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Constructive aspects of models for non-standard analysis

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Contents

Abstract

Reduced products are generalizations of ultraproducts where the filter used need not be an ultrafilter. With a suitable choice of filter, we can then get a more constructive model of non-standard analysis. We study properties of such reduced products and investigate what classical results are still valid in a constructive setting.

A boundedness principle **BD**, not derivable in pure constructive mathemetics **BISH**, is also studied. We show that certain theorems in classical mathematics related to reduced prdoucts or non-standard analysis are equivalent to or imply **BD** or **LLPO**, and thus not constructively provable.

1 Introduction

When Isaac Newton and Gottfried Leibniz introduced calculus in the 17th century, they did so with the help of so-called *infinitesimals*, numbers x so that $|x| < r$ for all finite rational numbers r. The concept didn't have a rigorous logical foundation, however, until Abraham Robinson [Rob66] used model theory to construct models of calculus with infinitesimals, called *non-standard analysis*. (An earlier attempt was made by Schmieden and Laugwitz [SL58].)

The methods used were non-constructive. Nevertheless, one often finds the reasoning in the non-standard model to be fairly constructive, as focus often has shifted from logical arguments (for each ϵ there is a δ etc) to straightforward computations. If we replace the non-constructive methods used to construct the non-standard model with constructive ones, we might hope to get the same constructivity, but with a model we can actually construct. The aim of this thesis is to see to what extent this is possible. For other attempts at constructive non-standard analysis, see [Pal07, Pal98, Pal96, Pal95, Sch00].

In Section 3, we study a non-constructive principle which will turn up multiple times in later sections. Section 4 is devoted to some nice properties of classical non-standard models, and whether they also may hold constructively. In Section 5, finally, we develop some constructive non-standard analysis with the tools we have acquired, and once again show that certain theorems cannot be proven constructively.

1.1 Main contributions of this thesis

The main contributions of this thesis are results showing that certain theorems cannot be proven constructively. The material in Section 3 is compiled from different sources, e. g. [Ish92, Lie05, TvD88a, TvD88b], many of them however not as detailed as our presentation. In Section 4, Theorem 4.6 connecting Palyutin's Theorem and **BD**, and Theorem 4.16, connecting the ω_1-saturation of reduced products with König's lemma are the main new contributions. In Section 5, we draw a new connection between **BD** and non-standard analysis in Theorem 5.13.

1.2 Acknowledgements

I would like to thank my advisor professor Erik Palmgren for introducing me to intuitionistic logic, helpful suggestions and mathematical guidance. I would also like to thank Olov Wilander and Anton Hedin for discussions and encouragement.

2 Preliminaries

2.1 Intuitionistic logic

By *classical logic*, we understand the logic which is usually used in mathematics. If we remove the *principle of the excluded middle* ($P \vee \neg P$ holds for every statement P), we instead get intuitionistic logic. This is equivalent to removing the inference rule *reductio ad absurdum* (if $\neg P$ leads to a contradiction, P must hold). Good introductions to intuitionistic logic are e.g. [TvD88a, TvD88b], [Bee85] and [Dum00].

In intuitionistic logic, existence is taken more seriously than in classical logic. $\exists x.P(x)$ holds if we can produce a t such that $P(t)$ holds, i.e. it is not enough that it is impossible for all t that $\neg P(t)$ holds. $P_1 \vee P_2$ can be read as $\exists i \in \{1, 2\}.P_i$, so from this reading we see that $P_1 \vee P_2$ holds if P_1 or P_2 holds, and, importantly, we can decide which one. As a consequence of this, every proof of an existential statement gives an effective method for constructing an element satisfying the statement.

If $P \vee \neg P$ would hold for all P, we would thus for each statement P have an algorithm for deciding the truth of this algorithm. There is of course no reason to believe this would be possible in general, and this is why we reject the law of the excluded middle.

Note that $P \vee \neg P$ holds for *decidable* P, so we also have *reductio ad absurdum* for such P: suppose $\neg P$ leads to a contradiction. We now have either P or $\neg P$, and we can decide which one holds. If P holds, we are done, and if $\neg P$ holds, we have a contradiction, from which everything follows, in particular P. Thus P holds in both cases. We will make use of this a few times, for example when quantifying over finite sets of natural numbers.

We define $\neg P$ as an abbreviation of $P \to \bot$.

Sometimes, we will state and prove a theorem using classical logic. Such theorems are marked with the symbol $\boxed{P \vee \neg P}$.

2.1.1 BISH, CLASS, RUSS and INT

We will mostly work in the system of Bishop's constructive mathematics **BISH**, i.e. mathematics done with intuitionistic logic. We can now extend **BISH** with different axioms to get other systems. If we add the principle of the excluded middle, we get classical mathematics **CLASS**.

If we instead add *Church's Thesis* – every operation from $\mathbb{N}$ to $\mathbb{N}$ is recursive – and *Markov's principle* – for a decidable A, $\neg\neg\exists x\ A(x) \to \exists x\ A(x)$ – we get the Russian school of constructive mathematics, denoted **RUSS**. One consequence of Church's Thesis is that there is an enumeration of all partial functions from $\mathbb{N}$ to $\mathbb{N}$ with countable domain. This enumeration is called a programming system. When working in **RUSS**, we will assume a fixed such system

$$\begin{array}{cccc} \varphi_0 & \varphi_1 & \varphi_2 & \cdots \\ D_0 & D_1 & D_2 & \cdots, \end{array}$$

where for each $m \in \mathbb{N}$, $\varphi_m : \mathbb{N} \rightharpoonup \mathbb{N}$ is a partial function and $(D_m(n))_{n=1}^{\infty}$ is a sequence of finite subsets of $\mathbb{N}$ such that $D_m(0) \subseteq D_m(1) \subseteq \ldots$, and $\bigcup_{n\in\mathbb{N}} D_m(n)$ is the domain of φ_m. (One may think of $D_m(n)$ as the set of inputs $x \in \mathbb{N}$ such that $\varphi_m(x)$ can be calculated in n steps.)

If we instead finally add the axioms "every $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is continuous", the axiom of choice $\mathbf{AC}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ and the *Fan Theorem* to **BISH**, we get Brouwer's intuitionistic mathematics **INT**.

As all other systems are obtained by adjoining axioms to **BISH**, results in **BISH** are of course also valid in all systems discussed. Hence, one might want to work in **BISH** instead of say **CLASS** for practical and not necessarily philosophical reasons; a theorem of **BISH** holds in much greater generality than a theorem that holds merely in **CLASS**.

A good reference for the different systems is [BR87].

2.1.2 Omniscience principles

There are of course sentences provable in classical logic which we cannot prove using intuitionistic logic. As every intuitionistic theorem also is a classical theorem, though, we can't hope to prove the negation of an unprovable sentence φ. So how can we tell when something is unprovable? One way is to reduce the provability of φ to some well-known non-constructive principle. We call those *omniscience principles*, as their truth would make one (more or less) omniscient. Most omniscience principles are weakenings of the law of the excluded middle.

The *Limited Principle of Omniscience* states the following:

Principle (LPO). For every binary sequence a, either $a_n = 0$ for all $n \in \mathbb{N}$ or there is $n \in \mathbb{N}$ such that $a_n = 1$,

i.e. given a sequence a, we can tell whether all its terms are zero, or find a term which is not. The *Lesser Limited Principle of Omniscience* says:

Principle (LLPO). For every binary sequence a containing at most one 1, $a_{2n} = 0$ for all $n \in \mathbb{N}$ or $a_{2n+1} = 0$ for all $n \in \mathbb{N}$.

2.2 Constructive mathematics

The main point of Bishop-style constructivism is to do ordinary mathematics using intuitionistic logic instead of classical logic. We don't want to study "constructive objects", we want to constructively study objects. However, since we want our theory to be constructive, we have to start over from the beginning and construct the objects we want to study in a way that is meaningful to us. Most of these definitions can be found in [BB85].

To construct a set, we must be able to tell what must be done to construct an element of the set, and what must be done to show that two elements of the set are equal (put another way, every set A comes with its own equivalence relation $=_A$, so that a set really is a pair $(A, =_A)$). A function $f : A \to B$ from a set A to a set B is an extensional operation from A to B, i.e. it associates elements a of A with elements $f(a)$ of B so that $f(a) =_B f(a')$ if $a =_A a'$. Formally, a subset of B is a pair (A, i) where A is a set and $i : A \to B$ is an inclusion function so that $i(a) = i(a')$ if and only if $a = a'$. (This way, we really have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.) In practice however, we will treat sets in the usual way.

We take the natural numbers and their order as given, and construct the integers and rational numbers in the usual way. We define a real number to be a sequence $(x_n)_{n=1}^{\infty}$ of rational numbers so that $|x_m - x_n| \leq m^{-1} + n^{-1}$ for all $n, m \in \mathbb{N} \setminus \{0\}$. This is similar to the usual construction with equivalence classes of Cauchy sequences, but simplifies things since we don't have to have an additional modulus sequence for x, telling how large n must be to approximate x to a specified accuracy. We define two real numbers (x_n) and (y_n) to be equal if $|x_n - y_n| \leq 2n^{-1}$ for all $n \in \mathbb{N} \setminus \{0\}$. (In the same way, just as we have constructed the complete space $\mathbb{R}$ from $\mathbb{Q}$, we can construct the completion $\widetilde{X}$ of an arbitrary metric space X.)

Arithmetic for real numbers can basically be done component-wise, as usual. For a real number z, a *canonical bound* for z is an integer K_z such that $|z_n| < K_z$ for all $n \in \mathbb{N}$. We can e.g. choose $K_z = \lfloor |z_1| + 2 \rfloor$, the least integer greater than $|z_1| + 2$. Fix two real numbers x, y and write $k = \max(K_x, K_y)$. Now we define

(a) $x + y = (x_{2n} + y_{2n})_{n=1}^{\infty}$

(b) $xy = (x_{2kn} y_{2kn})_{n=1}^{\infty}$

(c) $\max(x, y) = (\max(x_n, y_n))_{n=1}^{\infty}$

(d) $-x = (-x_n)_{n=1}^{\infty}$

One of course has to verify that this really defines real numbers, which can easily be done. It is also not hard to see that the operations $(x, y) \mapsto x + y$, $(x, y) \mapsto xy$ etc. are functions.

We define y to be less than x, $x > y$, if $|x_m - y_m| > 2m^{-1}$ for some $m \in \mathbb{N}$, and define $x \leq y$ as $\neg(x > y)$. Thus, we have $x = y \vee x < y \Rightarrow x \leq y$ but not necessarily $x = y \vee x < y \Leftarrow x \leq y$.

2.3 Finite and infinite sequences

We define $\mathbb{N}^*$ to be the set of finite sequences over $\mathbb{N}$. Every sequence $a \in \mathbb{N}^*$ can be coded as a natural number $\langle a \rangle$ by a coding function $\langle \cdot \rangle : \mathbb{N}^* \to \mathbb{N}$. There are many well-known choices for $\langle \cdot \rangle$. We could e.g. define $\langle a_0, \ldots, a_n \rangle := \prod_{i=0}^{n} p_i^{1+a_i}$, where $(p_n)_{n=0}^{\infty}$ is an enumeration of the prime numbers, or inductively define $\langle a_0, a_1, \ldots, a_n \rangle := \langle a_0, \langle a_1, \ldots, a_n \rangle \rangle'$ where $\langle \cdot, \cdot \rangle'$ is a pairing function, e.g. $\langle n, m \rangle' = 2^n(2m + 1)$ or the Cantor pairing function $\langle n, m \rangle' = \lfloor \frac{1}{2}(n + m)(n + m + 1) + m \rfloor$.

We won't care which one we use, as long as we have length, decoding, restriction and concatenation functions $\|\cdot\|$, $(\cdot)_i$, $\cdot \upharpoonright_i$ and $\cdot * \cdot$ respectively such that $\|\langle x_1, \ldots, x_n \rangle\| = n$, $(\langle x_1, \ldots, x_n \rangle)_i = x_i$, $\langle x_1, \ldots, x_n \rangle \upharpoonright_i = \langle x_1, \ldots, x_i \rangle$ for $i \leq n$ and $\langle x_1, \ldots, x_n \rangle * \langle y_1, \ldots, y_m \rangle = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$.

For an infinite sequence $\alpha : \mathbb{N} \to \mathbb{N}$, we write $\overline{\alpha}(n)$ for the finite sequence consisting of the n first terms of α, i.e. $\overline{\alpha}(n) = \langle a(0), \ldots, a(n-1) \rangle$. For $\beta : \mathbb{N} \to \mathbb{N}$ and $a \in \mathbb{N}^*$, we write $\beta \in a$ if $\overline{\beta}(\|a\|) = a$, i.e. we identify a with the set of infinite sequences extending it. We write $a \preceq b$ if $a = b \upharpoonright_k$ for some $k \leq \|b\|$, i.e. a is an initial segment of b.

2.4 Categorical logic

2.4.1 Category theory

Here, we define the categorical notions that we need for interpreting logic in a category. For a more complete introduction to category theory, see e.g. [ML98]. We start with the two most basic definitions.

Definition 2.1. A *category* $\mathcal{C}$ consists of a class of *objects* $\mathcal{C}$ and for any two objects A, $B \in \mathcal{C}$ a class $\mathcal{C}(A, B)$ of *morphisms* between A and B, together with a binary operation $\circ$, called *composition*, such that:

(i) for any $f \in \mathcal{C}(B, C)$ and $g \in \mathcal{C}(A, B)$, $f \circ g \in \mathcal{C}(A, C)$,

(ii) $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f \in \mathcal{C}(C, D)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(A, B)$.

(iii) for any object $x \in \mathcal{C}$, there is an *identity morphism* $\mathrm{id}_x \in \mathcal{C}(x, x)$ such that for every $f \in \mathcal{C}(A, B)$, $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

We often write $f : A \to B$ for $f \in \mathcal{C}(A, B)$. We call two objects A, B *isomorphic* if there are morphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a mapping

- associating to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$,
- associating to each morphism $f \in \mathcal{C}(A, B)$ a morphism $F(f) \in \mathcal{D}(F(A), F(B))$,

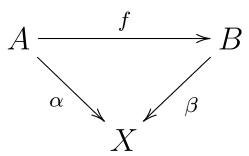
such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects $X \in \mathcal{C}$ and $F(f \circ g) = F(f) \circ F(g)$ for all morphisms f, g in $\mathcal{C}$.

Just like for morphisms, we can compose functors and for each category $\mathcal{C}$, there is an identity functor, denoted $1_{\mathcal{C}}$. (Thus, there is a category of small categories (small meaning that the objects and morphisms are sets and not proper classes) with small categories as objects and functors as morphisms.)

Definition 2.3. A morphism $f : A \to B$ is called a

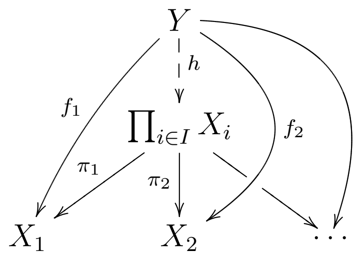
- *monomorphism* (*mono*) if for all $g_1, g_2 : X \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$,
- *epimorphism* (*epi*) if for all $g_1, g_2 : B \to X$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Definition 2.4. The *slice category* $\mathcal{C}/X$ has as objects morphisms in $\mathcal{C}$ into $X \in \mathcal{C}$, and a morphism $f : \alpha \to \beta$ is a morphism in $\mathcal{C}$ that makes the following diagram in $\mathcal{C}$ commute:



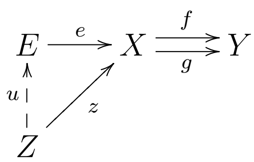
We now turn to some categorical constructions.

Definition 2.5. Let $\{X_i : i \in I\}$ be a collection of objects in a category $\mathcal{C}$. The *product* of $\{X_i : i \in I\}$ is an object $X = \prod_{i \in I} X_i$ together with *projection morphisms* $\pi_i : X \to X_i$ such that, for any object Y and any collection of morphisms $f_i : Y \to X_i$, there is a unique morphism $h : Y \to \prod_{i \in I} X_i$ so that $f_i = \pi_i \circ h$ for all $i \in I$, i.e. the following diagram commutes:



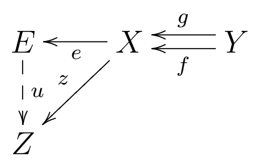
We write $X_1 \times \ldots \times X_n$ for finite products.

Definition 2.6. Let $f, g : X \rightrightarrows Y$ be morphisms in a category $\mathcal{C}$. The *equalizer* of f and g is an object E together with a morphism $e : E \to X$ such that $f \circ e = g \circ e$, and for any other object Z and morphism z such that $f \circ z = g \circ z$, there is a unique morphism $u : Z \to E$ so that $z = e \circ u$.



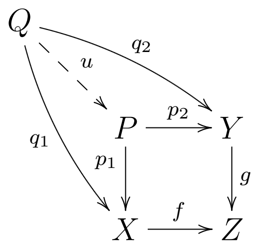
For every property, there is a so called dual property which we get by reversing all arrows. The only dual property we will use is the coequalizer.

Definition 2.7. The *coequalizer* of $f, g : Y \rightrightarrows X$ is an object E together with a morphism $e : X \to E$ such that $e \circ f = e \circ g$, and for any other object Z and morphism z such that $z \circ f = z \circ g$, there is a unique morphism $u : E \to Z$ such that

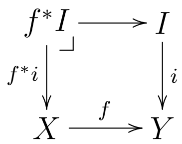


commutes.

Definition 2.8. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a category $\mathcal{C}$. The *pullback* of f and g is an object P together with two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ so that $p_1 \circ f = p_2 \circ g$, and if Q is any other object with morphisms $q_1 : Q \to X$ and $q_2 : Q \to Y$ so that $q_1 \circ f = q_2 \circ g$, then there is a unique morphism $u : Q \to P$ making the following diagram commute:



The pullback of f and g can be constructed as the equalizer (composed with appropriate projections) of $f \circ \pi_1, g \circ \pi_2 : X \times Y \rightrightarrows Z$, if products and equalizers exist. We will also consider functors $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ given by pullbacks along f, i.e. f^*i is the left vertical arrow in the following diagram:

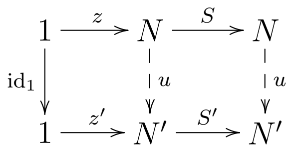


Note that if i is mono, so is f^*i.

