot q$ (transitivity) ➔ Constant paths $b$ (reflexivity)
Thinking of identities as paths

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- Path concatenation \( p \cdot q \) (transitivity)
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- Path concatenation $p \cdot q$ (transitivity)
- Constant paths $\text{refl}_b$ (reflexivity)
Identity Proofs are not Unique

There can be more than one path that connects two points, so we should not expect identity proofs to be unique.
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However, **not** because of this:
Identity Proofs are not Unique

There can be more than one path that connects two points, so we should not expect identity proofs to be unique.

However, not because of this:

since there is a path (homotopy) between the paths.
More problematic
More problematic
More problematic
More problematic

Stuck!
Formal rules for identity types
Identity type rules

Formation  If $a : A$ and $a' : A$ then $a =_A a'$ type.
Identity type rules

Formation  If $a : A$ and $a' : A$ then $a =_A a'$ type.

Introduction  If $a : A$ then $\text{refl}_a : a =_A a$. 

Elimination, informally  In order to do something with an arbitrary $p : a =_A a'$, it suffices to consider the case $\text{refl}_a : a =_A a$. 
Identity type rules

**Formation** If \( a : A \) and \( a' : A \) then \( a =_A a' \) type.

**Introduction** If \( a : A \) then \( \text{refl}_a : a =_A a \).

**Elimination** If

\[
\begin{align*}
&\quad \text{\( x : A, y : A, p : x =_A y \vdash C(x, y, p) \) type,} \\
&\quad \text{\( x : A \vdash d(x) : C(x, x, \text{refl}_x) \), and} \\
&\quad \text{\( a : A, a' : A \) and \( p : a =_A a' \)} \\
&\text{then \( \text{ind}_{=A}(C, d, a, a', p) : C(a, a', p) \).}
\end{align*}
\]
Identity type rules

Formation  If $a : A$ and $a' : A$ then $a =_A a'$ type.

Introduction  If $a : A$ then $\text{refl}_a : a =_A a$.

Elimination  If

- $x : A, y : A, p : x =_A y \vdash C(x, y, p)$ type,
- $x : A \vdash d(x) : C(x, x, \text{refl}_x)$, and
- $a : A, a' : A$ and $p : a =_A a'$

then $\text{ind}_{=A}(C, d, a, a', p) : C(a, a', p)$.

Computation  $\text{ind}_{=A}(C, d, a, a, \text{refl}_a) = d(a)$.
Identity type rules

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\end{align*}
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then \( \text{ind}_{=A}(C, d, a, a', p) : C(a, a', p). \)

Computation  \( \text{ind}_{=A}(C, d, a, a, \text{refl}_a) = d(a). \)

Elimination, informally

In order to do something with an arbitrary \( p : a =_A a' \), it suffices to consider the case \( \text{refl}_a : a =_A a. \).
Equality is symmetric

In practice: if you can write it down, it is trivial to prove it.
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**Theorem**

If $p : a =_A b$ then there is $p^{-1} : b =_A a$.  

**Second proof.**

By the elimination principle, we can assume $p$ is refl, in which case we need to give $\text{refl}^{-1} : a = a$. Obviously $\text{refl}^{-1} : a = a$ works.
Equality is symmetric

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**Theorem**

*If* \( p : a =_A b \) *then there is* \( p^{-1} : b =_A a \).*

**Proof.**

Consider elimination motive \( C(x, y, q) \equiv y =_A x \). We can give \( d(x) :\equiv \text{refl}_x : C(x, x, \text{refl}) \), hence by the elimination principle we can take \( p^{-1} :\equiv \text{ind}_{=_A}(C, d, a, b, p) : b =_A a \). □
Equality is symmetric

In practice: if you can write it down, it is trivial to prove it.

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Equality is transitive

**Theorem**

If $p : a = b$ and $q : b = c$ then there is $p \cdot q : a = c$. 
 Equality is transitive

**Theorem**

If \( p : a = b \) and \( q : b = c \) then there is \( p \cdot q : a = c \).

**Proof.**

We may assume \( b \) is \( a \) and \( p \) is \( \text{refl}_a \), in which case \( q : a = c \) has the right type, so \( \text{refl} \cdot q \equiv q \) works. \( \square \)
Theorem

If \( p : a = b \) and \( q : b = c \) then there is \( p \cdot q : a = c \).

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We may assume \( b \) is \( a \) and \( p \) is \( \text{refl}_a \), in which case \( q : a = c \) has the right type, so \( \text{refl} \cdot q : \equiv q \) works.

Second proof.

