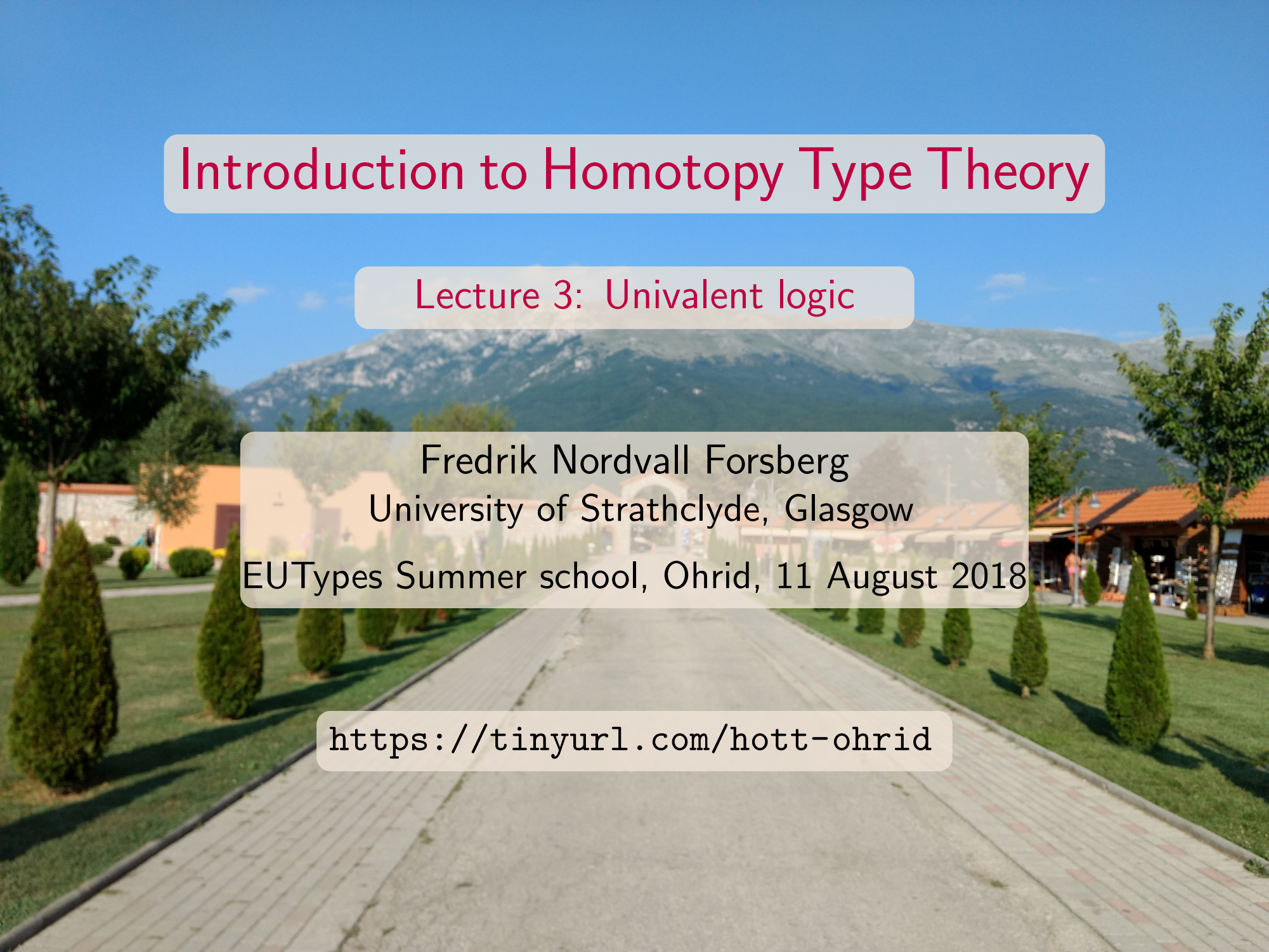

Introduction to Homotopy Type Theory

Lecture 3: Univalent logic

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EUTypes Summer school, Ohrid, 11 August 2018

`https://tinyurl.com/hott-ohrid`

Recap so far: paths

Operations on paths:

- $-^{-1} : a =_A b \to b =_A a$
- $\blacksquare : a =_A b \to b =_A c \to a =_A c$

Operations induced by paths:

- $\mathsf{transport}^B : a =_A a' \to B(a) \to B(a')$
- For $f : A \to B$, $\mathsf{ap}_f : a =_A a' \to f(a) =_B f(a')$

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- For $f : A \to B$, $\mathsf{ap}_f : a =_A a' \to f(a) =_B f(a')$
- For $f : (\Pi x : A)B(x)$, $\mathsf{apd}_f : (\Pi p : a =_A a')(f(a) \underset{p}{=} f(a'))$

E.g. $b \underset{p}{=} b' \simeq \mathsf{transport}^B(p, b) =_{B(a')} b'$

Recap so far: the Univalence Axiom and hlevels

Univalence Axiom

The canonical function idtoeqv : $(A =_{\mathcal{U}} B) \to (A \simeq B)$ is an equivalence.

In particular, gives an inverse ua : $(A \simeq B) \to (A =_{\mathcal{U}} B)$.

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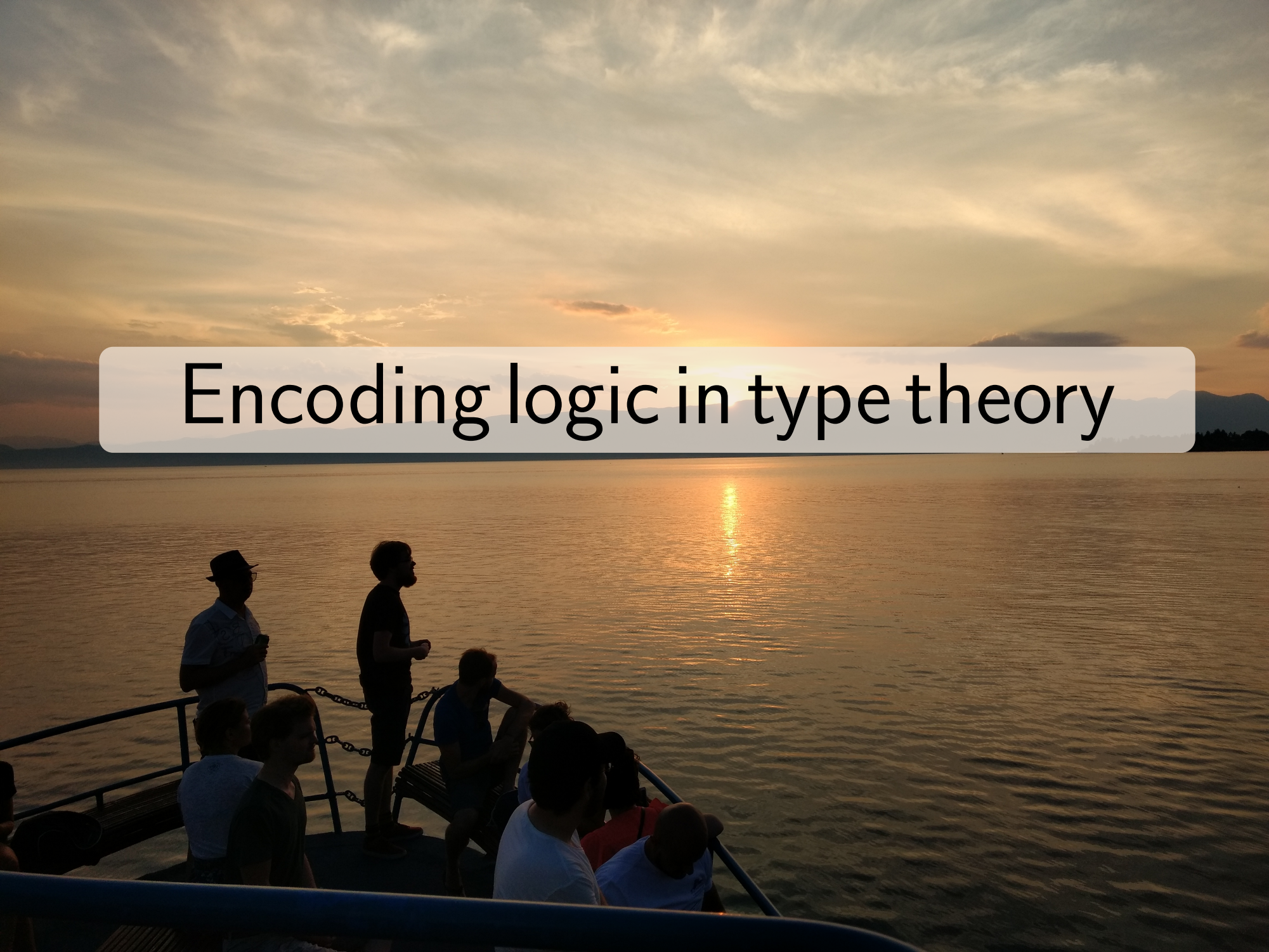
Homotopy levels: Hierarchy of "complexity" of the type A:

$$\mathsf{isContr}(A) \to \mathsf{isProp}(A) \to \mathsf{isSet}(A) \to \ \ldots$$


Encoding logic in type theory

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Curry-Howard logic:

- Propositions are types
- Proofs are terms

Univalent logic:

- Propositions are. . . propositions (subsingleton types)
- Proofs are terms

Note: Univalent propositions are not the same as Coq's propositions!

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Note: Univalent propositions are not the same as Coq's propositions!

Both constructive by default, but allows the "axiomatic freedom" to assume additional classical (or anti-classical) principles.

Curry-Howard logic

Conjunction $A \wedge B \equiv A \times B$

A proof of $A \wedge B$ is a proof of A and a proof of B.

Disjunction $A \vee B \equiv A + B$

A proof of $A \wedge B$ is a proof of A or a proof of B.

Implication $A \Rightarrow B \equiv A \to B$

A proof of $A \Rightarrow B$ is a method for transforming proofs of A into proofs of B.

Negation $\neg A \equiv A \to \mathbf{0}$

A proof of $\neg A$ is a proof that there are no proofs of A.

Existential quantification $(\exists x : A)B(x) \equiv (\Sigma x : A)B(x)$

A proof of $(\exists x : A)B(x)$ is a witness $a : A$ and a proof of $B(a)$.

Universal quantification $(\forall x : A)B(x) \equiv (\Pi x : A)B(x)$

A proof of $(\forall x : A)B(x)$ is a method for proving $B(a)$ for any given $a : A$.

Example

What does the following type express?

$$(\Pi n : \mathbb{N})(\Sigma p : \mathbb{N})\big(\mathsf{isPrime}(p) \times (p > n)\big)$$

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For every n, there is a prime greater than n — the infinitude of primes.

A proof of this is a function, which given n produces a natural number p, together with proofs that p is prime, and that $p > n$.

Assuming classical logic in Curry-Howard

The Law of Excluded Middle becomes $(\Pi A : \mathcal{U})(A + \neg A)$.

Of course, this is not provable, but we could assume it as an axiom.

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Lemma (exercise)

$(\Pi A : \mathcal{U})(A + \neg A) \leftrightarrow (\Pi A : \mathcal{U})(\neg\neg A \to A)$.

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Lemma (exercise)
$(\Pi A : \mathcal{U})(A + \neg A) \leftrightarrow (\Pi A : \mathcal{U})(\neg\neg A \to A)$.

- MLTT+LEM is consistent (by a model using classical logic).
- MLTT+UA is consistent (by the simplicial sets model).

But adding axioms is not modular. . .

MLTT+UA+LEM is inconsistent

Theorem
MLTT+UA+LEM proves $\mathbf{0}$.

Proof.
Let $f : (\Pi A : \mathcal{U})(\neg\neg A \to A)$ be given by LEM. We derive a contradiction from the fact that f must respect equivalences, by UA.

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Let $f : (\Pi A : \mathcal{U})(\neg\neg A \to A)$ be given by LEM. We derive a contradiction from the fact that f must respect equivalences, by UA.

