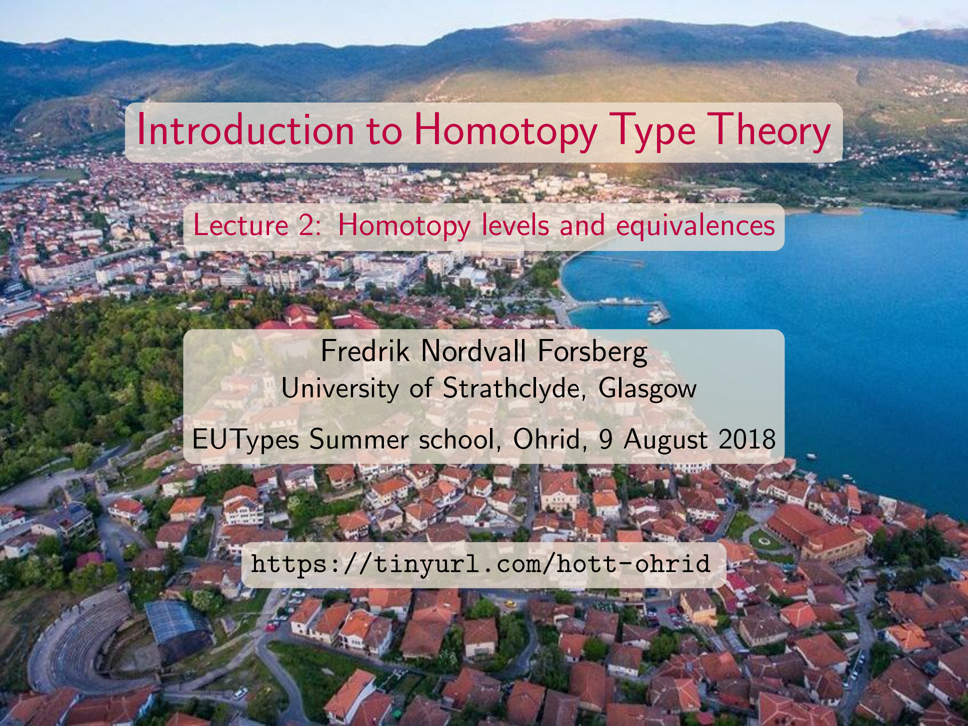

Introduction to Homotopy Type Theory

Lecture 2: Homotopy levels and equivalences

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https://tinyurl.com/hott-ohrid

Recap of yesterday

Martin-Löf Type Theory: type theory with dependent pair types (Σ), dependent function types (Π), a universe of small types, and identity types ($x =_A y$).

Homotopy Type Theory: new perspective on identity types based on intuitions from homotopical models.

Paths between paths: Proving properties of identity types using path induction.

New axioms: function extensionality and the Univalence Axiom.

The Univalence Axiom

Univalence Axiom

The canonical function idtoeqv : $(A =_{\mathcal{U}} B) \to (A \simeq B)$ is an equivalence.

Today's goal is to make sense of this axiom, and the notion of equivalence in particular.

Outline

Homotopy *n*-types

- A is contractible if we can prove

$$\mathrm{isContr}(A) :\equiv (\Sigma x : A)(\Pi y : A)(x =_A y)$$

- A is a proposition (subsingleton) if we can prove

$$\mathrm{isProp}(A) :\equiv (\Pi x, y : A)(x =_A y)$$

- A is a set if every type $x =_A y$ is a proposition:

$$\mathrm{isSet}(A) :\equiv (\Pi x, y : A)(\mathrm{isProp}(x =_A y))$$

Examples

- The type $\mathbf{0}$ is
- The type $\mathbf{1}$ is
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- The type $\mathbf{1}$ is contractible: there is a path from the only element to every element.
- The type $\mathbf{2}$ is a set: there are no non-trivial paths between elements.

Theorem (exercise)

$$\mathsf{isContr}(A) \to (A \to \mathsf{isContr}(A)) \leftrightarrow \mathsf{isProp}(A) \to \mathsf{isSet}(A)$$

Proving that types are sets

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Definition

A type A is decidable if there is a term $\mathsf{dec}_A : A + \neg A$.

Note: $\neg A :\equiv A \to \mathbf{0}$.

By normalising dec_A, we find out if A holds or not.

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Definition

A type A is has decidable equality if all path spaces $x =_A y$ are decidable, i.e. there is a term

$$\mathsf{decEq}_A : (\Pi x, y : A)\big((x =_A y) + \neg(x =_A y)\big)$$

Hedberg's Theorem

Theorem ([Hedberg 1998])

If a type A has decidable equality, then it is a set.