Definition 2.9. An object T in a category $\mathcal{C}$ is a *terminal object* if for every object $X \in \mathcal{C}$, there is exactly one morphism $f : X \to T$.

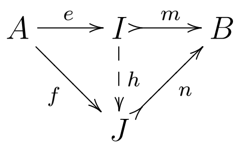
Between any two terminal objects in a category, there is a unique isomorphism, so we write 1 for "the" terminal object.

Definition 2.10. Let $\mathcal{C}$ be a category. A *natural number object* in $\mathcal{C}$ is an object $N \in \mathcal{C}$ together with morphisms $S : N \to N$ and $z : 1 \to N$ such that for any other object N' with morphisms $S' : N' \to N'$ and $z' : 1 \to N'$, there is a unique morphism $u : N \to N'$ so that the following diagram commutes:



One should think of $z : 1 \to N$ as picking out the zero element, and $S : N \to N$ as the successor function.

Definition 2.11. A category $\mathcal{C}$ *has images* if any morphism $g : A \to B$ may be factorized as $g = m \circ e$ where $e : A \to I$, and $m : I \rightarrowtail B$ is mono. Furthermore, if there is any other factorization $g = n \circ f$ with $f : A \to J$ and $n : J \rightarrowtail B$ mono, then there is a unique morphism $h : I \to J$ such that $n \circ h = m$.



The image factorization gives a functor $\mathbf{im} : \mathcal{C}/X \to \mathcal{C}/X$ defined by $\mathbf{im}(g) = \mathbf{im}(m \circ e) = m$, where $(e : A \to I, m : I \rightarrowtail B)$ is an image factorization of $g = m \circ e$.

Of course, these constructions might not exist in a general category. Since we need them to interpret logic, however, we are only interested in categories where they do exist. We call the categories we are interested in *regular* and *locally cartesian closed.*

Definition 2.12. A morphism $f : A \to B$ is *regular epi* if it arises as a coequalizer.

Definition 2.13. A category $\mathcal{C}$ is called *regular* if it has equalizers, binary products, a terminal object and images, and regular epis are stable under pullback (i.e. the pullback of a regular epi along any map is again a regular epi).

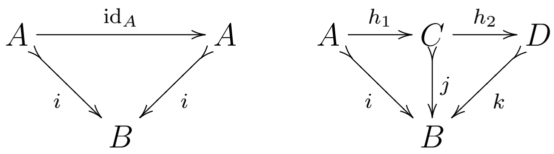
A regular category admits the interpretation of formulas with existential quantifiers. We need a locally cartesian closed category for universal quantification.

Definition 2.14. A category $\mathcal{C}$ is called *locally cartesian closed* if every slice category $\mathcal{C}/X$ is cartesian closed, i.e. it has binary products, exponentials and a terminal object.

Equivalently, a category $\mathcal{C}$ is locally cartesian closed if and only if the pullback functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint Π_f for each morphism $f : A \to B$ in $\mathcal{C}$. (For a proof, see [Joh02, A1.5.3].)

Consider two monos $i : A \rightarrowtail B$ and $j : C \rightarrowtail B$ in a category $\mathcal{C}$. We say that $i \leq j$ if i factors through j, i.e. if there is a morphism $h : A \to C$ such that $i = j \circ h$. The collection of all monos in $\mathcal{C}$ with fixed codomain B, denoted $\mathbf{Mon}(B)$, under the relation $\leq$ is thus a preorder: $i \leq i$ with

$h = \mathrm{id}_A$, and if $i \leq j$ and $j \leq k$, say $i = j \circ h_1$ and $j = k \circ h_2$, then $i = j \circ h_1 = (k \circ h_2) \circ h_1 = k \circ (h_2 \circ h_1)$ so that $i \leq k$.



We could however have $i \leq j$ and $j \leq i$ with $i \neq j$. As a remedy for this, we define the relation $\approx$ by $i \approx j$ iff $i \leq j$ and $j \leq i$. By the argument above, $\approx$ is an equivalence relation. The equivalence classes are called *subobjects* of B, and we write $\mathbf{Sub}(B) = \mathbf{Mon}(B)/\approx$ for the collection of all of them. The relation $\leq$ lifted to $\mathbf{Sub}(B)$ is thus a partial order. We will make much use of subobjects and the relation $\leq$ in our interpretation of formulas in categories.

2.4.2 Interpretation of many sorted predicate first order logic in a category $\mathcal{C}$

The following is based on [AB03] and [Pal06]. Another good source is [Joh02].

Let $\mathcal{C}$ be a regular and locally cartesian closed category. An interpretation of a multi-sorted lex logic with universal and existential quantifiers is given by the following data:

1. A sort A is interpreted as an object $[\![A]\!]$.

2. A typing context $x_1 : A_1, \ldots, x_n : A_n$ is interpreted as the product $[\![A_1]\!] \times \ldots \times [\![A_n]\!]$, and the empty context is interpreted as the terminal object 1.

3. A function symbol f with signature $(A_1, \ldots, A_m; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \ldots \times [\![A_m]\!] \to [\![B]\!]$.

4. A term in context $\Gamma | t : B$ is interpreted as a morphism $[\![\Gamma | t : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$, as follows:

   (a) A variable $x_1 : A_1, \ldots, x_n : A_n | x_i : A_i$ is interpreted as the ith projection $\pi_i : [\![A_1]\!] \times \ldots \times [\![A_n]\!] \to [\![A_i]\!]$.

   (b) A composite term $\Gamma | f(t_1, \ldots, t_m) : B$ is interpreted as the composition $[\![f]\!] \circ \langle [\![\Gamma | t_1 : A_1]\!], \ldots, [\![\Gamma | t_m : A_m]\!] \rangle$.

5. A relation symbol R with signature $(A_1, \ldots, A_m)$ is interpreted as a subobject $[\![R]\!] \in \mathbf{Sub}([\![A_1]\!] \times \ldots \times [\![A_m]\!])$.

6. A logical entailment $\Gamma|\psi_1, \ldots, \psi_k \vdash \phi$ is interpreted as an inequality $[\![\psi_1]\!] \wedge \ldots \wedge [\![\psi_k]\!] \leq [\![\phi]\!]$ in $\mathbf{Sub}([\![\Gamma]\!])$, where the empty hypothesis is interpreted as the maximal subobject $[\mathrm{id}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]]$.

7. A formula in context $\Gamma|\varphi$ is interpreted as a subobject $[\![\Gamma|\varphi]\!] \in \mathbf{Sub}([\![\Gamma]\!])$ as follows:

   (a) The true constant $\Gamma|\top$ is interpreted as the maximal subobject $[\![\Gamma|\top]\!] = [\mathrm{id}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]]$.

   (b) $\Gamma|t =_A u$ is interpreted as the equalizer of $[\![\Gamma|t]\!]$ and $[\![\Gamma|u]\!]$.

   (c) An atomic formula $\Gamma|R(t_1, \ldots, t_k)$ is interpreted as the pullback of $[\![R]\!]$ along $[\![\Gamma|\overrightarrow{t}]\!]$, i.e. $\langle [\![\Gamma|t_1]\!], \ldots, [\![\Gamma|t_k]\!] \rangle^*([\![R]\!])$.

   (d) $[\![\Gamma|\varphi \wedge \psi]\!] = [\![\Gamma|\varphi]\!] \wedge [\![\Gamma|\psi]\!]$, the meet in $\mathbf{Sub}([\![\Gamma]\!])$.

   (e) Assuming we have interpreted $\Gamma, x : A|\varphi(x)$, we set $[\![\Gamma|\exists x \in A.\varphi(x)]\!] = \exists_p [\![\Gamma, x : A|\varphi(x)]\!]$, where p is the projection $p : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$ and $\exists_p(\alpha) = \mathbf{im}(p \circ \alpha)$.

   (f) Supposing $\Gamma, x : A|\varphi(x)$ have been defined, we set $[\![\Gamma|\forall x \in A.\varphi(x)]\!] = \forall_p [\![\Gamma, x : A|\varphi(x)]\!]$, where p is the projection $p : [\![\Gamma]\!] \times [\![A]\!] \to [\![\Gamma]\!]$ and $\forall_p$ is the right adjoint of the pullback functor $p^* : \mathbf{Sub}([\![\Gamma]\!]) \to \mathbf{Sub}([\![\Gamma]\!] \times [\![A]\!])$.

   (g) The interpretation of $B \to C$, where $[\![B]\!]$ and $[\![C]\!]$ are interpreted as subobjects $[\beta : B \rightarrowtail X]$ and $[\gamma : C \rightarrowtail X]$ respectively, is $[\![B \to C]\!] = (\forall_\beta \circ \beta^*)(\gamma)$.

3 The Boundedness Principle BD

3.1 Definition and general properties

Definition 3.1. A subset $S \subseteq \mathbb{N}$ is *bounded*, if there is a $k \in \mathbb{N}$ such that for all $t \in S$, $t \leq k$.

Definition 3.2. A subset $S \subseteq \mathbb{N}$ is *pseudobounded*, if for all sequences $s \colon \mathbb{N} \to S$,

$$\lim_{n\to\infty} \frac{s(n)}{n} = 0.$$

Every bounded subset S of $\mathbb{N}$ is of course pseudobounded. (For a given positive ϵ, just choose $N > \sup S/\epsilon$. Then $s(n)/n \leq \sup S/n < \epsilon$ for all $n \geq N$.) It turns out that the following principle, stating the converse, is fruitful to study when doing constructive mathematics. It was introduced by Hajime Ishihara in [Ish92].

Principle (BD). Every inhabited pseudobounded subset of $\mathbb{N}$ is bounded.

Recall that a set X is called *countable* if there is a surjective function $f : \mathbb{N} \to X$ (in other words, we can enumerate the elements of X, possibly with repetitions). It seems very unlikely that we should be able to prove that every subset X of $\mathbb{N}$ is countable, as that would involve constructing a surjective $f : \mathbb{N} \to X$ for each $X \subseteq \mathbb{N}$. We thus get a slightly weaker form of **BD** if we only require countable pseudobounded subsets to be bounded:

Principle (BD − N). Every inhabited countable pseudobounded subset of $\mathbb{N}$ is bounded.

BD is equivalent to many theorems in analysis, e.g. Banach's Inverse Mapping Theorem, the Open Mapping Theorem, the Closed Graph Theorem and the Banach-Steinhaus Theorem [Ish01].

The following proposition by Richman [Ric09] sometimes makes **BD** easier to work with. We call a sequence (s_n) non-decreasing if we have $s_i \leq s_j$ for $i < j$.

Theorem 3.3 ([Ric09]). *Let (s_n) be a fixed non-decreasing sequence in $\mathbb{N}$ with $\lim s_n = \infty$. Then $A \subseteq \mathbb{N}$ is pseudobounded if and only if each sequence in A is eventually bounded by (s_n).*

Proof. ($\Rightarrow$) Suppose A is pseudobounded and let (a_i) be a sequence in A. We may assume that (a_i) is non-decreasing, since we can construct a non-decreasing sequence (a'_i) by $a'_0 = a_0$ and $a'_{n+1} = \max(a'_n, a_{n+1})$. Then $a_i \leq a'_i$ for all $i \in \mathbb{N}$ and thus if $a'_i \leq s_i$, then $a_i \leq s_i$.

Let $t_n = \min\{i\colon n \leq s_i\}$. Then (t_n) is non-decreasing and $t_k \leq i$ if and only if $k \leq s_i$. Consider the sequence $(b_n) = (a_{t_{n+1}})$ in A. As A is pseudobounded, there is a $k \in \mathbb{N}$ such that $\frac{b_n}{n} \leq 1$ for all $n \geq k$, that is $a_{t_{n+1}} \leq n$. Let $n \geq k$ and $t_n \leq i \leq t_{n+1}$. Then $a_i \leq a_{t_{n+1}}$ since a_i is non-decreasing, $a_{t_{n+1}} \leq n$ by the choice of n and $n \leq s_i$ since $t_n \leq i$. Thus $a_i \leq a_{t_{n+1}} \leq n \leq s_i$, i.e. $a_i \leq s_i$ for all $i \geq t_k$.

($\Leftarrow$) Suppose that each sequence in A is eventually bounded by (s_n), and let (a_i) be a sequence in A. Once again, we can assume that (a_i) is non-decreasing by constructing (a'_i) as above and noting that if $\lim \frac{a'_i}{i} = 0$, then $\lim \frac{a_i}{i} = 0$, since $a_i \leq a'_i$ for all $i \in \mathbb{N}$.

Given $m \in \mathbb{N}$, consider the sequence $(b_n) = (a_{ms_{n+1}})$ in A. By the hypothesis, there is a $k \in \mathbb{N}$ such that $b_n = a_{ms_{n+1}} \leq s_n$ for all $n \geq k$. Let $n \geq k$ and $ms_n \leq i \leq ms_{n+1}$, i.e. $s_n \leq \frac{i}{m} \leq s_{n+1}$. Since a_n is non-decreasing, $a_i \leq a_{ms_{n+1}}$ and $a_{ms_{n+1}} \leq s_n \leq \frac{i}{m}$. Thus $a_i \leq \frac{i}{m}$ for each $i \geq ms_k$, i.e. $\frac{a_i}{i} \leq \frac{1}{m}$ for all $m \in \mathbb{N}$ and $i \geq ms_k$, thus $\lim \frac{a_i}{i} = 0$. □

Corollary 3.4. *$A \subseteq \mathbb{N}$ is pseudobounded if and only if each sequence in A is eventually bounded by $n \mapsto n$.*

Proof. Set $s_n = n$, a non-decreasing sequence with $\lim s_n = \infty$ and apply the theorem. □

3.2 Invalidity of BD in BISH

To see that **BD** is not valid in **BISH**, we will exhibit a regular and locally cartesian closed category where **BD** can't hold. By a soundness result, we then have **BISH** $\nvdash$ **BD**. We follow the proof presented in [Lie05], but take our time to fill in the details.

We will first introduce the category in question, and show that it is regular and locally cartesian closed. The category is constructed from something called a typed partial combinatory algebra.

3.2.1 The typed partial combinatory algebra ωALat

Definition 3.5 (Typed partial combinatory algebra)**.** A *typed partial combinatory algebra* (tpca) is a non-empty set $\mathcal{T}$ of types with

(i) operations $\times, \longrightarrow\, :\ \mathcal{T}^2 \to \mathcal{T}$,

(ii) a set $|T|$ of realizers of type T for each $T \in \mathcal{T}$,

(iii) a partial application function $\cdot_{S,T} : |S \longrightarrow T| \times |S| \to |T|$ for all S, $T \in \mathcal{T}$ such that for all S, T, $U \in \mathcal{T}$ there are elements

$$\mathbf{k}_{S,T} \in |S \to T \to S| \qquad \mathbf{s}_{S,T,U} \in |(S \to T \to U) \to (S \to T) \to (S \to U)|$$
$$\mathbf{pair}_{S,T} \in |S \to T \to S \times T| \qquad \mathbf{fst}_{S,T} \in |S \times T \to S| \qquad \mathbf{snd}_{S,T} \in |S \times T \to T|$$

such that for all a, b, c of the right type,

$$\mathbf{k} \cdot a \cdot b = a \qquad \mathbf{s} \cdot a \cdot b \downarrow \qquad \mathbf{s} \cdot a \cdot b \cdot c \succeq a \cdot c \cdot (b \cdot c)$$
$$\mathbf{fst} \cdot (\mathbf{pair} \cdot a \cdot b) = a \qquad \mathbf{snd} \cdot (\mathbf{pair} \cdot a \cdot b) = b$$

where $e \downarrow$ means that e is defined, and $e \succeq e'$ that if e is defined, so is e' and $e = e'$.

We have a specific tpca in mind, namely the collection of ω-algebraic lattices.

Definition 3.6. (i) A partially ordered set P is a *lattice*, if every pair of elements x, $y \in P$ has an unique supremum $x \vee y$ and an unique infimum $x \wedge y$.

(ii) An element $c \in P$ of a poset is *compact* if for all directed $\emptyset \neq D \subseteq P$, if $\sup D$ exists and $c \leq \sup D$, then $c \leq d$ for some $d \in D$.

(iii) A lattice P is *ω-algebraic* if every element $x \in P$ is the supremum of some directed set of compact elements, and the set $\{c \in P \colon c \text{ is compact}\}$ is countable.

(iv) A map $f \colon P \to Q$ between posets is *Scott-continuous* if $f(\sup D) = \sup f(D)$ for all directed subsets $D \subseteq P$.

Let ωALat be the class of ω-algebraic lattices.

Lemma 3.7. *ωALat is a tpca.*

Proof. A type in ωALat is a lattice, and its realizers are all elements in the lattice. Given two lattices S and T, $S \times T := \{(x, y) \colon x \in S, y \in T\}$ with partial order $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$ is again an ω-algebraic lattice and so is $(S \longrightarrow T) := \{f \colon S \to T,\ f \text{ Scott-continuous }\}$, with partial order $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in S$. For a proof that these lattices are really ω-algebraic, see [Cro93, 2.8.3].

Let S, T, $U \in \omega$ALat. Define $\mathbf{k}_{S,T}$, $\mathbf{s}_{S,T,U}$, $\mathbf{pair}_{S,T}$, $\mathbf{fst}_{S,T}$, $\mathbf{snd}_{S,T}$ by the equations they have to satisfy. Routine calculations then show that they indeed are Scott-continuous and thus realizers. □

3.2.2 The category $\mathbf{Asm}(\mathcal{T})$ of assemblies over $\mathcal{T}$

Definition 3.8. Let $\mathcal{T}$ be a tpca. The category $\mathbf{Asm}(\mathcal{T})$ of *assemblies over* $\mathcal{T}$ is defined as follows.