Consider the non-identity equivalence $\mathsf{swap} : \mathbf{2} \to \mathbf{2}$. We have $\mathsf{ua}(\mathsf{swap}) : \mathbf{2} = \mathbf{2}$, and hence

$$\mathsf{apd}_f(\mathsf{ua}(\mathsf{swap})) : f(\mathbf{2}) \underset{\mathsf{ua}(\mathsf{swap})}{=} f(\mathbf{2})$$

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$$\mathsf{apd}_f(\mathsf{ua}(\mathsf{swap})) : f(\mathbf{2}) \underset{\mathsf{ua}(\mathsf{swap})}{=} f(\mathbf{2})$$

$$\simeq \mathsf{transport}^{z \mapsto \neg\neg z \to z}(\mathsf{ua}(\mathsf{swap}), f(\mathbf{2})) =_{\neg\neg\mathbf{2} \to \mathbf{2}} f(\mathbf{2})$$

Proof. (cont.)

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Need to know three facts:

1. How $\mathsf{transport}^{z \mapsto B(z)\to C(z)}(p, -)$ works.
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Transport in function spaces

Lemma

Let $p : x = y$ *and* $g : B(x) \to C(x)$.

$$\mathsf{transport}^{z \mapsto B(z) \to C(z)}(p, g) =$$

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Proof.

By path induction. □

Proof. (cont.)

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Transport in $A \mapsto A$ (coercion)

Let $p : A =_{\mathcal{U}} B$.

First attempt:

$$\mathsf{transport}^{z \mapsto z}(p, a) =_B \{? : B\}$$

Not much we can say... In particular

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This is one of the roundtrips of the Univalence Axiom!

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Finishing the proof

Proof. (cont.)

From the assumption that $f : (\Pi A : \mathcal{U})(\neg\neg A \to A)$, we derived

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But $\mathsf{swap}(x) \neq_{\mathbf{2}} x$ for every $x : \mathbf{2}$ by definition. This gives a contradiction. □

Propositions to the rescue

Conclusion: we cannot use Curry-Howard logic in HoTT if we want to have the freedom to assume classical logic.

To be clear: Curry-Howard logic is consistent with HoTT. But our options to extend it to classical logic are limited.

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Instead we will encode logic not using arbitrary types, but using propositions only.

Propositions to the rescue

Conclusion: we cannot use Curry-Howard logic in HoTT if we want to have the freedom to assume classical logic.

To be clear: Curry-Howard logic is consistent with HoTT. But our options to extend it to classical logic are limited.

Instead we will encode logic not using arbitrary types, but using propositions only.

Recall:

Definition
A type A is a proposition (subsingleton) if we can prove

$$\mathsf{isProp}(A) :\equiv (\Pi x, y : A)(x =_A y)$$

A proposition

$$(\Sigma n : \mathbb{N})\big(\mathsf{isOdd}(n) \times (\Pi a, b : \mathbb{N})(n! \neq_{\mathbb{N}} a^2 - b^2)\big)$$

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From any proof of the above proposition, the number 3 can be extracted.

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Propositions are not "proof-irrelevant"; they contain computational content.

From any proof of the above proposition, the number 3 can be extracted.

Another example: The type $\mathsf{isEquiv}(f)$ is always a proposition for any $f : A \to B$. From any proof of it, a function $g : B \to A$ can be extracted.

Closure of propositions under logical connectives

Theorem

- $\mathrm{isProp}(\mathbf{1})$.
- $\mathrm{isProp}(\mathbf{0})$.
- *If* $\mathrm{isProp}(P)$ *and* $\mathrm{isProp}(Q)$ *then* $\mathrm{isProp}(P \times Q)$.
- *If* $\mathrm{isProp}(Q)$ *then* $\mathrm{isProp}(P \to Q)$.
- *If* $\mathrm{isProp}(P(x))$ *for every* $x : A$, *then* $\mathrm{isProp}((\Pi x : A)P(x))$.

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Propositions are not closed under Σ:

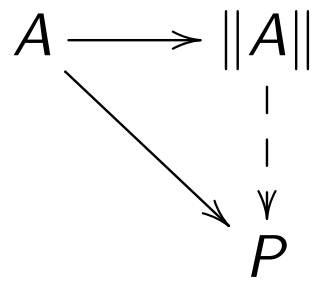
$$\mathbf{1} \text{ is a proposition, but } (\Sigma x : \mathbf{2})\mathbf{1} \text{ is not.}$$

We need a "best possible way" to map these types to propositions.