Elimination with motive \( C(x, y, r) \equiv (\prod s : y =_A c)(x =_A c) \) applied to \( p \) (for \( r \)) and \( q \) (for \( s \)).
Equality is unique?

Claim?

If \( p, q : a =_A b \), then \( p =_{a=_{A}b} q \)?
Equality is unique?

Claim?
If $p, q : a =_A b$, then $p =_{a =_A b} q$?

Proof?
We may assume $p$ and $q$ are refl; if so, $\text{refl}_{\text{refl}_a}$ obviously works. □
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Second proof?
Let’s do this formally: we want to prove

\[
(\Pi x, y : A)(\Pi r : x = y)(\Pi s : x = y)(r = s)
\]
Equality is unique?

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If $p, q : a =_A b$, then $p =_{a=_{A}b} q$?

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We may assume $p$ and $q$ are refl; if so, refl_{refl_a} obviously works.

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so our motive should be $C(x, y, r) \equiv (\Pi s : x = y)(r = s)$. 


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Equality is unique?

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If $p, q : a =_A b$, then $p =_{a=Ab} q$?

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$$d(x) : (\Pi s : x = x)(refl =_{x=x} s)$$

but we are stuck: the elimination rule does not apply!
Equality is unique?

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If \( p, q : a =_A b \), then \( p =_{a=_{A}b} q \)?:

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We could try to generalise the inner motive, but then refl does not type check anymore. We cannot write it down.
Equality is unique?

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If $p, q : a =_A b$, then $p =_{a = A} b$ $q$?

Proof?.
We may assume $p$ and $q$ are refl; if so, $\text{refl}_{\text{refl}a}$ obviously works.  

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but we are stuck: the elimination rule does not apply!  

We could try to generalise the inner motive, but then refl does not type check anymore. We cannot write it down.
Groupoid structure of paths

Theorem

- $p \cdot \text{refl}_b = p$
- $\text{refl}_a \cdot p = p$
- $p \cdot p^{-1} = \text{refl}_a$
- $p^{-1} \cdot p = \text{refl}_b$
- $(p^{-1})^{-1} = p$
- $p \cdot (q \cdot r) = (p \cdot q) \cdot r$

These equalities are not strict; they only hold up to paths, which in turn are coherent, but only up to higher paths, . . .
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These equalities are not strict; they only hold up to paths, which in turn are coherent, but only up to higher paths, ... 

Theorem (Lumsdaine [2010], van den Berg-Garner [2011])

For every type \( A \), \((A, =_A, =_=A, \ldots)\) form an \( \infty \)-groupoid.
Transporting along paths

Theorem

Let \( x : A \vdash B(x) \) type. If \( p : a = a' \) then there is transport \( B(p, -) : B(a) \rightarrow B(a') \) with transport \( B(\text{refl}_a, -) = \text{id}_{B(a)} \).
Transporting along paths

Theorem

Let $x : A \vdash B(x)$ type. If $p : a = A a'$ then there is transport $B(p, \_)$ : $B(a) \rightarrow B(a')$ with $\text{transport} B(\text{refl}_a, \_) = \text{id} B(a)$.
Theorem
Let \( x : A \vdash B(x) \) type. If \( p : a = a' \) then there is \( \text{transport}_{B}(p, -) : B(a) \rightarrow B(a') \) with \( \text{transport}_{B} \text{refl}_{a}, -) = \text{id}_{B(a)} \).
Transporting along paths

Theorem

Let \( x : A \vdash B(x) \) type. If \( p : a = A a' \) then there is \( \text{transport}^B(p, -) : B(a) \to B(a') \) with \( \text{transport}^B(\text{refl}_a, -) = \text{id}_{B(a)} \).
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Let \( x : A \vdash B(x) \) type. If \( p : a =_A a' \) then there is

\[
\text{transport}^B(p, -) : B(a) \rightarrow B(a')
\]

with \( \text{transport}^B(\text{refl}_a, -) = \text{id}_{B(a)} \).
Functions act on paths

\[ f: A \rightarrow B \]

**Theorem**

Let \( f: \Pi_{x, y: A} (x = A y \rightarrow f(x) = B f(y)) \) with \( \text{ap} f(x, y, \text{refl} x) : \equiv \text{refl} f(x) \).
Functions act on paths

Theorem
Let $f : A \to B$. There is $apf : (\Pi x, y : A) (x = A y \to f(x) = B f(y))$ with $apf(x, x, refl x) : \equiv refl f(x)$.

Theorem
Let $f : (\Pi x : A) B(x)$. There is $apd f : (\Pi x, y : A) (\Pi p : x = A y) (f(x) = p f(y))$ with $apd f(x, x, refl x) : \equiv refl f(x)$.  