- An object X of $\mathbf{Asm}(\mathcal{T})$ is a triple $X = (I, T, \Vdash_X)$ where I is a set, $T \in \mathcal{T}$ a type and $\Vdash_X \subseteq |T| \times I$ a realizability relation such that $(\forall i \in I)(\exists t \in |T|)\ (t \Vdash_X i)$.
- A morphism $f\colon X \to X'$ is a map $f\colon I \to I'$ which is tracked by some $a \in |T \longrightarrow T'|$, i.e. $(\forall i \in I)(\forall b \in |T|)\big(b \Vdash_X i \Rightarrow a \cdot b \downarrow \wedge\, a \cdot b \Vdash_{X'} f(i)\big)$.

Lemma 3.9. *A morphism f in $\boldsymbol{Asm}(\mathcal{T})$ is mono $\Leftrightarrow$ f is a tracked injective map.*

Proof. ($\Rightarrow$) Let $f : A \rightarrowtail B$ be mono, and let $C = (\{*\}, T, \Vdash)$ be a singleton set together with some tcpa T and relation $\Vdash$. Let x, $y \in A$, realized by t_x and t_y respectively, and assume $f(x) = f(y)$. Define g, $h : C \to A$ by $g(*) = x$, $h(*) = y$, tracked by $\mathbf{k}\ t_x$ and $\mathbf{k}\ t_y$ respectively. Thus $(f \circ g)(z) = f(x) = f(y) = (f \circ h)(z)$ for all $z \in \{*\}$, i.e. $f \circ g = f \circ h$ and hence, since f is mono, $g = h$, i.e. $x = y$, so f is injective, and all morphisms in $\mathbf{Asm}(\mathcal{T})$ are tracked.

($\Leftarrow$) Assume f is injective and tracked. Thus f is a morphism. Let $g, h\colon C \to A$ for some C and assume $f \circ g = f \circ h$, i.e. $f(g(z)) = f(h(z))$ for all z from C. But then, since f injective, $g(z) = h(z)$ for all z, i.e. $g = h$ and hence f is mono. □

Lemma 3.10. *A morphism f in $\boldsymbol{Asm}(\mathcal{T})$ is epi $\Leftrightarrow$ f is a tracked surjective map.*

Proof. ($\Rightarrow$) ([MRR88]) Let $f : A \twoheadrightarrow B$ be epi, and let $C = (\mathcal{P}(\{0\}), T, \Vdash_C)$ where there is a $* \in |T|$ such that $* \Vdash_C i$ for all $i \in \mathcal{P}(\{0\})$. Define $g, h\colon B \to C$ by $h(x) = \{0\}$ and $g(x) = \{y \in \{0\} \colon \exists k \in A. f(k) = x\}$, both tracked by the constant $\mathbf{k}\ *$. Then for all x from A, $h(f(x)) = \{0\} = g(f(x))$ (the predicate becomes $\exists k \in A. f(k) = f(x)$, so choose $k \coloneqq x$) thus since f is epi, $g = h$, i.e. $g(x) = \{0\}$ for all x, i.e. $\exists k \in A. f(k) = x$ for all x, so f is surjective.

($\Leftarrow$) Assume $f : A \to B$ is surjective and tracked. Let g, $h : B \to C$ for some C and assume $g \circ f = h \circ f$, i.e. $g(f(z)) = h(f(z))$ for all z from C. Let b be an element of B. Since f is surjective, there is a from A such that $b = f(a)$, i.e. $g(b) = g(f(a)) = h(f(a)) = h(b)$. Since b was arbitrary, we conclude that $g = h$ and thus that f is epi. □

Lemma 3.11. *In $\boldsymbol{Asm}(\mathcal{T})$, the product of $(I_A, T_A, \Vdash_A)$ and $(I_B, T_B, \Vdash_B)$ is $(I_A \times I_B, T_A \times T_B, \Vdash)$ where $t \Vdash (a, b) :\Leftrightarrow \mathbf{fst}\ t \Vdash_A a \wedge \mathbf{snd}\ t \Vdash_B b$.*

Proof. Straightforward. □

Lemma 3.12. *The terminal object in $\boldsymbol{Asm}(\mathcal{T})$ is $1 = (\{*\}, T, \Vdash)$, where $\{*\}$ is any singleton set and all $t \in |T|$ realize $*$, i.e. $t \Vdash x \Leftrightarrow t \in |T|$.*

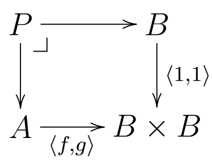
Proof. For any object $X = (I, T', \Vdash')$, the unique morphism from X to 1 is the one sending every element to $*$, tracked by an arbitrary $t \in |T \longrightarrow T'|$. □

Lemma 3.13. *A pullback of $f : A \to C$ and $g : B \to C$ in $\boldsymbol{Asm}(\mathcal{T})$ is the object $(\{\langle x, y\rangle \in I_A \times I_B \colon f(x) = g(y)\}, T_A \times T_B, \Vdash'_{A\times B})$, where $\Vdash'_{A\times B}$ is $\Vdash_{A\times B}$ restricted to $\{\langle x, y\rangle \in I_A \times I_B \colon f(x) = g(y)\}$, together with the projection morphisms.*

Proof. Straightforward. □

Lemma 3.14. *An equalizer of $f, g : A \rightrightarrows B$ in $\boldsymbol{Asm}(\mathcal{T})$ is the object $E :=$ $(\{x \in I_A \colon f(x) = g(x)\}, T_A, \Vdash'_A)$, where $\Vdash'_A$ is $\Vdash_A$ restricted to $\{x \in I_A \colon f(x) = g(x)\}$, together with the inclusion $e : E \to A$.*

Proof. The equalizer can be constructed as the pullback



which is isomorphic to E (with isomorphism $\langle a, b\rangle \mapsto a$, traced by **fst**, and inverse $a \mapsto \langle a, f(a)\rangle = \langle a, g(a)\rangle$, traced by $\lambda^* x.\mathbf{pair}\ (\mathbf{s}\ \mathbf{k}\ \mathbf{k}\ x)(t_f\ x)$, where t_f is a realizer for f (or g)). □

Lemma 3.15. *Let $A = (I, T, \Vdash_A)$ be an object in $\boldsymbol{Asm}(\mathcal{T})$ and $f \colon A \to B$ a morphism. Define $Q_f = (I/\sim, T, \Vdash')$, where $\sim$ is the relation defined by $x \sim y :\Leftrightarrow f(x) = f(y)$ and $t \Vdash' [x] :\Leftrightarrow t \Vdash_A y$ for some $y \sim x$.*

Then the image functor $\mathbf{im} : \boldsymbol{Asm}(\mathcal{T})/B \to \mathbf{Sub}(B)$ maps $f \colon A \to B$ to $[m \colon Q_f \rightarrowtail B]$, where $f = m \circ e$, $m([x]) = f(x)$ and $e \colon A \to Q_f$ is the canonical projection, mapping x to $e(x) := [x]$.

Proof.

- m is well-defined and mono: $[x] = [y] \Leftrightarrow m([x]) = f(x) = f(y) = m([y])$.

- m is tracked by the realizer t that tracks f: let $a \Vdash' [x]$. Then $a \Vdash y$ for some $y \sim x$ and hence $ta \Vdash_B f(y) = f(x) = m([x])$. ($e$ is of course tracked by the identity term **s k k**.)

- Let $f = m' \circ e'$ with $m' \colon Q' \rightarrowtail B$ be another factorization. Define $j \colon Q_f \to Q'$ by $j([x]) \coloneqq e'(x)$.

  - This is well-defined: if $[x] = [y]$, then $f(x) = f(y) \Leftrightarrow m'(e'(x)) = m'(e'(y))$ and since m' is mono, i.e. injective, we must have $e'(x) = e'(y)$, i.e. $j([x]) = j([y])$.
  - j is tracked by the realizer t' that tracks e': let $a \Vdash' [x]$, i.e. $a \Vdash y$ for some $y \sim x$. Again, since m' is mono and $f(y) = f(x)$, we must have $e'(x) = e'(y)$. Thus $t'\ a \Vdash_{Q'} e'(y) = e'(x) = j([x])$.
  - $m'(j([x])) = m'(e'(x)) = f(x) = m([x])$, so $m' \circ j = m$.

□

Lemma 3.16. *Let $f \colon A \to B$ be a morphism in $\boldsymbol{Asm}(\mathcal{T})$ and $[y \colon C \rightarrowtail A]$ a subobject. Then the universal quantifier $\forall_f y \colon \forall_f C \to B$ is the identity function from $\forall_f C$ to B, traced by* **fst***, where*

$$I_{\forall_f C} = \{b \in B | \forall a \in f^{-1}(b) \exists k \in C\big(y(k) = a\big) \wedge \exists t \in |T_B \times (T_A \longrightarrow T_C)| \, \big(t \Vdash b\big)\}$$

and

$$\forall_f C = (I_{\forall_f C}, T_B \times (T_A \longrightarrow T_C), \Vdash)$$

with $\Vdash$ defined by

$$\langle s, t\rangle \Vdash b :\Leftrightarrow s \Vdash_B b \wedge \forall a \in A \forall t_a \Vdash_A a \forall k \in C\big(f(a) = b \wedge y(k) = a \Rightarrow t\ t_a \Vdash_C k\big)$$

Proof. Note first that every $b \in I_{\forall_f C}$ is realized by the definition of $I_{\forall_f C}$. So $\forall_f C$ is indeed an object in $\mathbf{Asm}(\mathcal{T})$. Let $x \colon D \rightarrowtail B$. We show $x \leq \forall_f y \Leftrightarrow f^* x \leq y$.

($\Rightarrow$) Assume $x \leq \forall_f y$, i.e. there is a $g \colon D \to \forall_f C$ such that $x = \mathrm{id} \circ g = g$. Hence we have

$$(\forall d \in D)(\forall a \in A)\big(f(a) = x(d) \Rightarrow (\exists k \in C).y(k) = a\big)$$

and since $f^* D = \{\langle a, d\rangle \in A \times D | f(a) = x(d)\}$, there is for every $\langle a, d\rangle \in f^* D$ a unique (since y mono) $k \in C$ such that $y(k) = a$, i.e. $y(k) = f^* x(\langle a, d\rangle)$. So define $h \colon f^* D \to C$ to map $\langle a, d\rangle$ to this unique $k \in C$.

Let t trace g. Then $t' = \lambda^* x.(\mathbf{snd}\ (t\ (\mathbf{snd}\ x)))(\mathbf{fst}\ x)$ traces h: let $\langle t_a, t_d\rangle \Vdash \langle a, d\rangle \in f^* D$. $t' \langle t_a, t_d\rangle = (\mathbf{snd}\ (t\ t_d))\ t_a \Vdash_C h(\langle a, d\rangle)$ since $y(h(\langle a, d\rangle)) = a$ and $f(a) = x(d)$.

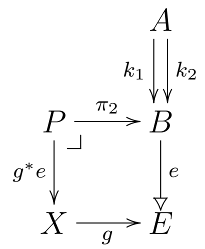
($\Leftarrow$) Suppose $f^*x \leq y$, i.e. $f^*x = y \circ g$. Then $x(d) \in \forall_f C$ for all $d \in D$: $x(d) \in B$, and if $a \in A$ with $f(a) = x(d)$, then $\langle a, d \rangle \in f^*D \Rightarrow y(g(\langle a, d \rangle)) = f^*x(\langle a, d \rangle) = a$. Hence $k = g(\langle a, d \rangle)$ works.

Let s trace g, and consider the λ-abstraction $\hat{s} \in |T_D \longrightarrow (T_A \longrightarrow T_C)|$ of s, i.e. $(\hat{s}\ d)\ a = s\ \langle a, d \rangle$. Let t_d be a realizer for an arbitrary $d \in D$, $a \in A$ and $t_a \Vdash_A a$. Further suppose $k \in C$ such that $y(k) = a$ and $f(a) = x(d)$. Then $(\hat{s}\ t_d)\ t_a = s\ \langle t_a, t_d \rangle \Vdash_C g(\langle a, d \rangle) = k$ since y mono, and $y(g(\langle a, d \rangle)) = a$. Thus $\lambda^* z.\mathbf{pair}\ (u\ z)(\hat{s}\ z)$ traces $x' \colon D \to \forall_f C$, $x'(d) \coloneqq x(d)$, where u traces x. In particular, every $x(d)$ is realized by $\langle u\ t_d, \hat{s}\ t_d \rangle$ where $t_d \Vdash_D d$, which completes the proof that $x(d) \in \forall_f C$ for all $d \in D$. □

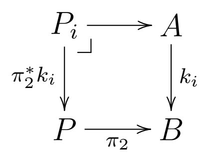
Lemma 3.17. ***Asm***$(\mathcal{T})$ *is regular and locally cartesian closed.*

Proof. Local cartesian closedness is well-known. For a proof, see e.g. [Bau00, 4.1.5].

Asm$(\mathcal{T})$ has binary products, a terminal object, equalizers and images by the Lemmas 3.11, 3.12, 3.14 and 3.15. Hence, for regularity, it remains to show that regular epis are stable under pullback. For this, let $e : B \to E$ be the coequalizer of $k_1, k_2 : A \rightrightarrows B$. Construct the pullback of e along an arbitrary morphism $g : X \to E$:



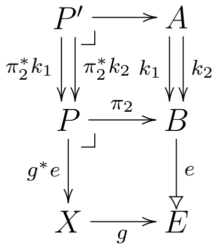
Now construct the pullback of k_1 and k_2 along π_2:



We know that

$$\begin{aligned} P_i =& \{((x, y), z) \in P \times A : \pi_2(x, y) = k_i(z)\} \\ =& \{((x, y), z) \in (X \times B) \times A : g(x) = e(y) \wedge y = k_i(z)\} \\ =& \{((x, k_i(z)), z) \in (X \times B) \times A : g(x) = e(k_i(z))\}. \end{aligned}$$

Since e is the coequalizer k_1 and k_2, we have $e(k_1(z)) = e(k_2(z))$ for all $z \in A$, so that $P_1 = P_2$. Write $P' = P_1 = P_2$. We then have the following diagram:



But now, $g^*e(\pi_2^*k_1(x,y,z)) = g^*e(x,y) = x = g^*e(\pi_2^*k_2(x,y,z))$, so that g^*e is the coequalizer of $\pi_2^*k_1, \pi_2^*k_2 : P' \rightrightarrows P$, hence regular epi. $\square$

Now we are ready to look at our interpretation from Section 2.4.2 in **Asm**(ωALat):

1. A sort S is interpreted as an object $(S, T_S, \Vdash_S)$.

2. A typing context $x_1 : A_1, \ldots, x_n : A : n$ is interpreted as the product $[\![A_1]\!] \times \ldots \times [\![A_n]\!]$, with the empty context interpreted as the terminal object $(\{*\}, \{*\}, \Vdash_*)$.

3. A function symbol f with signature $(A_1, \ldots, A_m; B)$ is interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \ldots \times [\![A_m]\!] \to [\![B]\!]$, i.e. a map

$$[\![f]\!] \colon I_{A_1} \times \ldots \times I_{A_m} \to I_B$$

which is tracked by some Scott-continuous $a \in |T_{A_1} \times \ldots \times T_{A_m} \longrightarrow T_B|$.

4. A term in context $\Gamma|t : B$ is interpreted as a morphism $[\![\Gamma|t : B]\!] : [\![\Gamma]\!] \to [\![B]\!]$ in the standard way: a variable is interpreted as a projection function (which is Scott-continuous) and a composite term as a composition.

5. A relation symbol R with signature $(A_1, \ldots, A_m)$ is interpreted as a monomorphism $[\![R]\!] \rightarrowtail [\![A_1]\!] \times \ldots \times [\![A_m]\!]$, i.e. an tracked injective map $I_R \to I_{A_1} \times \ldots \times I_{A_m}$.

6. A logical entailment $\Gamma|\Psi \vdash \phi$ is interpreted as an inequality $[\![\Psi]\!] \leq [\![\phi]\!]$ in **Sub**($[\![\Gamma]\!]$) as usual.

7. A formula in context $\Gamma|\varphi$ is interpreted as follows:

   (a) $[\![\Gamma|\top]\!]$ is the identity function $[\mathrm{id}_{[\![\Gamma]\!]} : [\![\Gamma]\!] \rightarrowtail [\![\Gamma]\!]]$, tracked by **s s k**.

(b) $\Gamma | t =_A u$ is interpreted as the equalizer $[\mathrm{id} : E \rightarrowtail [\![A]\!]]$, where $E = (\{x \in I_A : t(x) = u(x)\}, T_A, \Vdash_A)$, tracked by $\mathbf{s}\,\mathbf{s}\,\mathbf{k}$.

(c) An atomic formula $\Gamma | R(t_1, \ldots, t_k)$ is interpreted as the pullback of $[r : [\![R]\!] \rightarrowtail [\![A_1]\!] \times \ldots \times [\![A_k]\!]]$ along $[t : [\![\Gamma]\!] \rightarrowtail [\![A_1]\!] \times \ldots \times [\![A_k]\!]]$, i.e. the projection

$$[p : (\{(\overrightarrow{x}, \overrightarrow{y}) \in [\![R]\!] \times [\![\Gamma]\!] : r(\overrightarrow{x}) = t(\overrightarrow{y})\}, T_R \times T_\Gamma, \Vdash) \rightarrowtail [\![\Gamma]\!]].$$

(d) If $[p : [\![\Gamma]\!] \rightarrowtail A]$ represent $\Gamma | \varphi_1$ and $[q : [\![\Gamma]\!] \rightarrowtail A]$ represent $\Gamma | \psi$, then the pullback of p along q, composed with p, represent $\Gamma | \varphi \wedge \psi$.

(e) Assuming we have interpreted $\Gamma, x : X | \varphi(x)$ as $[\alpha : A \rightarrowtail [\![\Gamma]\!] \times [\![X]\!]]$, we have $[\![\Gamma | \exists x \in X. \varphi(x)]\!] = \mathbf{im}(p \circ \alpha)$ where p is the projection $p : [\![\Gamma]\!] \times [\![X]\!] \to [\![\Gamma]\!]$, with $\mathbf{im}$ as in Lemma 3.15.