Propositional truncation

A propositional truncation $\|A\|$ of a type A is a universal solution to mapping A into a proposition, in the following sense:

arbitrary type



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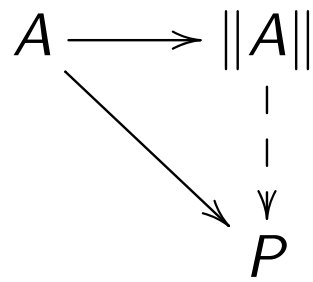

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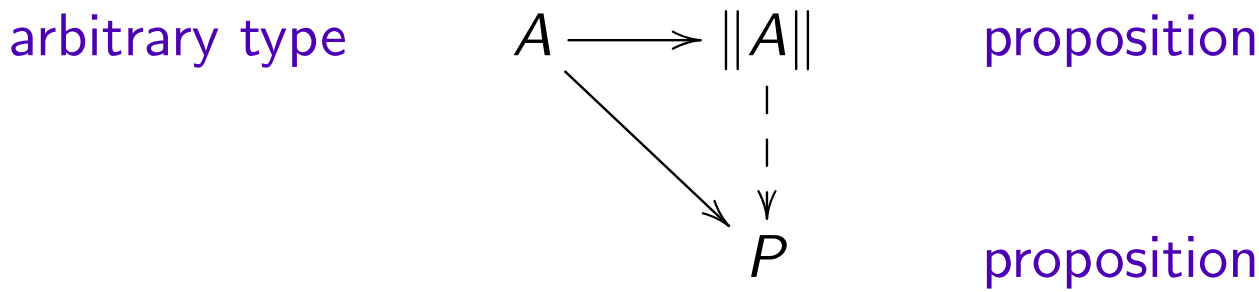

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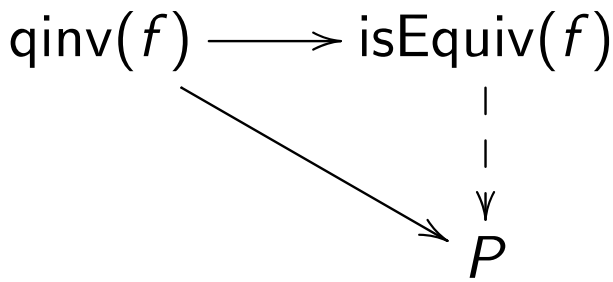
Some types can be shown to be propositional truncations of other types.

But usually the theory is extended to get propositional truncations for all types.

Example

not always proposition



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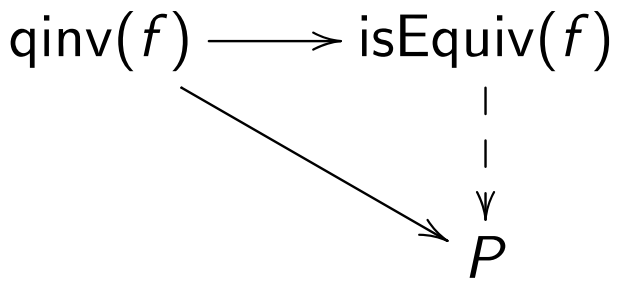

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isEquiv(f) is a canonical way of making qinv(f) into a proposition.

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Tomorrow: can be defined as a higher inductive type.

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2. If there is a canonical choice, e.g. for a decidable family $P : \mathbb{N} \to \text{Prop}$, we can factor

$$\|A\| \to (\Sigma x : \mathbb{N})(P(n) \times "x \text{ is least sat. } P") \to (\Sigma x : \mathbb{N})P(n)$$

Univalent logic

Curry-Howard logic, except:

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Consequence: All formulas are interpreted as propositions.
Since atomic formulas are propositions, and all connectives now preserve propositions.

Σ vs $\exists$: Unexpected Curry-Howard images

Image for a function $f : A \to B$ translated into Curry-Howard:

$$\mathsf{image}_{CH}\, f :\equiv (\Sigma y : B)(\Sigma x : A)(f\, x =_B y)$$

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Similarly, the right notion of surjectivity is

$$\mathrm{isSurjective}(f) :\equiv (\Pi b : B)\|(\Sigma x : A)(f\, x =_B y)\|$$

— with untruncated Σ, every "surjection" would come with a choice of an inverse.