Functions act on paths

\[
\begin{align*}
\text{Theorem} & \quad \text{Let } f : A \to B. \text{ There is } \text{ap}\ f : (\prod x, y : A) (x = A y \to f(x) = B f(y)) \\
\text{Theorem} & \quad \text{Let } f : (\prod x : A) B(x). \text{ There is } \text{apd}\ f : (\prod x, y : A) (\prod p : x = A y) (f(x) = p f(y))
\end{align*}
\]
Theorem

Let $f : A \rightarrow B$. There is

$$ap_f : (\prod x, y : A)(x =_A y \rightarrow f(x) =_B f(y))$$

with $ap_f(x, x, refl_x) \equiv refl_{f(x)}$. 
Functions act on paths

Theorem
Let $f : A \to B$. There is

$$ap_f : (\prod x, y : A) (x =_A y \to f(x) =_B f(y))$$

with $ap_f(x, x, \text{refl}_x) :\equiv \text{refl}_{f(x)}$.

Theorem
Let $f : (\prod x : A) B(x)$. There is

$$apd_f : (\prod x, y : A) (\prod p : x =_A y) (f(x) =_{p} f(y))$$

with $apd_f(x, x, \text{refl}_x) :\equiv \text{refl}_{f(x)}$. 
Paths over paths (dependent paths)

Formation  If $a, a' : A$ and $p : a =_A a'$, and $b : B(a)$, $b' : B(a')$ then $b = b'$ type.

Introduction If $b : B(a)$ then $\text{refl}_b : b = b$(using path induction).

Elimination . . .

Computation . . .

Can by implemented by e.g. $(b = p b') : \equiv (\text{transport}_{B(b)}(p, b) = B(a')b')$ or $(b = \text{refl}_a b') : \equiv (b = B(a)b')$
Paths over paths (dependent paths)

**Formation** If $a, a' : A$ and $p : a =_A a'$, and $b : B(a)$, $b' : B(a')$ then $b = b'$ type.

**Introduction** If $b : B(a)$ then $\text{refl}_b : b = b$.
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Elimination  …

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Paths over paths (dependent paths)

**Formation** If \( a, a' : A \) and \( p : a =_A a' \), and \( b : B(a), b' : B(a') \) then \( b = b' \) type.

**Introduction** If \( b : B(a) \) then \( \text{refl}_b : b = \text{refl}_a b \).

**Elimination** ...

**Computation** ...

Can by implemented by e.g.

\[(b = b') \equiv (\text{transport}^B(p, b) =_{B(a')} b')\]

or

\[(b = b') \equiv (b =_{B(a)} b') \quad \text{(using path induction)}\]
Characterising path spaces
Transporting in Cartesian products

Theorem

\[
\text{transport}_{z \mapsto A(z) \times B(z)}(p, x) = \\
(\text{transport}^A(p, \text{fst}(x)), \text{transport}^B(p, \text{snd}(x)))
\]
Theorem

\[
\text{transport} \overset{z \mapsto A(z) \times B(z)}{\rightarrow} (p, x) = \quad \\
(\text{transport}^A(p, \text{fst}(x)), \text{transport}^B(p, \text{snd}(x)))
\]

Proof.
It is enough to consider \( p \equiv \text{refl}_x \), in which case the problem reduces to \( x = (\text{fst}(x), \text{snd}(x)) \).
Transporting in Cartesian products

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\text{transport}^{z \mapsto A (z) \times B (z)} (p, x) = \\
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\]

Proof.

It is enough to consider \( p \equiv \text{refl}_x \), in which case the problem reduces to \( x = (\text{fst}(x), \text{snd}(x)) \). True by the η-rule (or an induction on \( x \)).
Paths in Cartesian products

Given \( p : (a, b) =_{A \times B} (a', b') \), we have

\[ (ap_{\text{fst}}(p), ap_{\text{snd}}(p)) : (a =_{A} a') \times (b =_{B} b') \]

Conversely:

**Theorem**

There is a function

\[ \text{pair}^= : (a =_{A} a') \times (b =_{B} b') \to (a, b) =_{A \times B} (a', b') \]
Paths in Cartesian products

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Conversely:

**Theorem**

There is a function

\[
\text{pair}^- : (a =_A a') \times (b =_B b') \rightarrow (a, b) =_{A \times B} (a', b')
\]

These two maps are inverse to each other in a precise sense; more tomorrow, but for now, this can be summarised by:

**Theorem**

\[
\left( (a, b) =_{A \times B} (a', b') \right) \simeq \left( (a =_A a') \times (b =_B b') \right)
\]