(f) Assuming we have interpreted $\Gamma, x : X | \varphi(x)$ as $[\alpha : A \rightarrowtail [\![\Gamma]\!] \times [\![X]\!]]$, we have $[\![\Gamma | \forall x \in X. \varphi(x)]\!] = [\mathrm{id} : (I, T, \Vdash) \rightarrowtail [\![\Gamma]\!]$ where p is the projection $p : [\![\Gamma]\!] \times [\![X]\!] \to [\![\Gamma]\!]$, and

$$I = \{b \in [\![\Gamma]\!] | \forall a \in p^{-1}(b) \exists k \in I\big(\alpha(k) = a\big) \wedge \exists t \in |T| \,\big(t \Vdash b\big)\}$$

with $T = T_\Gamma \times (T_{\Gamma \times X} \longrightarrow T_A)$ and $\langle s, t \rangle \Vdash b$ if

$$s \Vdash_{\Gamma \times X} b \wedge \forall a \in A \forall t_a \Vdash_A a \forall k \in A\big(f(a) = b \wedge y(k) = a \Rightarrow t\, t_a \Vdash_A k\big).$$

(g) The interpretation of $B \to C$, where $[\![B]\!]$ and $[\![C]\!]$ are interpreted as subobjects $[\beta : B \rightarrowtail X]$ and $[\gamma : C \rightarrowtail X]$ respectively, is $[\mathrm{id} : (I, T_X \times (T_B \longrightarrow T_B \times T_C), \Vdash) \rightarrowtail X]$ where

$$I = \{x \in X : \forall b \in B. \beta(b) = x \Rightarrow \exists c \in C. \gamma(c) = \beta(b) = x\}$$

and $\langle t_\to, t_x \rangle \Vdash x$ if

$$t_x \Vdash_X x \wedge \forall b \in B\, \forall t_b \vdash_B b\, \forall c \in C\big(\beta(b) = \gamma(c) \Rightarrow t_\to\, t_b \Vdash_{B \times X} \langle b, c \rangle\big).$$

3.2.3 Separating models showing the invalidity of BD in BISH

We will now show that **BD** cannot hold in $\mathbf{Asm}(\omega\mathrm{ALat})$. For this, we will consider some continuity principles. For metric spaces (X, d) and (X', d'),

($\mathbf{CP}(X, X')$) "Every function from X to X' is ϵ-δ-continuous"

state that all functions between them are continuous.

Recall that for $\alpha, \beta : \mathbb{N} \to \mathbb{N}$, "$\beta \in \bar{\alpha}(m)$" means that $\beta(n) = \alpha(n)$ for all $n \leq m$. The weak principle of continuity

$$
\begin{aligned}
(\mathbf{WC}-\mathbf{N}) \quad & \big(\forall\alpha : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}. A(\alpha, n)\big) \\
& \to \big(\forall\alpha : \mathbb{N} \to \mathbb{N}\ \exists m, n \in \mathbb{N}\ \forall\beta : \mathbb{N} \to \mathbb{N}. \beta \in \bar{\alpha}(m) \to A(\beta, n)\big)
\end{aligned}
$$

is a combination of choice and continuity of every operation from $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}$. We will also consider a weakening of $\mathbf{WC}-\mathbf{N}$. Write $2 := \{0, 1\}$. We then have the continuity principle

$$
\begin{aligned}
(\mathbf{WC_{cp}}-\mathbf{N}) \quad & \big(\forall\alpha : \mathbb{N} \to 2\ \exists n \in \mathbb{N}. A(\alpha, n)\big) \\
& \to \big(\forall\alpha : \mathbb{N} \to 2\ \exists m, n \in \mathbb{N}\ \forall\beta : \mathbb{N} \to 2. \beta \in \bar{\alpha}(m) \to A(\beta, n)\big)
\end{aligned}
$$

stating that every operation from the full binary fan $\mathbb{N} \to \{0, 1\}$ to $\mathbb{N}$ is continuous.

Lemma 3.18. $\mathbf{WC}-\mathbf{N} \Rightarrow \mathbf{WC_{cp}}-\mathbf{N}$.

Proof. Consider the spreads T of all (codes for) finite sequences of natural numbers and T' of all finite binary sequences. If $\langle x_1, \ldots, x_n\rangle \in T$, then $\langle x_1, \ldots, x_m\rangle \in T'$ for some maximal $m \leq n$ since $T' \subseteq T$ are trees (m might be 0, so that $\langle x_1, \ldots, x_m\rangle = \langle\rangle$, the empty sequence). Since T' is a spread, there are $y_{m+1}, \ldots, y_n$ such that $\langle x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\rangle \in T'$. Define a mapping $\Gamma : T \to T'$ by $\Gamma(\langle x_1, \ldots, x_n\rangle) = \langle x_1, \ldots, x_m, y_{m+1}, \ldots, y_n\rangle$. Then $\|\Gamma(x)\| = \|x\|$ for all $x \in T$, and $\Gamma(x') = x'$ for all $x' \in T'$. Extend Γ to infinite sequences by $\Gamma(\alpha) = \lambda n. \left(\Gamma(\bar{\alpha}(n + 1))\right)_n$.

Assume now that $\mathbf{WC}-\mathbf{N}$ and the premise to $\mathbf{WC_{cp}}-\mathbf{N}$ holds for some predicate $A(\alpha, n)$. With $A'(\alpha, n) \equiv A(\Gamma(\alpha), n)$, we satisfy the premise to $\mathbf{WC}-\mathbf{N}$ since $\Gamma(\alpha) : \mathbb{N} \to 2$ for all α and $A(\beta, n)$ holds for all $\beta : \mathbb{N} \to 2$ by assumption. But $\Gamma(\alpha) = \alpha$ for all $\alpha : \mathbb{N} \to 2$, so the conclusion in $\mathbf{WC}-\mathbf{N}$ for $A'(\alpha, n)$ is actually the conclusion in $\mathbf{WC_{cp}}-\mathbf{N}$ for $A(\alpha, n)$. □

We will study $\mathbf{CP}(X, X')$ mainly in the special case when $X' = \mathbb{N}$ and $X = \mathbb{N} \to \mathbb{N}$ or $X = \mathbb{N} \to 2$. For this to make sense, we must give $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$ and $\mathbb{N} \to 2$ the structure of metric spaces. Equip $\mathbb{N}$ and 2 with the discrete metric

$$
d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}.
$$

Recall that the product topology of a cartesian product $\prod_{i \in I} X_i$ is the topology generated by basic open sets $\prod U_k$ where U_k is open in X_k, and

$U_k \neq X_k$ for only finitely many $k \in I$ (this coincides with the box topology for finite products). We will equip $\mathbb{N} \to \mathbb{N} = \prod_{i\in\mathbb{N}} \mathbb{N}$ and $\mathbb{N} \to 2 = \prod_{i\in\mathbb{N}} 2$ with the product topology, and see that this indeed makes $\mathbb{N} \to \mathbb{N}$ and $\mathbb{N} \to 2$ into metric spaces.

Lemma 3.19. *Let $(X_k, d_k)_{i\in\mathbb{N}}$ be a countable collection of metric spaces. Then the product topology on $X = \prod_{i\in\mathbb{N}} X_k$ is induced by the metric*

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k,y_k)}{1+d_k(x_k,y_k)}$$

Proof. We first show that d indeed is a metric on X. Positive definiteness and symmetry follows directly from the positive definiteness and symmetry of each d_k. For the triangle inequality, let $x, y, z \in X$. In general, we have for $a, b \geq 0$

$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}, \tag{1}$$

and if $a < b$ then $a/(1+a) < b/(1+b)$ as $x \mapsto \frac{x}{1+x}$ is strictly increasing (the derivative is $1/(1+x)^2 > 0$ for all $x \neq -1$). Thus, for each $k \in \mathbb{N}$, we have

$$\frac{d_k(x_k,z_k)}{1+d_k(x_k,z_k)} \leq \frac{d_k(x_k,y_k)+d_k(y_k,z_k)}{1+d_k(x_k,y_k)+d_k(y_k,z_k)}$$

by the triangle inequality for d_k, and furthermore by (1)

$$\frac{d_k(x_k,y_k)+d_k(y_k,z_k)}{1+d_k(x_k,y_k)+d_k(y_k,z_k)} \leq \frac{d_k(x_k,y_k)}{1+d_k(x_k,y_k)} + \frac{d_k(y_k,z_k)}{1+d_k(y_k,z_k)}.$$

Combining this, we get

$$\frac{d_k(x_k,z_k)}{1+d_k(x_k,z_k)} \leq \frac{d_k(x_k,y_k)}{1+d_k(x_k,y_k)} + \frac{d_k(y_k,z_k)}{1+d_k(y_k,z_k)}.$$

Thus each term of the sum is a metric on X_k, which implies that the whole sum is a metric on X.

Let $\mathcal{T}$ denote the product topology on X and $\mathcal{T}_d$ the topology induced by d. We show $\mathcal{T}_d \subseteq \mathcal{T}$ and $\mathcal{T}_d \supseteq \mathcal{T}$.

For $\mathcal{T}_d \subseteq \mathcal{T}$, consider the basic open set $B = \{x \in X : d(x,z) < \epsilon\} \in \mathcal{T}_d$ where $z \in X$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{-N} < \frac{\epsilon}{2}$. Define

$$U_k = \begin{cases} \{y_k \in X_k : d(y_k,z_k) < \frac{\epsilon}{2N}\} & \text{if } k \leq N \\ X_k & \text{if } k > N \end{cases}.$$

Then $U = \prod_{k=1}^{\infty} U_k$ is a basic open set in $\mathcal{T}$. We have to show that $U \subseteq B$. Suppose that $y \in U$. Then

$$\begin{aligned} d(y,z) &= \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(y_k,z_k)}{1+d_k(y_k,z_k)} \\ &= \sum_{k=1}^{N} 2^{-k} \frac{d_k(y_k,z_k)}{1+d_k(y_k,z_k)} + \sum_{k=N+1}^{\infty} 2^{-k} \frac{d_k(y_k,z_k)}{1+d_k(y_k,z_k)} \\ &\leq \sum_{k=1}^{N} d_k(y_k,z_k) + \sum_{k=N+1}^{\infty} 2^{-k} < \sum_{k=1}^{N} \frac{\epsilon}{2N} + 2^{-N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $y \in B$ and $U \subseteq B$.

For $\mathcal{T}_d \supseteq \mathcal{T}$, let $U = \prod_{k=1}^{\infty} U_k$ be a basic open set in $\mathcal{T}$, and $z \in U$. Then there is a finite set $I \subseteq \mathbb{N}$ such that $U_k = X_k$ for all $k \in \mathbb{N} \setminus I$. For every $k \in I$, let $B_k = \{x_k \in X_k : d_k(x_k, z_k) < \epsilon_k\} \subseteq U_k$ be an open subset of U_k, centered at z_k with some radius ϵ_k (these exist by the definition of open sets in X_k). Let $\epsilon \coloneqq \min\{2^{-k}\frac{\epsilon_k}{1+\epsilon_k} : k \in I\}$. This is well-defined since I is finite.

Now consider $B \coloneqq \{x \in X : d(x,z) < \epsilon\}$. Suppose that $x \in B$. We want to show that $x \in U$. For all $k \in \mathbb{N}$, we have

$$2^{-k} \frac{d_k(x_k,z_k)}{1+d_k(x_k,z_k)} \leq \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k,z_k)}{1+d_k(x_k,z_k)} = d(x,z) < \epsilon,$$

or equivalently (since the inverse of $x \mapsto \frac{x}{1+x}$ is $x \mapsto \frac{x}{1-x}$)

$$d_k(x_k,z_k) < \frac{2^k \epsilon}{1-2^k\epsilon} \leq \frac{2^k 2^{-k} \frac{\epsilon_k}{1+\epsilon_k}}{1 - 2^k 2^{-k} \frac{\epsilon_k}{1+\epsilon_k}} = \epsilon_k.$$

Thus for each $k \in \mathbb{N}$, $x_k \in B_k \subseteq U_k$, so $x \in U$ and $B \subseteq U$. □

So what does it mean for a function $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ to be ϵ-δ-continuous, when $\mathbb{N}$ is given the discrete metric and $\mathbb{N} \to \mathbb{N}$ the product topology? It turns out that f only uses a finite amount of information:

Lemma 3.20. *A function $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is continuous at $\alpha : \mathbb{N} \to \mathbb{N}$ $\iff \exists m \in \mathbb{N} \forall \beta : \mathbb{N} \to \mathbb{N}. \beta \in \bar{\alpha}(m) \Rightarrow f(\alpha) = f(\beta)$.*

Proof. ($\Rightarrow$) f is continuous at α if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\beta : \mathbb{N} \to \mathbb{N}$, if $d(\alpha, \beta) < \delta$ then $d_{\mathbb{N}}(f(\alpha), f(\beta)) < \epsilon$. But now, $\mathbb{N}$ is equipped with the discrete metric, so $d_{\mathbb{N}}(f(\alpha), f(\beta)) < \epsilon$ is the same as

$f(\alpha) = f(\beta)$ for $\epsilon < 1$, i.e. we can find $\delta > 0$ such that if $d(\alpha, \beta) < \delta$, then $f(\alpha) = f(\beta)$. Now choose N such that $\sum_{k=N}^{\infty} 2^{-k} < \delta$. If $\beta \in \bar{\alpha}(N)$, we have

$$\begin{aligned} d(\alpha, \beta) &= \sum_{k=1}^{\infty} 2^{-k} \frac{d_{\mathbb{N}}(\alpha(k), \beta(k))}{1 + d_{\mathbb{N}}(\alpha(k), \beta(k))} \\ &= \sum_{k=1}^{N-1} 2^{-k} \frac{d_{\mathbb{N}}(\alpha(k), \beta(k))}{1 + d_{\mathbb{N}}(\alpha(k), \beta(k))} + \sum_{k=N}^{\infty} 2^{-k} \frac{d_{\mathbb{N}}(\alpha(k), \beta(k))}{1 + d_{\mathbb{N}}(\alpha(k), \beta(k))} \\ &= \quad 0 + \sum_{k=N}^{\infty} 2^{-k} \frac{d_{\mathbb{N}}(\alpha(k), \beta(k))}{1 + d_{\mathbb{N}}(\alpha(k), \beta(k))} \leq \sum_{k=N}^{\infty} 2^{-k} < \delta, \end{aligned}$$

so $f(\alpha) = f(\beta)$.

($\Leftarrow$) Suppose m is such that $f(\beta) = f(\alpha)$ if $\beta \in \bar{\alpha}(m)$. Choose $\delta = 2^{-m}$. Then, if $d(\alpha, \beta) < \delta$, we also have

$$2^{-k} \frac{d_{\mathbb{N}}(\alpha(k), \beta(k))}{1 + d_{\mathbb{N}}(\alpha(k), \beta(k))} < d(\alpha, \beta) < \delta = 2^{-m}$$

for all $k \leq m$, i.e. $d_{\mathbb{N}}(\alpha(k), \beta(k)) = 0$ for all $k \leq m$, so $\beta \in \bar{\alpha}(m)$ and $f(\alpha) = f(\beta)$ by assumption. Hence f is continuous at α. $\square$

We can now give an easy proof of the following relationship between $\mathbf{WC} - \mathbf{N}$ and $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$. With a little more work, one can actually show that $\mathbf{WC} - \mathbf{N}$ implies $\mathbf{CP}(X, X')$ for all complete separable metric spaces X and X' (see [TvD88b, 7.2.7]).

Lemma 3.21.

(i) $\mathbf{WC} - \mathbf{N} \Rightarrow \mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$

(ii) $\mathbf{WC_{cp}} - \mathbf{N} \Rightarrow \mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$

Proof. We prove (i). (ii) can be proven in exactly the same way.

Let $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$. From Lemma 3.20, we know it is enough to prove

$$\forall \alpha : \mathbb{N} \to \mathbb{N}\ \exists m \in \mathbb{N}\ \forall \beta : \mathbb{N} \to \mathbb{N}. \beta \in \bar{\alpha}(m) \Rightarrow f(\alpha) = f(\beta).$$

Now define $A(\alpha, n) :\Leftrightarrow f(\alpha) = n$. Since f is a total function, we have $\forall \alpha : \mathbb{N} \to \mathbb{N}\ \exists n. f(\alpha) = n$, so $\mathbf{WC} - \mathbf{N}$ gives

$$\forall \alpha : \mathbb{N} \to \mathbb{N}\ \exists m, n \in \mathbb{N}\ \forall \beta : \mathbb{N} \to \mathbb{N}. \beta \in \bar{\alpha}(m) \Rightarrow f(\beta) = n,$$

and since $\alpha \in \bar{\alpha}(m)$, in particular we have $f(\alpha) = n$ so that we get

$$\forall \alpha : \mathbb{N} \to \mathbb{N}\ \exists m \in \mathbb{N}\ \forall \beta : \mathbb{N} \to \mathbb{N}. \beta \in \bar{\alpha}(m) \Rightarrow f(\beta) = f(\alpha),$$

i.e. f is continuous. $\square$

Assuming the axiom of choice, we can also reverse the implication:

Lemma 3.22.

(i) $\mathbf{CP}(\mathbb{N} \to \mathbb{N},\ \mathbb{N})$ *and* $\mathbf{AC}(\mathbb{N} \to \mathbb{N},\ \mathbb{N}) \Rightarrow \mathbf{WC} - \mathbf{N}$

(ii) $\mathbf{CP}(\mathbb{N} \to 2,\ \mathbb{N})$ *and* $\mathbf{AC}(\mathbb{N} \to 2,\ \mathbb{N}) \Rightarrow \mathbf{WC_{cp}} - \mathbf{N}$

Proof. Once again, we only prove (i) as (ii) can be proven in exactly the same way.