Classical principles in univalent logic

The right formulation of the Law of Excluded Middle in univalent logic is now $(\prod P : \mathcal{U})(\mathsf{isProp}(P) \to P + \neg P)$.

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These axioms are not provable, but they are consistent with UA (since they are true in the simplicial sets model).

The Axiom of Choice

The type-theoretic Axiom of Choice

$$\big((\Pi x : A)(\Sigma y : B) R\, x\, y\big) \to (\Sigma f : A \to B)(\Pi x : A)(R\, x\, (f\, x))$$

is provable.

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This axiom is consistent with UA (simplicial sets model again).

The univalent Axiom of Choice is non-constructive

Theorem (after Diaconescu [1975])

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Proof.

We assume AC and instantiate

$$\begin{aligned} A &:\equiv (\Sigma Q : \mathbf{2} \to \mathrm{Prop})\|(\Sigma b : \mathbf{2}) Q(b)\| \\ B &:\equiv \mathbf{2} \\ R\,(Q, q)\, b &:\equiv Q(b) \end{aligned}$$

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The premise of AC says "every inhabited predicate is inhabited", which is clearly true (proof term $\lambda(Q, q).\, q$).

Proof. (cont.)

Hence we get

$$\|(\Sigma f : A \to B)(\Pi x : A)(R\, x\, (f\, x))\|$$

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From this, we are trying to prove the proposition $P + \neg P$, so we can forget about the truncation:

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$$T\, b :\equiv \|(b =_{\mathbf{2}} \mathsf{true}) + P\|$$
$$F\, b :\equiv \|(b =_{\mathbf{2}} \mathsf{false}) + P\|$$

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There is $f : A \to \mathbf{2}$ such that $T\, (f\ T)$ and $F\, (f\ F)$.

Proof. (cont.)

$\mathbf{2}$ has decidable equality, so there are four possibilities:

$f\ T$	$f\ F$	
false	false	$T(f\ T) \equiv \|\mathsf{false} = \mathsf{true} + P\| \simeq \|\mathbf{0} + P\| \simeq P$
false	true	$F(f\ F) \equiv \|\mathsf{true} = \mathsf{false} + P\| \simeq \|\mathbf{0} + P\| \simeq P$
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In three cases we clearly have P. If $f\ T \equiv \mathsf{true}$ and $f\ F \equiv \mathsf{false}$, we prove $\neg P$:

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Assume P. Then $T\ b \simeq F\ b$ for every $b : \mathbf{2}$, hence $T = F$ by univalence.

But if $T = F$ then $f\ T = f\ F$, i.e. $\mathsf{true} = \mathsf{false}$, a contradiction. Hence we have proven $\neg P$.

Hence in each case we have proven $P + \neg P$. □

Exercises

1. Prove $(\Pi A : \mathcal{U})(A + \neg A) \leftrightarrow (\Pi A : \mathcal{U})(\neg\neg A \to A)$.

2. State and show that a more traditional dependent elimination rule is equivalent to the given elimination rule for propositional truncation on slide 22.

3. Show that if isProp(A) and isProp(B) and $\neg(A \times B)$, then isProp$(A + B)$. In particular, no truncation is needed around $P + \neg P$ in the statement of univalent LEM.

4. Prove

$$\big((\Pi x : A)(\Sigma y : B)R\,x\,y\big) \simeq (\Sigma f : A \to B)(\Pi x : A)(R\,x\,(f\,x))$$

5. Show that
$(\Sigma x : \mathbb{N})(f(x) = 0 \times ((\Pi y : \mathbb{N})(f(y) = 0 \to x \leq y))$ is the propositional truncation of $(\Sigma x : \mathbb{N})f(x) = 0$.

Summary

Univalent logic: propositions as subsingleton types.

Propositional truncation: used to interpret $\vee$ and $\exists$.

Makes it possible to assume non-constructive principles such as the Law of Excluded Middle and the Axiom of Choice without giving up the Univalence Axiom.

Σ used for structure, $\exists$ for properties. In general an art to decide which one to use.

Coming up tomorrow:

- Higher Inductive Types
- Synthetic homotopy theory

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