*In particular, we have* \( \text{isEquiv}((\text{ap}_{\text{fst}}(-), \text{ap}_{\text{snd}}(-))) \).
Paths in sigma types

Suppose \( a, a' : A \) and \( b : B(a) \) and \( b' : B(a') \). A path

\[
(a, b) = (\Sigma x : A) B(x) (a', b')
\]

should consist of:
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▶ a path \(p : a =_A a'\)
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Suppose $a, a' : A$ and $b : B(a)$ and $b' : B(a')$. A path

$$(a, b) = (\Sigma x : A)B(x) (a', b')$$

should consist of:

- a path $p : a =_A a'$
- a path $q : b =_{B(a)} b'$
 Paths in sigma types

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- a path $q : b = b'$

In particular, we have

$$\text{isEquiv}((ap \text{fst} (-), apd \text{snd} (-)))$$
Paths in sigma types

Suppose \( a, a' : A \) and \( b : B(a) \) and \( b' : B(a') \). A path

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should consist of:

- a path \( p : a =_A a' \)
- a path \( q : b =_{B(a)} b' \) not well-typed!
- a path \( q : b = b' \)

Theorem

\[
((a, b) = (\Sigma x : A) B(x) (a', b')) \simeq (\Sigma p : a =_A a')(b = b')
\]

In particular, we have isEquiv((ap_{\text{fst}}(\_), ap_{\text{snd}}(\_))).
Transporting in path types

A prime example of “if you can write it down, it will be trivial to prove it”.

Lemma

Let $a: A$ and $p: x = A x'$. Let $f, g: A \rightarrow B$.

$\triangleright$ transport $z \mapsto \overset{\rightarrow}{a} = z (p, q) = q$

$\triangleright$ transport $z \mapsto \overset{\rightarrow}{z} = a (p, q) = p^{-1} q$

$\triangleright$ transport $z \mapsto f (z) = g (z) (p, q) = ap f (p^{-1} q)$

We don't expect a general characterisation of paths in $= A$—this will depend on $A$.23
Transporting in path types

A prime example of “if you can write it down, it will be trivial to prove it”.

**Lemma**

Let $a : A$ and $p : x =_A x'$.

- $\text{transport}^{z \mapsto a=z}(p, q) = q \cdot p$
- $\text{transport}^{z \mapsto z=a}(p, q) = p^{-1} \cdot q$
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- $\text{transport}_{z \mapsto z = z} (p, q) =$
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Let $a : A$ and $p : x =_A x'$. Let $f, g : A \to B$.

- $\text{transport}^{z \mapsto a = z}(p, q) = q \cdot p$
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- $\text{transport}^{z \mapsto z = z}(p, q) = p^{-1} \cdot q \cdot p$
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Lemma

Let \( a : A \) and \( p : x =_A x' \). Let \( f, g : A \to B \).

\[
\begin{align*}
\text{transport}^{z \mapsto a = z} (p, q) &= q \cdot p \\
\text{transport}^{z \mapsto z = a} (p, q) &= p^{-1} \cdot q \\
\text{transport}^{z \mapsto z = z} (p, q) &= p^{-1} \cdot q \cdot p \\
\text{transport}^{z \mapsto f(z) = g(z)} (p, q) &= \text{ap}_f(p^{-1}) \cdot q \cdot \text{ap}_g(p)
\end{align*}
\]

We don't expect a general characterisation of paths in \( =_A \) — this will depend on \( A \).
Transporting in path types

A prime example of “if you can write it down, it will be trivial to prove it”.

Lemma

Let \( a : A \) and \( p : x =_A x' \). Let \( f, g : A \to B \).

\[
\begin{align*}
\quad & \text{transport}_{z \mapsto a = z}(p, q) = q \cdot p \\
\quad & \text{transport}_{z \mapsto z = a}(p, q) = p^{-1} \cdot q \\
\quad & \text{transport}_{z \mapsto z = z}(p, q) = p^{-1} \cdot q \cdot p \\
\quad & \text{transport}_{z \mapsto f(z) = g(z)}(p, q) = ap_f(p^{-1}) \cdot q \cdot ap_g(p)
\end{align*}
\]

We don’t expect a general characterisation of paths in \( =_A \) — this will depend on \( A \).
Suppose $f, g : (\Pi x : A)B(x)$. What should a path $f = (\Pi x : A)B(x) \ g$ consist of?
Paths in pi types

Suppose \( f, g : (\Pi x : A)B(x) \). What should a path

\[
f = (\Pi x : A)B(x) \ g
\]

consist of?