Assume the premise to $\mathbf{WC} - \mathbf{N}$, i.e. for all $\alpha : \mathbb{N} \to \mathbb{N}$ there is a $n \in \mathbb{N}$ such that $A(\alpha, n)$ holds. By $\mathbf{AC}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$, we then have a choice function $\Phi : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ such that $A(\alpha, \Phi(\alpha))$ holds for all $\alpha : \mathbb{N} \to \mathbb{N}$.

Let $\alpha : \mathbb{N} \to \mathbb{N}$, and choose $n := \Phi(\alpha)$. Let m be given from $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ for Φ. Let $\beta : \mathbb{N} \to \mathbb{N}$ be such that $\beta \in \bar{\alpha}(m)$. We have to show that $A(\beta, n)$ holds. But $n = \Phi(\alpha)$, and since $\beta \in \bar{\alpha}(m)$, $\Phi(\alpha) = \Phi(\beta)$ so that $n = \Phi(\beta)$. Hence $A(\beta, n)$ holds, since we have $A(\beta, \Phi(\beta))$ by $\mathbf{AC}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$. This proves the lemma. □

As we of course have the implication $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N}) \Rightarrow \mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$ (every function $f : \mathbb{N} \to 2$ is also a function $f : \mathbb{N} \to \mathbb{N}$), we have the following situation:

$$\begin{array}{ccc}
\mathbf{WC} - \mathbf{N} & \Longrightarrow & \mathbf{WC_{cp}} - \mathbf{N} \\
\Downarrow & & \Downarrow \\
\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N}) & \Longrightarrow & \mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})
\end{array}$$

We will now show that the implications from left to right cannot be reversed in ωALat, but that **BD** would do exactly that. Hence **BD** can't hold in ωALat.

Lemma 3.23. *In **Asm**(ωALat), the axiom of choice holds for all finite types over* $\mathbb{N}$.

Proof. We interpret the axiom of choice

$$(\forall z \in U)\big[(\forall x \in X)(\exists y \in Y)R(x, y, z) \to (\exists f : X \to Y)(\forall x \in X)R(x, f(x), z)\big]$$

in $\mathbf{Asm}(\omega\mathrm{ALat})$, but restrict ourselves to the no parameter case $U = \{*\}$ for simplicity. The general result still holds, of course, as can for example be seen from the more abstract proof in [Lie05, 2.3.3(i)].

The natural number object in $\mathbf{Asm}(\omega\mathrm{ALat})$ is $(\mathbb{N}, \mathbb{N}_\perp, \Vdash_\mathbb{N})$ where $n \Vdash_\mathbb{N} m$ if and only if $n = m$. For finite types σ and τ, the interpretation in

$\mathbf{Asm}(\omega\text{ALat})$ of the type of functions between them becomes the object $(\sigma \to \tau, A_\sigma \longrightarrow A\tau, \Vdash_{\sigma\to\tau})$ where A_σ and A_τ are the lattices of realizers for σ and τ respectively and $f \Vdash_{\sigma\to\tau} g$ if and only if $f = g$. Thus, for every finite type over $\mathbb{N}$, we have that every element of it has exactly one realizer, namely itself.

Let two finite types σ and τ be given. The interpretation of the premise of $\mathbf{AC}(\sigma, \tau)$ becomes

$$A = (\{* \in 1 \colon \forall x \in \sigma\ \exists y \in \tau\ R(x,y)\}, T_1 \times (T_\sigma \longrightarrow T_A \times T_B), \Vdash)$$

where $t \Vdash *$ if and only if $\mathbf{fst}\ t \Vdash_1 *$ and for all $x \in \sigma$ and $t_x \Vdash_\sigma x$ (i.e. just $t_x = x$ in this case), there is $y \in \tau$ such that $(\mathbf{snd}\ t)\ t_x \Vdash_{\sigma\times\tau} (x,y)$.

Suppose $t \Vdash *$. We can then define a choice function $f : \sigma \to \tau$ by $f(x) := \mathbf{snd}\ ((\mathbf{snd}\ t)\ x)$, traced by itself. The interpretation of the conclusion of $\mathbf{AC}(\sigma,\tau)$ is

$$B = (\{[f] \colon \forall\, x \in \sigma R(x, f(x))\}, T_{\sigma\longrightarrow\tau} \times (T_{(\sigma\longrightarrow\tau)\times\sigma} \longrightarrow (T_\sigma \times T_\sigma) \times (T_{\sigma\longrightarrow\tau} \times T_\sigma)), \Vdash')$$

where $t \Vdash' [f] \Leftrightarrow$ there is $g : \sigma \to \tau$ such that $R(x, g(x))$ holds for all $x \in \sigma$, $\mathbf{fst}\ t \Vdash_{\sigma\longrightarrow\tau}$ and for all $x \in \sigma$ and all $t_{g,x} \Vdash_{(\sigma\longrightarrow\tau)\times\sigma}$,

$$\mathbf{snd}\ t\ t_{g,x} \Vdash_{\sigma\times\sigma\times(\sigma\longrightarrow\tau)\times\sigma} (a, g(a), g, a).$$

f constructed from the interpretation of the premise is such a $g : \sigma \to \tau$, and we can map a realizer t of the premise to a realizer

$$\mathbf{pair}\ (\lambda^* x.\, \mathbf{snd}(t\, x))\ (\lambda^* x.\, \mathbf{pair}\ (\mathbf{pair}\ (\mathbf{snd}\, x)((\mathbf{fst}\, x)\cdot(\mathbf{snd}\, x)))(\mathbf{pair}\ (\mathbf{fst}\, x)\ (\mathbf{snd}\, x)))$$

of the conclusion, so the morphism from A to B, sending $*$ to $[f]$ is indeed traced and $\mathbf{AC}(\sigma,\tau)$ holds in $\mathbf{Asm}(\omega\text{ALat})$. $\square$

Lemma 3.24. *In* $\mathbf{Asm}(\omega\text{ALat})$, $\mathbf{WC_{cp}} - \mathbf{N}$ *holds, but* $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ *does not.*

Proof. Since $\mathbf{AC}(\mathbb{N} \to 2, \mathbb{N})$ holds by the previous lemma, it is enough to show that $\mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$ holds by Lemma 3.22. To establish $\mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$, it is by Lemma 3.20 enough to see that the following sentence holds in ωALat:

$$\forall\ f : (\mathbb{N} \to 2) \to \mathbb{N} \exists\ m \in \mathbb{N} \forall \alpha, \beta : \mathbb{N} \to 2. \big(\beta \in \bar{\alpha}(m) \to f(\alpha) = f(\beta)\big).$$

Interpreting this, we see that it holds if there is a realizer t transforming realizers for f to realizers for a m such that m is a uniform modulus of continuity for f. But that is exactly what the fan functional is, and the fan functional is continuous, so $\mathbf{WC_{cp}} - \mathbf{N}$ holds in $\mathbf{Asm}(\omega\text{ALat})$.

For the second part, assume that $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ holds in $\mathbf{Asm}(\omega\text{ALat})$. Thus by Lemma 3.22, we have that $\mathbf{WC}-\mathbf{N}$ holds. We will construct a modulus of continuity at $\lambda x.0$ functional as in [TvD88b, 9.6.10(i)] and show that this leads to a contradiction.

Since every $f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is a total function, we of course have

$$\forall\, f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}\ \forall\, \alpha : \mathbb{N} \to \mathbb{N}\ \exists x \in \mathbb{N}\big(f(\alpha) = x\big),$$

so by $\mathbf{WC}-\mathbf{N}$

$$\forall\, f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}\ \forall\, \alpha : \mathbb{N} \to \mathbb{N}\ \exists m \in \mathbb{N}\ \forall\, \beta \in \bar{\alpha}(m)\big(f(\alpha) = f(\beta)\big).$$

In particular, for $\alpha = \lambda x.0$ we get

$$\forall\, f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}\ \exists m \in \mathbb{N}\ \forall\, \beta \in \overline{\lambda x.0}(m)\big(f(\beta) = f(\lambda x.0)\big)$$

and with $\mathbf{AC}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ we find $\Phi : ((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \to \mathbb{N}$ such that

$$\forall\, f : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}\ \forall\, \beta \in \overline{\lambda x.0}(\Phi(f))\big(f(\beta) = f(\lambda x.0)\big).$$

Define $\Phi^* \coloneqq \lambda f.\ (\overline{\lambda x.0}(\Phi(f)))$ (i.e. we have $\beta \in \Phi^*(f) \Rightarrow f(\beta) = f(\lambda x.0)$), and let $n_0 \coloneqq \Phi^*(\lambda\alpha.0)$.

We construct a neighborhood function γ_β for each $\beta : \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned}
&\gamma_\beta(m) = 0 \quad \text{for } m \preceq n_0;\\
&\gamma_\beta(n_0 * \langle 0\rangle * z) = 1 \quad \text{(for z a code for an arbitrary sequence)};\\
&\gamma_\beta(n_0 * \langle x\rangle * z) = \begin{cases} 1 & \text{if } \beta(y) = 0 \text{ for all } 0 < y \le x\\ 2 & \text{otherwise;}\end{cases}\\
&\gamma_\beta(m) = 1 \quad \text{if } \neg(n_0 \preceq m \vee n_0 \succeq m).
\end{aligned}$$

Every γ_β is indeed a neighborhood function, so we have

$$\forall \alpha, \beta\ \exists x, y\ \big(\gamma_\beta(\overline{\alpha}(x)) = y + 1\big).$$

Since we have the axiom of choice for all finite types over $\mathbb{N}$, we might as well use it and we get a $\Psi : (\mathbb{N} \to \mathbb{N}) \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ so that

$$\forall \alpha, \beta\ \exists x\ \big(\gamma_\beta(\overline{\alpha}(x)) = \Psi(\alpha, \beta) + 1\big).$$

Define $\Psi_\beta \coloneqq \lambda\alpha.\Psi(\alpha, \beta)$.

If now $\beta(y) = 0$ for all $y > 0$, we have that $\gamma_\beta(\overline{\alpha}(x)) = 1$ for all α and all x large enough, which forces $\Psi(\alpha, \beta)$ to be 0 for all α, i.e. $\Psi_\beta = \lambda\alpha.0$.

Thus, since everything in our finite type hierarchy is extensional, we get $\Phi^*(\Psi_\beta) = \Phi^*(\lambda\alpha.0) = n_0$.

On the other hand, if $\beta(y) \neq 0$ for some $y > 0$, then $\gamma_\beta(n_0 * \langle y \rangle) = 2$ so that $\Psi(\alpha, \beta)$ must be 1 for $\alpha \in n_0 * \langle y \rangle$. At the same time, we have that $\Psi(\lambda x.0, \beta) = 0$ since $(\lambda x.0) \in n_0 * \langle 0 \rangle$ by the definition of n_0. Hence there are $\alpha \in n_0$ such that $\Psi_\beta(\alpha) \neq \Psi_\beta(\lambda x.0)$, and we conclude that $\Phi^*(\Psi_\beta) \neq n_0$.

Hence, even though $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ demands that every operation from $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}$ is continuous, we cannot compute the value of Ψ_β continuously in β at $\lambda x.0$ (if we could, we could determine whether $\beta(x) = 0$ for all x from an initial segment of β) and we have arrived at our contradiction. Thus $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ cannot hold in ωALat. □

To see that **BD** can't hold in **Asm**(ωALat), we will consider yet another metric space, this time a restriction of $\mathbb{N} \to 2$. Let

$$\mathbb{N}^\star := \{\alpha : \mathbb{N} \to 2 | \forall m \in \mathbb{N}\big(\alpha(m) = 0 \implies \forall n > m.\ \alpha(n) = 0\big)\},$$

i.e. $\mathbb{N}^\star$ consists of all non-increasing binary sequences. $\mathbb{N}^\star$ can be seen as the one-point compactification of $\mathbb{N}$ (see [BB85, 4.6.6]) if we identify $n \in \mathbb{N}$ with the sequence κ_n whose first n terms are 1, and all other 0. The constant one sequence $\kappa_\infty = \lambda m.\ 1$ then becomes a point at infinity.

As before, we of course have that $\mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$ implies $\mathbf{CP}(\mathbb{N}^\star, \mathbb{N})$. We will see that $\mathbb{N}^\star$ have some nice properties when it comes to sequential continuity. For metric spaces (X, d) and (X', d'), we consider the continuity principle

($\mathbf{CP_{seq}}(X, X')$) Every function from X to X' is sequentially continuous

stating that for every $f : X \to X'$, if $a_n \to a$ in X, then $f(a_n) \to f(a)$ in X' for every sequence $(a_n)_{n=1}^\infty$ in X. The following two lemmas are proved in [BS04] as propositions 4.3 and 4.4.

Lemma 3.25. $\mathbf{CP}(\mathbb{N}^\star, Y) \Leftrightarrow \mathbf{CP_{seq}}(\mathbb{N}^\star, Y)$ *for any metric space* Y.

Proof. ($\Rightarrow$) This of course holds in general: let $f : \mathbb{N}^\star \to Y$ be continuous and $a_n \to a$ in $\mathbb{N}^\star$. Let $\epsilon > 0$ be given. We shall find $m \in \mathbb{N}$ so that $d(f(a_n), f(a)) < \epsilon$ for $n > m$. But since f is continuous, there is $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ if $d(x, a) < \delta$. Furthermore, since $a_n \to a$, there is $m \in \mathbb{N}$ so that $d(a_n, a) < \delta$ for $n > m$, hence $d(f(a_n), f(a)) < \epsilon$ if $n > m$ and $f(a_n) \to f(a)$, so that f is sequentially continuous.

($\Leftarrow$) Assume that every $f : \mathbb{N}^\star \to Y$ is sequentially continuous. In particular, since $\kappa_n \to \kappa_\infty$, we then have that $f(\kappa_n) \to f(\kappa_\infty)$ for every $f : \mathbb{N}^\star \to Y$, i.e.

$$\forall f : \mathbb{N}^\star \to Y, \epsilon > 0\ \exists m \in \mathbb{N}\ \forall n \geq m.\ d(f(\kappa_n), f(\kappa_\infty)) < \epsilon. \quad (2)$$

Let $f : \mathbb{N}^\star \to Y$, $\alpha \in \mathbb{N}^\star$ and $\epsilon > 0$ be given. By (2), there is $m \in \mathbb{N}$ such that if $n \geq m$ then $d(f(\kappa_n), f(\kappa_\infty)) < \epsilon/4$. Let $\delta \coloneqq 2^{-(m+1)}$. We show that $d(f(\alpha), f(\alpha')) < \epsilon$ if $d(\alpha, \alpha') < \delta$, so that f is pointwise continuous.

Since there are only finitely many $i < m$, we either have $\alpha_i = 1$ for all $i < m$, or there is a least $i < m$ such that $\alpha_i = 0$. In the latter case, $\alpha = \kappa_i$, and the only $\alpha' \in \mathbb{N}^\star$ with $d(\alpha, \alpha') < \delta < 2^{-(i+1)}$ is $\alpha' = \alpha$, so that $d(\alpha, \alpha') < \delta \implies d(f(\alpha), f(\alpha')) = 0 < \epsilon$.

Otherwise, if $\alpha_i = 1$ for all $i < m$, define $g : \mathbb{N}^\star \to \mathbb{N}^\star$ by $g(\beta)_i = \alpha_i \cdot \beta_i$. Then $g(\kappa_\infty) = \alpha$ and for any $m' \geq m$, $g(\kappa_{m'}) = \kappa_n$ for a unique $m \leq n \leq m'$. If we apply (2) to $f \circ g$, we find $m' \in \mathbb{N}$ such that $d(f(g(\kappa_{n'})), f(\alpha)) = d(f(g(\kappa_{n'})), f(g(\kappa_\infty))) < \epsilon/4$ for all $n' \geq m'$. Now $g(\kappa_{\max(m,m')}) = \kappa_{n_0}$ for some $m \leq n_0 \leq \max(m, m')$ and $d(f(\kappa_{n_0}), f(\alpha)) = d(f(g(\kappa_{\max(m,m')})), f(\alpha)) < \epsilon/4$ since trivially $\max(m, m') \geq m'$. But we also have $d(f(\kappa_{n_0}), f(\kappa_\infty)) < \epsilon/4$ by (2) since $n_0 \geq m$. Combining this, we get

$$d(f(\alpha), f(\kappa_\infty)) \leq d(f(\alpha), f(\kappa_{n_0})) + d(f(\kappa_{n_0}), f(\kappa_\infty)) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Now, if $d(\alpha, \alpha') < \delta = 2^{-(m+1)}$, then $\alpha'_i = 1$ for all $i < m$ and we can repeat the argument in the previous paragraph (with $g'(\beta)_i = \alpha'_i \cdot \beta_i$) to get $d(f(\alpha'), f(\kappa_\infty)) < \epsilon/2$. Thus, for α' with $d(\alpha, \alpha') < \delta$, we have

$$d(f(\alpha), f(\alpha')) \leq d(f(\alpha), f(\kappa_\infty)) + d(f(\kappa_\infty), f(\alpha')) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so f is pointwise continuous. □

Lemma 3.26. $\mathbf{CP_{seq}}(\mathbb{N}^\star, \mathbb{N}) \Leftrightarrow \mathbf{CP_{seq}}(X, \mathbb{N})$ *for any complete separable metric space* X*.*

Proof. ($\Leftarrow$) Trivial, as $\mathbb{N}^\star$ is separable and complete.

($\Rightarrow$) Consider any $f : X \to \mathbb{N}$. Let $(x_n)_{n=1}^\infty$ be a sequence in X with $x_n \to x$. Define $g : \mathbb{N}^\star \to X$ by $g(\alpha) = \lim_{n\to\infty} h(\alpha, n)$, where

$$h(\alpha, n) = \begin{cases} x & \text{if } \alpha(k) = 1 \text{ for all } k \leq n \\ x_m & \text{if } \alpha(k) = 1 \text{ for } 0 \leq k \leq m \text{ and } \alpha(k) = 0 \text{ for } m+1 \leq k \leq n. \end{cases}$$

We then have $g(\kappa_n) = x_n$ for all $n \in \mathbb{N}$ and $g(\kappa_\infty) = x$. By assumption, $f \circ g$ is sequentially continuous, so for every $\epsilon > 0$, there is $m \in \mathbb{N}$ such that $d(f(x_n), f(x)) = d(f(g(\kappa_n)), f(g(\kappa_\infty))) < \epsilon$ for all $n \geq m$, so $f : X \to \mathbb{N}$ is sequentially continuous. □

Lemma 3.27. **BD** *and* $\mathbf{CP}(\mathbb{N} \to 2, \mathbb{N}) \Rightarrow \mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$.