Examples

- $\mathsf{isSet}(\mathbf{2})$
- $\mathsf{isSet}(\mathbb{N})$
- $\mathsf{isSet}(A) \to \mathsf{isSet}(\mathsf{List}\ A)$
- ...

Properties of contractible types

Theorem ("Vacuum cord principle")

Singletons are contractible: for $a : A$, $\mathsf{isContr}((\Sigma x : A)(x = a))$.

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Singletons are contractible: for $a : A$, $\mathrm{isContr}((\Sigma x : A)(x = a))$.

Theorem

- *A is contractible if and only if* $A \simeq \mathbf{1}$.
- *If A is contractible then* $\big((\Sigma x : A)B(x)\big) \simeq B(x_0)$, *where x_0 is the centre of contraction.*
- *If $B(x)$ is contractible for all $x : A$ then* $\big((\Sigma x : A)B(x)\big) \simeq A$.

Equivalences

What is an equivalence?

Naive first guess:

Definition

$f : A \to B$ is a quasi-equivalence if we can prove

$$\mathsf{qinv}(f) :\equiv (\Sigma g : B \to A)\big((\Pi x : A)(g(f\,x) =_A x) \times (\Pi y : B)(f(g\,y) =_B y)\big)$$

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Problem: $\mathsf{qinv}(f)$ is not always a proposition.

Surprisingly(?), this could have disastrous consequences.

Quasi-equivalences with coherence

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Quasi-equivalence proofs (g, η, ϵ) with a coherence condition.

We define $A \simeq B :\equiv (\Sigma f : A \to B)(\mathsf{isEquiv}(f))$.

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Good news: $\mathsf{qinv}(f) \leftrightarrow \mathsf{isEquiv}(f)$.

Better news: $\mathsf{isEquiv}(f)$ is always a proposition.

Proof of better news

Theorem

For any $f : A \to B$, *we have* $\mathsf{isProp}(\mathsf{isEquiv}(f))$.

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Univalence formulated with quasi-inverses is false, i.e. the function

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is not a quasi-equivalence.

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If idtoqinv is a quasi-equivalence, then every quasi-equivalence $f : A \to B$ is coherent:

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In particular, every $(\mathsf{id}_A, \mathsf{id}_A, \eta, \epsilon)$ for any $\eta, \epsilon : (\Pi x : A)(x =_A x)$ is coherent, i.e. $\eta = \epsilon$.

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But there are A such that $(\Pi x : A)(x =_A x)$ is not a proposition (Sunday); a contradiction. □

Morals of the story

- Univalence states that we may treat every equivalence as the identity equivalence.

- This is consistent because there are no properties to distinguish the identity equivalence from any other — not true for quasi-equivalences.

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- Shows the importance of constructing a model to ensure consistency. . .

Special cases

- If isProp(P) and isProp(Q) and $P \leftrightarrow Q$, then $P \simeq Q$ (hence $P =_{\mathcal{U}} Q$ by Univalence).
- If isSet(A) and isSet(B) and there is a quasi-equivalence $A \to B$, then $A \simeq B$.

Equivalent formulations of equivalences

That $f : A \to B$ is an equivalence can be defined in different ways:

- Voevodsky: $(\Pi y : B)(\mathrm{isContr}((\Sigma x : A)(f\,x =_B y)))$
  The preimage of each y is a singleton

- Joyal: $(\Sigma g, h : B \to A)\big((g \circ f = \mathrm{id}) \times (f \circ h = \mathrm{id})\big)$
  The function f has both a left inverse g and a right inverse h

- Lumsdaine: $(A \simeq B)$ is equivalent to the type

$$(\Sigma R : A \to B \to \mathcal{U})((\Pi a : A)(\mathrm{isContr}((\Sigma b : B)R\,a\,b)) \times (\Pi b : B)(\mathrm{isContr}((\Sigma a : A)R\,a\,b)))$$

  There exists a “bi-functional” relation R

Exercises

See `lec2-exercises.agda`.

Summary

- Hierarchy of homotopy levels:

$$\mathsf{isContr}(A) \to \mathsf{isProp}(A) \to \mathsf{isSet}(A) \to \ \ldots$$

- Coherent notion of equivalence

$$A \simeq B :\equiv (\Sigma f : A \to B)(\mathsf{isEquiv}(f))$$

  Properties:

  - $\mathsf{qinv}(f) \to \mathsf{isEquiv}(f)$
  - $\mathsf{isEquiv}(f) \to \mathsf{qinv}(f)$
  - $\mathsf{isProp}(\mathsf{isEquiv}(f))$

- Proving equivalences by chaining together basic ones.

Coming up on Saturday:

- Logic in HoTT: Propositions-as-some-types

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