**Theorem (using the Univalence Axiom)**

\[
(f = (\Pi x : A)B(x) \ g) \simeq (\Pi x : A)(f(x) =_{B(x)} g(x))
\]

*In particular, we have isEquiv(happly), where*

\[
happly : (f = g) \to (\Pi x : A)(f(x) =_{B(x)} g(x))
\]

*is defined by happly\( (p, x) = \text{ap}_{h \mapsto h(x)}(p) \).*
Before HoTT, it was common to assume as an axiom a term

\[
\text{funext} : (\Pi x : A)(f(x) =_{B(x)} g(x)) \rightarrow (f = g)
\]

(the non-trivial direction of \((f = g) \simeq (\Pi x : A)(f(x) =_{B(x)} g(x))\)).
Strong function extensionality from weak

Before HoTT, it was common to assume as an axiom a term

\[
\text{funext} : (\prod x : A)(f(x) =_{B(x)} g(x)) \rightarrow (f = g)
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(the non-trivial direction of \((f = g) \simeq (\prod x : A)(f(x) =_{B(x)} g(x))\)).

Surprisingly, this weaker statement implies the stronger one:

**Theorem (Voevodsky [Lumsdaine, HoTT blog])**

*If there is a term funext as above, then isEquiv(happly), i.e.*

\[
(f =_{(\prod x : A)B(x)} g) \simeq (\prod x : A)(f(x) =_{B(x)} g(x))
\]
Strong function extensionality from weak

Before HoTT, it was common to assume as an axiom a term

\[ \text{funext} : (\Pi x : A)(f(x) =_{B(x)} g(x)) \rightarrow (f = g) \]

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**Theorem (Voevodsky [Lumsdaine, HoTT blog])**

*If there is a term \(\text{funext}\) as above, then \(\text{isEquiv(happly)}\), i.e.*

\[ \left(f =_{(\Pi x : A)B(x)} g\right) \simeq (\Pi x : A)(f(x) =_{B(x)} g(x)) \]

In cubical type theory, \(\text{funext}\) is trivial to define.
Paths in the universe

Suppose $A, B : \mathcal{U}$. What should a path

$$A =_\mathcal{U} B$$

consist of?
Paths in the universe

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Paths in the universe

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A =_\mathcal{U} B
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consist of? Only sensible notion(?): \( (A =_\mathcal{U} B) \simeq (A \simeq B) \).

Again we can be more precise: we can define

\[
idtoequiv : (A =_\mathcal{U} B) \to (A \simeq B)
\]

by path induction: if \( p : A = B \) is \( \text{refl}_A \), we let

\[
idtoequiv(\text{refl}_A) \equiv \text{id}_A \equiv (\text{id}_A, \text{id}_A, \ldots)
\]

Univalence Axiom

\[
(A =_\mathcal{U} B) \simeq (A \simeq B)
\]

in particular, we have \( \text{isEquiv}(\text{idtoequiv}) \).
Paths in the universe

Suppose $A, B : \mathcal{U}$. What should a path $A =_\mathcal{U} B$ consist of? Only sensible notion(?): $(A =_\mathcal{U} B) \simeq (A \simeq B)$.

Again we can be more precise: we can define

$$\text{idtoeqv} : (A =_\mathcal{U} B) \to (A \simeq B)$$

by path induction: if $p : A = B$ is $\text{refl}_A$, we let

$$\text{idtoeqv}(\text{refl}_A) \equiv \text{id}_A \equiv (\text{id}_A, \text{id}_A, \ldots)$$

Univalence Axiom

$$(A =_\mathcal{U} B) \simeq (A \simeq B)$$

in particular, we have $\text{isEquiv}(\text{idtoeqv})$.

In cubical type theory, Univalence is a theorem, not an axiom.
Consequences of Univalence

▶ “Isomorphic structures are equal”

▶ Propositional extensionality: \((P \leftrightarrow Q) \simeq (P = Q)\) for propositions \(P, Q\).

▶ Function extensionality

▶ Large quotients exists

▶ Homotopy theory is non-trivial (there are two paths \(2 =_U 2\))

▶ Enough slack for large elimination of higher inductive types (Sunday)

▶ …
Summary

New perspective on identity types based on intuitions from homotopical models.

Lack of uniqueness of identity proofs leads to path algebra: “if you can write it down, it is trivial to prove it”.

Important characterisations/axioms: function extensionality and Univalence (more tomorrow).
Exercises

1. How does transport$^B$ interact with the groupoid structure of paths? What about ap$_f$? Prove your claims.

2. State and prove lemmas for decomposing a transport in function types and sigma types (the latter is messier).

3. Use paths over paths to state and prove that the empty vector is a unit for vector concatenation, and that vector concatenation is associative. (Hint: you will need to generalise ap$_f$ to paths over paths.)
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