Proof. We now have the following situation:

$$\begin{array}{ccccc}
\mathbf{CP}(\mathbb{N}\to\mathbb{N},\mathbb{N}) & \Longrightarrow & \mathbf{CP}(\mathbb{N}\to 2,\mathbb{N}) & \Longrightarrow & \mathbf{CP}(\mathbb{N}^\star,\mathbb{N}) \\
\textbf{BD}\Big\Updownarrow & & & & \Big\Updownarrow\text{Lemma 3.25} \\
\mathbf{CP_{seq}}(\mathbb{N}\to\mathbb{N},\mathbb{N}) & & \overset{\text{Lemma 3.26}}{\Longleftrightarrow} & & \mathbf{CP_{seq}}(\mathbb{N}^\star,\mathbb{N})
\end{array}$$

where the leftmost vertical equivalence holds because **BD** is equivalent to that every sequentially continuous function is pointwise continuous (see Theorem 5.12).

Thus, under the assumption of **BD**, we can follow the implications all the way from $\mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$ to $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$. □

Summing up everything, we finally get:

Theorem 3.28 (Peter Lietz [Lie05])**.** **BD** *does not hold in the category* ***Asm****(ωALat).*

Proof. Assume that **BD** holds in **Asm**(ωALat). By Lemma 3.24, $\mathbf{WC_{cp}} - \mathbf{N}$ holds in **Asm**(ωALat) and thus $\mathbf{CP}(\mathbb{N} \to 2, \mathbb{N})$. But then $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ holds in **Asm**(ωALat) by Lemma 3.27, contradicting Lemma 3.24. □

3.3 Validity of BD in CLASS, INT, RUSS

Even though **BD** can't be proven constructively, it can be proven from the extra axioms introduced in Brouwer's intuitionism or under a recursive interpretation. Using classical logic, **BD** can of course easily be proved by using contraposition:

Theorem 3.29. **BD** *is valid in* ***CLASS****.*

Proof. Assume that A is not bounded. Then, for each $n \in N$, there is a $s_n \in A$ such that $s_n > n$ (if not, A would be bounded by n). But then (s_n) is a sequence in A which is not eventually bounded by n, so A is not pseudobounded by Corollary 3.4. □

The following two theorems are proved by Ishihara in [Ish92].

Theorem 3.30. **BD − N** *is valid in* ***INT***.

Proof. Since $\mathbf{AC}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ and $\mathbf{CP}(\mathbb{N} \to \mathbb{N}, \mathbb{N})$ are axioms of **INT**, **WC − N** holds in **INT** by Lemma 3.22.

Let $A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ be a countable pseudobounded subset of $\mathbb{N}$. For every $\alpha : \mathbb{N} \to \mathbb{N}$, $(a_{\alpha(n)})_{n=1}^{\infty}$ is then a sequence in A, so that $a_{\alpha(n)}n^{-1} \to 0$ by pseudoboundedness. Choosing $\epsilon = 1$, we thus find

$$\forall \alpha : \mathbb{N} \to \mathbb{N}\ \exists k \in \mathbb{N}\ \forall n \geq k\big(\left|a_{\alpha(n)}n^{-1}\right| < 1\big)$$

which becomes

$$\forall \alpha : \mathbb{N} \to \mathbb{N}\ \exists k, \ell \in \mathbb{N}\ \forall \beta : \mathbb{N} \to \mathbb{N}.\beta \in \overline{\alpha}(\ell) \to \forall n \geq k\big(\left|a_{\beta(n)}n^{-1}\right| < 1\big)$$

with an application of **WC − N**. Choose $\alpha = \lambda u.0$, the constant zero sequence, and let $m = \max(k, \ell)$, where k and ℓ are given for α. We then have

$$\forall \beta \in \overline{\lambda u.0}(\ell)\ \forall n \geq k\big(\left|a_{\beta(n)}n^{-1}\right| < 1\big).$$

For each $i \in \mathbb{N}$, construct $\beta : \mathbb{N} \to \mathbb{N}$ such that $\beta(n) = 0$ for $n < m$ and $\beta(n) = i$ for $n \geq m$. Then $\beta \in \overline{\lambda u.0}(\ell)$ (because $m \geq \ell$), and for $n = m \geq k$ we have $\left|a_{\beta(m)}m^{-1}\right| < 1$, i.e. $a_i < m$ (since $\beta(m) = i$) and thus A is bounded. □

Theorem 3.31. **BD − N** *is valid in* ***RUSS***.

Proof. Let $A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ be a countable pseudobounded subset of $\mathbb{N}$. By our recursive interpretation, there exists a partial function $\psi : \mathbb{N} \rightharpoonup \mathbb{N}$ such that if $(\varphi_s(n))_{n=1}^{\infty}$ is a sequence in A, then $\psi(s)$ is defined and $|\varphi_s(n)n^{-1}| < 1$ for all $n \geq \psi(s)$.

Define $A_n := \{a \in A : a \geq n\}$, and let $\theta : \mathbb{N} \rightharpoonup \mathbb{N}$ be the partial function $\theta(n) = a_{\min_i(a_i \geq n)}$, i.e. $\theta(n) \in A_n$ if A_n is inhabited, otherwise $\theta(n)$ is undefined. With the *s-m-n*-theorem (see [BR87, 3.1.7]), construct a total function $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_{\sigma(k)}(n) = \begin{cases} a_0 & \text{if } \neg\exists i \leq n\big(k \in D_k(i) \land \varphi_k(k) \leq i\big) \\ \theta\Big(\min\limits_{i \leq n}(k \in D_k(i) \land \varphi_k(k) \leq i)\Big) & \text{if } \exists i \leq n\big(k \in D_k(i) \land \varphi_k(k) \leq i\big). \end{cases}$$

(Recall that $D_k(i)$ intuitively is the set of inputs $x \in \mathbb{N}$ on which $\varphi_k(x)$ can be computed in i steps.)

Consider the partial function $k \mapsto \psi(\sigma(k))$. It has some index k_0, i.e. $\varphi_{k_0}(k) = \psi(\sigma(k))$. Now suppose $\varphi_{k_0}(k_0)$ is undefined, i.e. $k_0 \notin D_{k_0}(n)$ for all $n \in \mathbb{N}$. Then $\varphi_{\sigma(k_0)}(n) = a_0$ for all $n \in \mathbb{N}$, so $(\varphi_{\sigma(k_0)}(n))_{n=1}^{\infty}$ is a sequence in

A. Thus $\psi(\sigma(k_0))$ is defined. But $\varphi_{k_0}(k_0) = \psi(\sigma(k_0))$, so $\varphi_{k_0}(k_0)$ is defined, which contradicts that it was undefined. Hence $\varphi_{k_0}(k_0)$ is not undefined and so $\varphi_{k_0}(k_0)$ *is* defined by Markov's principle.

Let $\nu := \min_n(k_0 \in D_{k_0}(n) \wedge \varphi_{k_0}(k_0) \leq n)$. Suppose that A_ν is inhabited. Then $(\varphi_{\sigma(k_0)}(n))_{n=1}^{\infty}$ is a sequence in A (for $n < \nu$, $\varphi_{\sigma(k_0)}(n) = a_0 \in A$ since ν was minimal, and for $n \geq \nu$, $\varphi_{\sigma(k_0)}(n) = \theta(\nu) \in A_\nu \subseteq A$) and $\varphi_{k_0}(k_0) = \psi(\sigma(k_0)) \leq \nu$, so that $1 > \varphi_{k_0}(\nu)\nu^{-1} = \theta(\nu)\nu^{-1} \geq 1$ (since $\theta(\nu) \in A_\nu$), which is a contradiction. Hence A_ν is not inhabited and $a_i < \nu$ for all $a_i \in A$. $\square$

4 Constructive Aspects of Reduced Products

4.1 Palyutin's Theorem

4.1.1 Filters, filter bases and reduced products

We recall that a *filter* F on a set S is a subset of $\mathcal{P}(S)$ such that $S \in F$, $\emptyset \notin F$, for all $X \in F$, if $X \subseteq Y$ then $Y \in F$ and if $X, Y \in F$ then $X \cap Y \in F$. An *ultrafilter* is a filter U where either $A \in U$ or $S \setminus A \in U$ for each $A \subseteq S$. If U is of the form $U = \{X : A \subseteq X\}$ for some $A \subseteq S$, it is called *principal*. Ultrafilters which are not principal are called *non-principal* or *free*.

Given a family $(\mathcal{A}_i)_{i \in I}$ of models, indexed by a set I, and a filter $F \subseteq \mathcal{P}(I)$ on I, we can construct the *reduced product* $\prod_{i \in I} \mathcal{A}_i/F$. An atomic formula φ holds in $\prod_{i \in I} \mathcal{A}_i/F$ if the set $\{i \in I : \mathcal{A}_i \models \varphi\}$ belongs to the filter F. In particular, this means that constants and function symbols are interpreted "compentent-wise", i.e. $c^{\mathcal{A}} := [\lambda i.c^{\mathcal{A}_i}]$ and $g(x)^{\mathcal{A}} := [\lambda i.g(x)^{\mathcal{A}_i}]$ (we write $[f]$ to denote the equivalence class of $f : I \to \cup A_i$ under the equivalence relation $[f] = [g] \Leftrightarrow \{i \in I : \mathcal{A}_i \models f(i) = g(i)\} \in F$, i.e. $[f] = [g] \Leftrightarrow \prod_{i \in I} \mathcal{A}_i/F \models f = g$).

If F is in fact a free ultrafilter, we call the product an *ultraproduct*, and if all $\mathcal{A}_i$ are the same, say $\mathcal{A}_i = \mathcal{A}$, we call $\prod_{i \in I} \mathcal{A}/F$ a *reduced power* or *ultrapower* if F is a free ultrafilter.

Ultrapowers have the nice property of being elementary equivalent to their original models. This follows from the fundamental theorem of ultraproducts, known as Łos's Theorem (see [CK73, 4.1.9]). The proof of it is constructive, at least when $I = \mathbb{N}$ (being a simple induction over the complexity of formulas), but the ultrafilter itself is not constructive. Thus, when working constructively, we instead turn to reduced products. The corresponding theorem for reduced products is called Palyutin's Theorem (Theorem 4.2). To make things smoother from a constructive point of view, we consider filter bases instead of filters.

Definition 4.1 (Filter base)**.** Let S be a set. $B \subseteq \mathcal{P}(S)$ is a *filter base* if the following holds:

(i) $X, Y \in B \Rightarrow \exists Z \in B \big(Z \subseteq X \cap Y\big)$,

(ii) B is inhabited and $\emptyset \notin B$.

Every filter is a filter base, and from every filter base B over S, a filter B' $= \{X \subseteq S : \exists Z \in B \big(Z \subseteq X\big) \}$ can be constructed. An important filter base is $Fr := \{\mathbb{N} \setminus \{0, \ldots, n\} : n \in \mathbb{N}\}$ which is a base for the *Frechét filter* consisting of all cofinite subsets of $\mathbb{N}$.

4.1.2 Palyutin's Theorem

Classically, the following theorem holds:

$\boxed{P \vee \neg P}$ **Theorem 4.2** (Palyutin [Pal80])**.** *Let φ be a h-formula (i.e. φ is built up from $\wedge$, $\exists$, $\forall$, $\exists x\ \psi_1(x) \wedge \forall x\ (\psi_1(x) \to \psi_2(x))$ and atomic formulas), and let F be a filter base. Then*

$$\exists B \in F\big(B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi(\overline{a}(i))\}\big) \Longleftrightarrow \prod_{i \in I} \mathcal{A}_i/F \models \varphi(\overline{a})$$

This is as much as we can hope for, since we have (constructive) counterexamples for $\vee$ and unrestricted $\to$ (note that $\neg$ is a special case of $\to$, since we define $\neg P$ to be $P \to \bot$):

Let $\varphi(a) \equiv a \bmod 2 = 0 \vee a \bmod 2 = 1$, $A_i = \mathbb{N}$, $F = Fr$, and $a_i = i$. We then have $\mathcal{A}_i \models \varphi(a_i)$ for each $i \in \mathbb{N}$, since mod 2 and equality on $\mathbb{N}$ is decidable. In the reduced product, though, we have

$$\begin{aligned} & \prod_{i \in \mathbb{N}} \mathcal{A}_i/Fr \models \varphi(a) \Leftrightarrow \\ \Leftrightarrow\quad & \prod_{i \in \mathbb{N}} \mathcal{A}_i/Fr \models a \bmod 2 = 0 \text{ or } \prod_{i \in \mathbb{N}} \mathcal{A}_i/Fr \models a \bmod 2 = 1 \\ \Leftrightarrow\quad & \exists k \forall n \geq k\big(a_n \bmod 2 = 0\big) \text{ or } \exists k \forall n \geq k\big(a_n \bmod 2 = 1\big) \\ \Leftrightarrow\quad & \exists k \forall n \geq k\big(n \bmod 2 = 0\big) \text{ or } \exists k \forall n \geq k\big(n \bmod 2 = 1\big) \end{aligned}$$

which is obviously false (n and $n+1$ can't both be or not be divisible by 2). Thus the implication

$$\exists B \in F\big(B \subseteq \{i \in \mathbb{N} \colon \mathcal{A}_i \models \varphi(\overline{a}(i))\}\big) \Longrightarrow \prod_{i \in \mathbb{N}} \mathcal{A}_i/F \models \varphi(\overline{a})$$

is false, since $\{i \in \mathbb{N} \colon \mathcal{A}_i \models \varphi(\overline{a}(i))\} = \mathbb{N}$ and F is inhabited.

For a counterexample of $\to$, let $\varphi(a) \equiv \neg(a \bmod 2 = 1) \to a \bmod 2 = 0$, $A_i = \mathbb{N}$, $F = Fr$, and $a_i = i$. Once again, we then have $\mathcal{A}_i \models \varphi(a_i)$ for each $i \in \mathbb{N}$: Suppose $\mathcal{A}_i \models (a_i \bmod 2 = 1) \to \bot$. Equality on $\mathbb{N}$ is decidable, so we know $a_i \bmod 2 = 0 \vee a_i \bmod 2 = 1$. But $a_i \bmod 2 = 1$ gives a contradiction, hence we must have $a_i \bmod 2 = 0$.

In the reduced product, we get in the same way as before that

$$\prod_{i \in \mathbb{N}} \mathcal{A}_i/Fr \not\models a \bmod 2 = 0, \text{ and } \prod_{i \in \mathbb{N}} \mathcal{A}_i/Fr \not\models a \bmod 2 = 1$$

Thus

$$\prod_{i \in \mathbb{N}} \mathcal{A}_i/Fr \models \neg\big(a \bmod 2 = 1\big)$$

but

$$\prod_{i\in\mathbb{N}} \mathcal{A}_i/Fr \not\models a \bmod 2 = 0,$$

so

$$\prod_{i\in\mathbb{N}} \mathcal{A}_i/Fr \not\models \underbrace{\neg\big(a \bmod 2 = 1\big) \to a \bmod 2 = 0}_{\varphi(a)}.$$

Constructively, Theorem 4.2 instead becomes the following. We will show that this is as much as we can get by constructing a Brouwerian counterexample.

Theorem 4.3. *Let F be a filter base consisting of decidable sets, and I a projective set (i.e. the axiom of choice holds on I).*

(i) If φ is geometric (i.e. φ is built up from atomic formulas and $\wedge$, $\vee$, $\exists$), then

$$\prod_{i\in I} \mathcal{A}_i/F \models \varphi(\overline{a}) \Longrightarrow \exists B \in F\big(B \subseteq \{i \in I\colon \mathcal{A}_i \models \varphi(\overline{a}(i))\}\big)$$

(ii) If φ is constructive Horn (i.e. φ is built up from atomic formulas and $\wedge$, $\forall$, $\exists$, $\varphi_1 \to \varphi_2$ where φ_1 is geometric), then

$$\prod_{i\in I} \mathcal{A}_i/F \models \varphi(\overline{a}) \Longleftarrow \exists B \in F\big(B \subseteq \{i \in I\colon \mathcal{A}_i \models \varphi(\overline{a}(i))\}\big)$$

Proof. We follow [Pal92] and proceed by induction on formulae both in (i) and (ii). We have $\prod_{i\in I} \mathcal{A}_i/F \models \bot \Leftrightarrow \exists B \in F\big(B \subseteq \{i \in I\colon \mathcal{A}_i \models \bot\}\big)$ since the right hand side just reduces to $\exists B \in F.B \subseteq \emptyset$, i.e. $\emptyset \in F$ which is false by the filter base axioms.

The other atomic cases follow by definition and the following lemma:

Lemma 4.4. *Let $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i/F$ be a reduced product of L-structures and $t(x_1, \dots, x_k)$ an L-term. For all $[f_1], \dots, [f_k] \in \mathcal{A}$ we then have*

$$t([f_1], \dots, [f_k])^{\mathcal{A}} = [\lambda i.t(f_1(i), \dots, f_k(i))^{\mathcal{A}_i}].$$

Proof of the lemma. Induction on the complexity of terms. If $t(\overline{x}) \equiv c$ is a constant, $c^{\mathcal{A}} = [\lambda i.c^{\mathcal{A}_i}]$ by definition, and if $t(x_1, \dots, x_k) \equiv x_j$ is a variable, $t([f_1], \dots, [f_k])^{\mathcal{A}} = [f_j] = [\lambda i.f_j(i)] = [\lambda i.t(f_1(i), \dots, f_k(i))^{\mathcal{A}_i}]$.

For the induction step, assume $t(x_1, \dots, x_k) \equiv g(t_1(\overline{x}), \dots, t_n(\overline{x}))$ where $t_j(\overline{[f]})^{\mathcal{A}} = [\lambda i.t_j(\overline{f(i)})^{\mathcal{A}_i}]$ for $1 \le j \le n$. By definition, we have $g(t_1(\overline{[f]}), \dots, t_n(\overline{[f]}))^{\mathcal{A}} = [\lambda i.g(t_1(\overline{[f]})^{\mathcal{A}}(i), \dots, t_n(\overline{[f]})^{\mathcal{A}}(i))]$ which by the induction hypothesis is the same as $[\lambda i.g(t_1(\overline{f(i)}), \dots, t_n(\overline{f(i)}))^{\mathcal{A}_i}]$, and we are done. ∎

We now continue with the proof of Theorem 4.3. Write $\mathcal{A} := \prod_{i\in I} \mathcal{A}_i/F$.

(i) **$\wedge$-case** Suppose $\varphi = \varphi_1 \wedge \varphi_2$ and $\mathcal{A} \models \varphi$. Then $\mathcal{A} \models \varphi_1$ and $\mathcal{A} \models \varphi_2$ and hence by the induction hypothesis, $\exists B_j \in F\ (B_j \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_j\})$ for $j = 1, 2$. Since F is a filter base, there is $B \in F$ such that $B \subseteq B_1 \cap B_2 \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_1\} \cap \{i \in I \colon \mathcal{A}_i \models \varphi_2\} = \{i \in I \colon \mathcal{A}_i \models \varphi\}$.

$\vee$-case Suppose $\varphi = \varphi_1 \vee \varphi_2$ and $\mathcal{A} \models \varphi$. Then $\mathcal{A} \models \varphi_1$ or $\mathcal{A} \models \varphi_2$ and hence by the induction hypothesis, $\exists B_j \in F\ (B_j \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_j\})$ for $j = 1$ or $j = 2$. Now, for this j, $\{i \in I \colon \mathcal{A}_i \models \varphi_j\} \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_1 \text{ or } \mathcal{A}_i \models \varphi_2\ \}$ and hence B_j suffices.

$\exists$-case Suppose $\varphi = \exists x \psi(x)$ and $\mathcal{A} \models \varphi$, i.e. there is $a \in \mathcal{A}$ such that $\mathcal{A} \models \psi(a)$. By the induction hypothesis, there is a $B \in F$ such that $B \subseteq \{i \in I \colon \mathcal{A}_i \models \psi(a_i)\} \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi\}$.

(ii) **$\wedge$-case** Suppose $\varphi = \varphi_1 \wedge \varphi_2$ and $B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi\} = \{i \in I \colon \mathcal{A}_i \models \varphi_1 \text{ and } \mathcal{A}_i \models \varphi_2\} \Rightarrow B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_j\}$ and hence $\mathcal{A} \models \varphi_j$ by the induction hypothesis, for $j = 1, 2$. Thus $\mathcal{A} \models \varphi$.

$\exists$-case Suppose $\varphi = \exists x \psi(x)$ and $B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi\}$. For each $i \in B$, we can by the projectivity of I choose an a_i such that $\mathcal{A}_i \models \psi(a_i)$, i.e. $B \subseteq \{i \in I \colon \mathcal{A}_i \models \psi(a_i)$ and hence by the induction hypothesis $\mathcal{A} \models \psi(a)$ for $a(n) = a_n$ if $n \in B$, $a(n) = b_n$ for some $b_n \in \mathcal{A}_n$ otherwise (this works, since B is decidable). Thus $\mathcal{A} \models \varphi$.

$\forall$-case Suppose $\varphi = \forall x \psi(x)$ and $B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi\}$. Let $a \in \mathcal{A}$. If $\mathcal{A}_i \models \forall x \psi(x)$, we must in particular have $\mathcal{A}_i \models \psi(a_i)$, so that $\{i \in I \colon \mathcal{A}_i \models \varphi\} \subseteq \{i \in I \colon \mathcal{A}_i \models \psi(a_i)\}$ and hence $B \subseteq \{i \in I \colon \mathcal{A}_i \models \psi(a_i)\}$, so that $\mathcal{A} \models \psi(a)$ by the induction hypothesis. Since $a \in \mathcal{A}$ was arbitrary, we conclude $\mathcal{A} \models \forall x \psi(x)$.

$\to$-case Suppose $\varphi = \varphi_1 \to \varphi_2$, where φ_1 is geometric, and $B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi\}$. Suppose $\mathcal{A} \models \varphi_1$. By (i), we then have a $B' \in F$ with $B' \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_1\}$. There is a $Z \subseteq B \cap B' \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_1 \wedge (\varphi_1 \to \varphi_2)\} \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi_2\}$. Hence, by the induction hypothesis, $\mathcal{A} \models \varphi_2$ and $\mathcal{A} \models \varphi_1 \to \varphi_2 \Leftrightarrow \mathcal{A} \models \varphi$.

□

Corollary 4.5.

$$\prod_{i\in I} \mathcal{A}_i/F \models \varphi(\overline{a}) \iff \exists B \in F\big(B \subseteq \{i \in I \colon \mathcal{A}_i \models \varphi(\overline{a}(i))\}\big)$$

holds for subgeometric formulas φ (i.e. φ is built up from atomic formulas, $\wedge$ and $\exists$).

Proof. Just combine (i) and (ii) from the theorem. □

Theorem 4.6. *Theorem 4.2* ⇒ **BD**.

Proof. Assume that Theorem 4.2 holds for formulas containing ∀ and consider the ⇐ direction. Let $S \subset \mathbb{N}$ be pseudobounded, i.e. for all sequences $s \colon \mathbb{N} \to S$ we have $(\exists k)(\forall n \geq k)\big(s(n) < n\big)$. Choose $I = \mathbb{N}$, $F = Fr$, $\mathcal{A}_i = (S, <)$, $\phi(a) \equiv \forall x\big(x < a\big)$ and $a(n) = n$. Then

$$\prod_{i\in\mathbb{N}} S/Fr \models \forall x\big(x < a\big) \Longrightarrow \{i \in \mathbb{N} \colon S \models \forall x\big(x < a\big)\} \in Fr$$
$$\Longleftrightarrow (\forall s : \mathbb{N} \to S)\exists k(\forall n \geq k)\big(s(n) < n\big) \Longrightarrow \exists k(\forall n \geq k)\forall s \in S\big(s < n\big)$$

Now, the premise says that S is pseudobounded and is therefore by assumption true. Thus, we get

$$\exists k \forall n \geq k \forall s \in S\big(s < n\big)$$

and in particular choosing $n = k$

$$\exists k \forall s \in S\big(s < k\big),$$

i.e. S is bounded, and **BD** is proved. □

If we allow formulas containing $\exists x\ \psi_1(x) \wedge \forall x\ (\psi_1(x) \to \psi_2(x))$, we can once again derive **BD** since $\forall x \varphi(x) \Longleftrightarrow \exists x\ x = x \wedge \forall x\ (x = x \to \varphi(x))$. Thus, the only formulas among the ones mentioned in Theorem 4.2 for which the theorem is also constructively valid are formulas built up from ∃, ∧ and atomic formulas.

4.2 Saturation of reduced products

In this section, we will investigate to what extent the following theorem, proved classically by Jónsson and Olin [JO68], is constructively valid.

$\boxed{P \vee \neg P}$ **Theorem 4.7.** *Let Fr be the Frechét filter on* $\mathbb{N}$*. Then* $\prod_{i\in\mathbb{N}} \mathcal{A}_i/Fr$ *is* ω_1*-saturated.*

Definition 4.8 (κ-saturated)**.** A structure $\mathcal{A}$ is *κ-saturated* if for each $X \subseteq A$ with cardinality $|X| < \kappa$, and for each set Y consisting of formulas over the signature of $(\mathcal{A}, x)_{x\in X}$, if every finite subset of Y is satisfiable then Y is satisfiable.

Thus, a structure $\mathcal{A}$ is ω_1-saturated if every set Y of formulas over $(\mathcal{A}, x)_{x\in X}$ is satisfiable whenever it is finitely satisfiable holds for every *countable* X.

4.2.1 Fréchet products are ω_1-saturated for subgeometric formulas

Instead of working with realizers of sets of formulas directly, we will work with something which we call internal sets.

Definition 4.9 (Internal set). Let $\mathcal{A} = \prod \mathcal{A}_i/Fr$ be a Fréchet product. $S \subseteq A$ is called *internal* if there is a sequence (S_n) of sets $S_n \subseteq A_n$ such that

$$[x] \in S \Leftrightarrow (\exists k)(\forall n \geq k)x(n) \in S_n.$$

The set S given by the sequence (S_n) is denoted $(S_n)^+$.

Note that the set

$$S = \{x \in \prod A_n/Fr\colon \prod \mathcal{A}_n/Fr \models \varphi(x)\}$$

of realizers of a subgeometric formula φ is internal with sequence

$$S_n = \{y \in A_n \colon \mathcal{A}_n \models \varphi(y)\}$$

by Theorem 4.5, since

$$[x] \in S \Leftrightarrow (\exists k)(\forall n \geq k)\big(\mathcal{A}_n \models \varphi(x(n))\big) \Leftrightarrow (\exists k)(\forall n \geq k)\big(x(n) \in S_n\big).$$

Suppose that two internal sets $S \subseteq T$ are given by the sequences (S_n) and (T_n) respectively. Classically, we then have $(\exists k)(\forall n \geq k)\big(S_n \subseteq T_n\big)$ (if not, for all k there is a $n \geq k$ and a $a_n \in S_n$ such that $a_n \notin T_n$. But then $[\lambda n.a_n] \in S \setminus T$, contradicting $S \subseteq T$). Constructively, it is not so clear that this is always the case. Instead, we use the following stronger inclusion relation:

Definition 4.10 (Strongly included). Let (A_n) and (B_n) be sequences of sets. Then (A_n) is *strongly included* in (B_n), written $(A_n) \preceq (B_n)$, if $(\exists k)(\forall n \geq k)$ $\big(A_n \subseteq B_n\big)$.

Theorem 4.11 ([Pal92]). *Let $\prod_{n \in \mathbb{N}} \mathcal{A}_n/Fr$ be a Fréchet product, and let $(S_n^0) \succeq (S_n^1) \succeq (S_n^2) \succeq \ldots$ be a strongly decreasing chain of inhabited sequences, where $(S_n^i) \subseteq A_n$. Then $\cap_i (S_n^i)^+$ is inhabited.*

Proof. Let $[x_i] \in (S_n^i)^+$. Choose successively $k_0 < k_1 < k_2 < \ldots$ so that $\forall n \geq k_m$

(i) $x_m(n) \in S_n^m$

(ii) $S_n^m \supseteq S_n^{m+1}$.

This is possible since for each $m \in \mathbb{N}$, (i) $[x_m] \in (S_n^m)^+$ and thus $(\exists k)(\forall n \geq k)$ $\big(x_m(n) \in S_n^m \big)$, and (ii) $(S_n^m) \succeq (S_n^{m+1})$, i.e. $(\exists k')(\forall n \geq k')$ $\big(S_n^{m+1} \subseteq S_n^m \big)$, so let $k_m = \max(k, k')$.

Now, for $k_m \leq n < k_{m+1}$, define $y(n) = x_m(n)$. For $n < k_0$, let $y(n) = x_0(n)$. We now have to prove that $[y] \in (S_n^i)^+$ for each i. To do this, we have to find k such that $y(n) \in S_n^i$ for all $n \geq k$. But consider $n \geq k_i$, say $k_{i+\ell} \leq n \leq k_{i+\ell+1}$. By (i) we then have $y(n) = x_{i+\ell}(n) \in S_n^{i+\ell}$, and from (ii) it follows that $S_n^{i+\ell} \subseteq S_n^i$, thus $y(n) \in S_n^i$ and $[y] \in \cap_i (S_n^i)^+$. □

4.2.2 König's Lemma

Let $\langle \cdot \rangle : \mathbb{N}^* \to \mathbb{N}$ be a coding function with length, decoding and restriction functions $\|\cdot\|$, $(\cdot)_i$ and $\cdot \restriction_i$ respectively as described in Section 2.3. We will further assume that our coding fuction is bijective.

Definition 4.12 (Finitary tree)**.** A set $T \subseteq \mathbb{N}$ of codes for sequences of natural numbers is a *finitary tree* if

(i) For each $s \in \mathbb{N}$, $s \in T \vee s \notin T$,

(ii) $\langle x_0, \ldots, x_n \rangle \in T \Rightarrow \langle x_0, \ldots, x_{n-1} \rangle \in T$,

(iii) For all $x = \langle x_0, \ldots, x_n \rangle \in T$, there is $m_x \in \mathbb{N}$ such that $\langle x_0, \ldots, x_n, z \rangle \in T$ implies $z \leq m_x$.

If m_x in (iii) is constantly 1 for all x, T is called a *binary tree.*

Lemma 4.13. *If $w \in T$, then $w \restriction_k \in T$ for all $k \leq \|w\|$.*

Proof. Induction on $\|w\| - k$. For $k = \|w\|$, $w \restriction_{\|w\|} = w \in T$ by assumption. For $\|w\| - k \rightsquigarrow \|w\| - k + 1 \Leftrightarrow k \rightsquigarrow k - 1$, suppose $w \restriction_k \in T$. Then $w \restriction_{k-1}$ $\in T$ by (ii) in the definition. □

$\boxed{P \vee \neg P}$ **Theorem** (König's Lemma)**.** *If T is an infinite finitary tree (i.e. T has arbitrarily large finite subsets), then there is a sequence $(x_i)_{i=0}^{\infty}$ such that $\langle x_0, \ldots, x_n \rangle \in T$ for each n.*

Proof. Consider all x with $\langle x \rangle \in T$. There is at most $m_{\langle x \rangle}$ of them, so if the subset $S_x = \{w \in T \colon w \restriction_1 = x\}$ would be finite for all of them, T wouldn't have arbitrarily large finite subsets. Thus, with an application of the law of the excluded middle, S_x must be infinite for at least one x. Let x_0 be the smallest such x.

Having chosen $x_0, \ldots, x_{n-1}$, repeat the argument with $x' = \langle x_0, \ldots, x_{n-1}, x \rangle$, and we get the desired sequence by induction. □

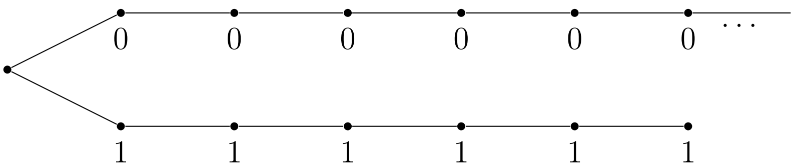


Figure 1: The tree T from the proof of Theorem 4.14. A path of consisting of x ($x \in \{0, 1\}$) stops after k steps if $a_{2k+x} = 1$. Otherwise, the path will have infinite length.

If we restrict König's Lemma to binary trees, we get *Weak König's Lemma.* König's Lemma implies LLPO and is hence not constructive:

Theorem 4.14. *(Weak) König's Lemma ⇒ LLPO.*

Proof. Assume that König's Lemma holds. Let $(a_n)_{n=0}^{\infty}$ be a binary sequence with at most one 1, and let

$$T_n = \{\underbrace{\langle 0, \dots, 0\rangle}_{n \text{ times}} \colon a_{2k} = 0 \text{ for } 0 \leq k \leq n\} \cup \{\underbrace{\langle 1, \dots, 1\rangle}_{n \text{ times}} \colon a_{2k+1} = 0 \text{ for } 0 \leq k \leq n\}$$

and

$$T = \bigcup_{n \in \mathbb{N}} T_n.$$

Then T is an binary tree. Furthermore, T is infinite, since $\{\langle 0\rangle, \langle 0, 0\rangle, \dots, \langle 0, \dots, 0\rangle\} \subseteq T$ or $\{\langle 1\rangle, \langle 1, 1\rangle, \dots, \langle 1, \dots, 1\rangle\} \subseteq T$ for arbitrarily large such subsets (and we can decide which one holds by looking at finitely many values a_k). By König's Lemma, we then find a sequence $(x_i)_{i=0}^{\infty}$ such that $\langle x_0, \dots, x_n\rangle \in T$ for each n. Now just inspect x_0: If it is 0, then $a_{2n} = 0$ for all $n \in \mathbb{N}$, otherwise $a_{2n+1} = 0$ for all $n \in \mathbb{N}$ and **LLPO** is proved. □

4.2.3 ω_1-saturation of Fréchet products for geometric formulas is equivalent to König's Lemma

Theorem 4.15. $\prod_{i \in \mathbb{N}} \mathcal{A}_i / Fr$ *is* ω_1*-saturated for* $\{\vee, \wedge\}$*-formulas ⇒ Weak König's Lemma.*

Proof. Let $\mathcal{A}_i = \mathbb{N}$ and T be an infinite binary tree. Define

$$\psi_w(x) = (\|x\| \geq \|w\| \wedge x \, 1_{\|w\|} = w)$$

and

$$\varphi_n(x) = \bigvee_{\substack{\|w\| = n, \\ w \in T}} \psi_w(x).$$

Let $Y = \{\varphi_n(x)\colon n \in \mathbb{N}\}$ and note that we only use a countable number of constants. We show that any finite subset $Y_f = \{\varphi_{n_1}(x), \dots, \varphi_{n_\ell}(x)\} \subseteq Y$ is satisfiable. It is enough to find a term satisfying $\varphi_{\max\{n_i\}}(x)$, since if a satisfies $\psi_w(x)$ with $\|w\| = m := \max\{n_i\}$, a will satisfy $\psi_{w\upharpoonright_{n_j}}(x)$ and hence $\varphi_{n_j}(x)$ by Lemma 4.13 and the fact that $w \in T$.

Let $U \subset T$ be a subset containing 2^m elements. Since U is finite, we have $\exists x \in U(\|x\| \geq m) \vee \neg\exists x \in U(\|x\| \geq m)$. If $\neg\exists x \in U(\|x\| \geq m)$ would be the case, each of the 2^m elements in U would have length less than m. But there are only

$$\sum_{i=0}^{m-1} 2^i = 2^{m-1+1} - 1 = 2^m - 1$$

elements with length less than m, so we arrive at a contradiction. Hence there is $a \in U$ with $\|a\| \geq m$, so $\psi_{a\upharpoonright_m}(a)$ is satisfied and since $a \in U \subset T$, $\psi_{a\upharpoonright_m}(x)$ is a disjunct in $\varphi_m(x)$. Thus $\varphi_m(a)$ is satisfied. Using our inclusion $\mathbb{N} \hookrightarrow \prod \mathbb{N}/Fr$, we hence find $d(a) \in \prod \mathbb{N}/Fr$ satisfying Y_f, which (under the assumption that $\prod \mathbb{N}/Fr$ is ω_1-saturated for $\{\vee, \wedge\}$-formulas) gives us a $c \in \prod \mathbb{N}/Fr$ satisfying all of Y.

So we know

$$\forall m \prod \mathbb{N}/Fr \models \varphi_m(c),$$

i.e.

$$\forall m \bigvee_{\substack{\|w\|=m,\\ w\in T}} \exists k_m \forall n \geq k_m \big(\psi_w(c(n))\big).$$

Define the sequence $\{x_i\}_i$ by $x_n = (c(k_n))_n$, i.e. x_n is the nth component of $c(k_n)$. It remains to prove that $\langle x_0, \dots, x_\ell\rangle \in T$ for each $\ell \in \mathbb{N}$. Since $c(k_\ell)\upharpoonright_\ell \in T$ for each $\ell \in \mathbb{N}$, it is enough to show that $x_i = (c(k_j))_i$ for each $i \leq j$ and hence $\langle x_0, \dots, x_\ell\rangle = c(k_\ell)\upharpoonright_\ell \in T$.

Let $i \leq j$. For $m = i$, we know that there is a w with $\|w\| = i$ such that $\exists k_i \forall n \geq k_i\big(\psi_w(c(n))\big)$, i.e. $c(n)\upharpoonright_i = w$ for all $n \geq k_i$. For $m = j$, we likewise get a w' with $\|w'\| = j$ such that $c(n)\upharpoonright_j = w'$ for all $n \geq k_j$. In particular, for $n = \max(k_i, k_j)$, we have $c(n)\upharpoonright_i = w = c(k_i)\upharpoonright_i$ since $n, k_i \geq k_i$. We also have $c(n)\upharpoonright_j = w' = c(k_j)\upharpoonright_j$ in the same way. But then $(c(k_i))_i = (c(n))_i = (c(k_j))_i$ where the last equality holds because $c(n)$ and $c(k_j)$ agree up to position at least j, and $i \leq j$. By the remark above, we are done. □

If we modify our proof a little, we can actually prove the full König's Lemma:

Theorem 4.16. *$\prod_{i\in\mathbb{N}} \mathcal{A}_i/Fr$ is ω_1-saturated for $\{\vee,\wedge\}$-formulas $\Rightarrow$ König's Lemma.*

Proof. Define

$$W_n = \{w * \langle z\rangle \in T : w \in W_{n-1}, z \le m_w\}$$

with $W_0 = \{\langle\rangle\}$ and m_w taken from definition 4.12(iii). Note that W_n is decidable with definition 4.12(i) and replace φ_n with

$$\varphi'_n(x) = \bigvee_{w\in W_n} \psi_w(x). \qquad \square$$

We conclude that there can be no constructive proof of a general version of Theorem 4.7, since this then would imply **LLPO** by Theorem 4.16 together with Theorem 4.14. We can, however, say more:

Theorem 4.17. *Under the assumption that all decreasing sequences of internal sets consisting of realizers of subgeometric formulas are strongly decreasing, we have:*

$\prod_{i\in\mathbb{N}} \mathcal{A}_i/Fr$ is ω_1-saturated for geometric formulas in a countable language $\iff$ König's Lemma.

Proof. ($\Rightarrow$) is clear by Theorem 4.16. For ($\Leftarrow$), we follow ideas from [Pal92]. Assume that König's Lemma holds. Let $\Sigma(x) = \{\varphi_0(x), \varphi_1(x), \ldots\}$ be a finitely satisfiable set ($\Sigma(x)$ is countable since the language is), and define $G^n = \{x \in \prod A_i/Fr\colon \prod \mathcal{A}_i/Fr \models \varphi_0(x) \wedge \ldots \wedge \varphi_n(x)\}$. Then $G^0 \supseteq G^1 \supseteq \ldots$ is a decreasing sequence of inhabited sets.

Each geometric formula can be written as a disjunction of subgeometric formulae, so each G^n can be written as a union of internal sets (possibly in different ways) – this is called a *splitting*. We will inductively define inhabited internal sets $I^{n,j}$ $(0 \le j \le \nu(n))$ so that $G^n = I^{n,0} \cup \ldots \cup I^{n,\nu(n)}$ and each $I^{n+1,j}$ is included in some $I^{n,k}$.

Let $G^0 = I^{0,0} \cup \ldots \cup I^{0,\nu(0)}$ be any splitting. Suppose that we have found splittings $G^n = I^{n,0} \cup \ldots \cup I^{n,\nu(n)}$ satisfying the conditions for $n \le m$. Let $G^{m+1} = J^{m+1,0} \cup \ldots \cup J^{m+1,\nu'(m+1)}$ be a splitting of G^{m+1}, and construct a new splitting

$$G^{m+1} = \bigcup_{\substack{0\le i\le \nu(m)\\ 0\le j\le \nu'(m+1)}} I^{m,i} \cap J^{m+1,j}$$

(see figure 2 for a more intuitive picture). This really is a splitting of G^{m+1} since $G^{m+1} \subseteq G^m$. Every $I^{m,i} \cap J^{m+1,j}$ is an intersection of internal sets and thus an internal set, and $I^{m,i} \cap J^{m+1,j} \subseteq I^{m,i}$ by construction.

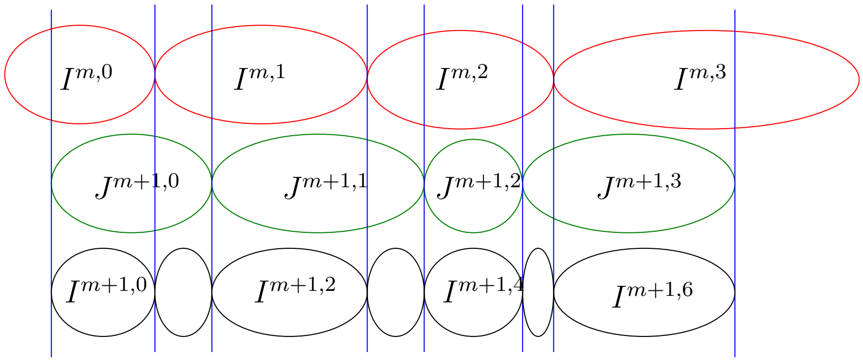


Figure 2: How to construct $I^{m+1,0} \cup \ldots \cup I^{m+1,\nu(m+1)}$ from $I^{m,0} \cup \ldots \cup I^{m,\nu(m)}$ and $J^{m+1,0} \cup \ldots \cup J^{m+1,\nu'(m+1)}$

Now, the branchings among the $I^{n,j}$ define a finitely branching tree T. Since each G^n is inhabited, T has a branch of length n for every $n \in \mathbb{N}$ and thus an infinite branch by König's Lemma. But this infinite branch $(I^{n,j_n})_{n=0}^{\infty}$ is a decreasing sequence of internal sets (strongly decreasing by assumption), so Theorem 4.11 applies and gives us an element $[y] \in \cap_n I^{n,j_n} \subseteq \cap_n G^n$, i.e. $\prod \mathcal{A}_i/Fr$ is ω_1-saturated. $\square$

5 Some Constructive Non-standard Analysis

Now, let us do some constructive non-standard analysis to see what can be done with the tools we have developed. We will once again eventually see what can't be done by constructing certain Brouwerian counterexamples.

As our non-standard model, we will take a Fréchet power of the real numbers, where we assume that our language has a sufficient amount of function symbols and constants as usual. The main tool is Theorem 4.3, here restated for our situation with the filter base $Fr = \{\mathbb{N} \setminus \{0, \ldots, n\} : n \in \mathbb{N}\}$:

Theorem 5.1. *Let $\mathbb{R}^+ = \prod_{i \in \mathbb{N}} \mathbb{R}/Fr$ be the Fréchet power of $\mathbb{R}$.*

(i) If φ is constructive Horn (i.e. φ is built up from atomic formulas and $\wedge$, $\forall$, $\exists$, $\varphi_1 \to \varphi_2$ where φ_1 is geometric), then

$$(\exists k)(\forall n \geq k)\ \mathbb{R} \models \varphi(\overline{a}(n)) \Longrightarrow \mathbb{R}^+ \models \varphi(\overline{a})$$

(ii) If φ is geometric (i.e. φ is built up from atomic formulas and $\wedge$, $\vee$, $\exists$), then

$$\mathbb{R}^+ \models \varphi(\overline{a}) \Longrightarrow (\exists k)(\forall n \geq k)\ \mathbb{R} \models \varphi(\overline{a}(n))$$

One has to be a bit careful with $\mathbb{R}^+$. On the one hand, almost all basic standard results on the arithmetic of the real numbers can be lifted to $\mathbb{R}^+$, since they can be formulated as constructive Horn formulas ($\forall x\big(x + 0 = x\big)$, $\forall x\big(x > 0 \to \neg(x < 0)\big)$ etc). On the other hand, $\mathbb{R}^+$ is not even a field, since $[(0, 1, 0, 1, \ldots)] \cdot [(1, 0, 1, 0, \ldots)] = [(0, 0, \ldots)] = 0$, i.e. $\mathbb{R}^+$ has zero divisors. This is because the sentence expressing that every non-zero element has a multiplicative inverse ($\forall x\big(\neg\big(x = 0\big) \to \exists y\big(x \cdot y = 1\big)\big)$) can not be expressed as a constructive Horn formula.

Definition 5.2. Let $a, b \in \mathbb{R}^+$.

(i) a is *finite*, if there is $k \in \mathbb{R}$ such that $|a| < k$.

(ii) a is *infinite*, if for all $k \in \mathbb{R}$, $k < |a|$.

(iii) a is *infinitesimal*, if for all real $k > 0$, $|a| < k$.

(iv) a and b are *infinitely close*, $a \simeq b$, if $a - b$ is infinitesimal.

Let us start with some well-known non-standard characterizations of properties of sequences of real numbers.

Definition 5.3. Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$.

(i) $(x_n) \to L \in \mathbb{R}$ if for all real $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq k$.

(ii) $L \in \mathbb{R}$ is a *limit point* of (x_n) if for all real $\varepsilon > 0$ and all $k \in \mathbb{N}$ there is a $n \geq k$ such that $|x_n - L| < \varepsilon$.

(iii) (x_n) is *bounded* if there is $k \in \mathbb{R}$ such that $|x_n| < k$ for all $n \in \mathbb{N}$.

Theorem 5.4. *Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$.*

(i) $(x_n) \to L \in \mathbb{R}$ if and only if $x_\mu \simeq L$ for all infinite μ.

(ii) $L \in \mathbb{R}$ is a limit point of (x_n) if and only if $x_\mu \simeq L$ for some infinite μ.

(iii) (x_n) is bounded if and only if x_μ is finite for all infinite μ.

Proof. (i) ($\Rightarrow$) Assume $(\forall \varepsilon > 0)(\exists k)(\forall n \geq k)\big(|x_n - L| < \varepsilon\big)$ in $\mathbb{R}$. We have to show that $|x_\mu - L| < k$ for all standard $k > 0$ and all infinite μ. Let $k > 0$ and μ be given. We have $(\forall n \geq \ell)\big(|x_n - L| < k\big)$ for some $\ell \in \mathbb{N}$. This is a constructive Horn formula, and can thus be lifted with Theorem 5.1(i), giving $(\forall n \geq \ell)\big(|x_n - L| < k\big)$. But $\mu \geq \ell$ since μ is infinite, so $|x_\mu - L| < k$ and $x_\mu \simeq L$.

($\Leftarrow$) If $x_\mu \simeq L$ for all infinite μ, we in particular have $x_\Omega \simeq L$, where $\Omega = [\lambda n.n]$, i.e. $|x_\Omega| < k$ for all positive $k \in \mathbb{R}$. For each such k, we can use the downward transfer from Theorem 5.1(ii), and get $(\exists \ell)(\forall n \geq \ell)\, |x_n| < k$, i.e. $(x_n) \to L$.

(ii) ($\Rightarrow$) Suppose that $(\forall \varepsilon > 0, k \in \mathbb{N})(\exists n \geq k)\, |x_n - L| < \varepsilon$. This can be lifted, giving $(\forall \varepsilon > 0, k \in \mathbb{N}^+)(\exists n \geq k)\, |x_n - L| < \varepsilon$. For ε infinitesimal and k infinite, we get a $\mu \geq k$ (so μ is infinite), with $|x_\mu - L| < \varepsilon$, hence $|x_\mu - L| < \ell$ for all standard $\ell > 0$ (since $\varepsilon < \ell$ for all such ℓ), i.e. $x_\mu \simeq L$.

($\Leftarrow$) Suppose that $|x_\mu - L| < \ell$ for all standard $\ell > 0$. Let standard $\varepsilon > 0$ and $k \in \mathbb{N}$ be given. Downward transfer applied to $\varphi(a) \equiv |x_a - L| < \varepsilon$ gives a k' such that $\left|x_{\mu(i)} - L\right| < \varepsilon$ for all $i \geq k'$. Since μ is infinite, we must have $\mu(i) > k$ for some $i \geq k'$, so that $|x_n - L| < \varepsilon$ holds for $n = \mu(i)$ for this i.

(iii) (⇒) Suppose that $|x_n| < k$ for a $k \in \mathbb{R}$. Lifting this gives $\forall n \in \mathbb{N}^+\big(\, |x_n| < k\big)$, so in particular, this holds for all infinite n.

(⇐) If $|x_\mu| < k$, with $k \in \mathbb{R}$, for all infinite μ, we in particular have $|x_\Omega| < k$, where $\Omega = [\lambda n.n]$. Transfer downwards now gives $(\exists \ell)(\forall n \geq \ell)\left|x_{\Omega(n)}\right| < k$, i.e. $|x_n| < k$ for all $n \geq \ell$, so (x_n) is bounded by $\max\{|x_0|, |x_1|, \ldots, |x_{\ell-1}|, k\}$.

□

In the proof, we see that some properties hold for *all* infinite numbers as soon as they hold for $\Omega = [\lambda n.n]$. We make this precise in the following theorem.

Theorem 5.5. *Let $P(n)$ be a subgeometric formula. Then $\mathbb{R}^+ \models P(\Omega)$ if and only if $\mathbb{R}^+ \models P(\mu)$ for all infinite μ.*

Proof. (⇐) is obvious, since Ω is infinite.

(⇒) Suppose $\mathbb{R}^+ \models P(\Omega)$. By downward transfer, we then have a standard k such that $\mathbb{R} \models P(n)$ for all $n \geq k$. Let μ be infinite. Then, in particular, $\mu > k$, so there is an ℓ such that $\mu(i) > k$ for all $i \geq \ell$. We thus have $\mathbb{R} \models P(\mu(i))$ for all $i \geq \ell$, so by lifting we get $\mathbb{R}^+ \models P(\mu)$. □

We now turn to functions.

Definition 5.6. Let $X \subseteq \mathbb{R}$. A mapping $f\colon X \to \mathbb{R}$ is *uniformly continuous*, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $x, y \in X$.

The following is also well-known:

Theorem 5.7. *$f : X \to \mathbb{R}$ is uniformly continuous $\Rightarrow \forall x, y \in X^+$ $\big(\; x \simeq y \Rightarrow f(x) \simeq f(y) \;\big)$.*

Proof. Assume that f is uniformly continuous, and that $x \simeq y$. Let a standard $\varepsilon > 0$ be given. Since f is uniformly continuous, there is a standard $\delta > 0$ such that $(\forall z, w \in X)\ \big(\; |z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon \;\big)$ holds in $\mathbb{R}$. By lifting, this also holds in $\mathbb{R}^+$. Since $x \simeq y$, $|x - y| < \delta$ and thus $|f(x) - f(y)| < \varepsilon$. Since ε was arbitrary, $f(x) \simeq f(y)$. □

Classically, the converse also holds. We will now construct a Brouwerian counterexample, showing that this is not possible in **BISH**, however true in **INT** and **RUSS**.

Definition 5.8. Let $X \subseteq \mathbb{R}$. A mapping $f\colon X \to \mathbb{R}$ is *uniformly sequentially continuous*, if $x_n - y_n \to 0 \Rightarrow f(x_n) - f(y_n) \to 0$ for all sequences $(x_n), (y_n)$ in X.

Theorem 5.9. *Let $f : X \to \mathbb{R}$ be a standard function. Then $\forall x, y \in X^+$ $\big(x \simeq y \Rightarrow f(x) \simeq f(y)\big) \Leftrightarrow f : X \to \mathbb{R}$ is uniformly sequentially continuous.*

Proof. ($\Rightarrow$) Let $(x_n), (y_n)$ be sequences in X and assume $x_n - y_n \to 0$. By Theorem 5.4(i), we then have $x_\mu - y_\mu \simeq 0$, i.e. $x_\mu \simeq y_\mu$ for all infinite μ. By the assumption, we then have $f(x_\mu) \simeq f(y_\mu)$, hence $f(x_n) - f(y_n) \to 0$ again by Theorem 5.4(i), so f is uniformly sequentially continuous.

($\Leftarrow$) Let $x \simeq y$, where $[x], [y] \in X^+$. By downward transfer (of the formula $\varphi(a) \equiv a \in X$, where we have "$\in X$" in the language), there is a $k \in \mathbb{N}$ such that $x(n), y(n) \in X$ for all $n \geq k$. Let x', y' be modifications of x and y so that $x'(n) = x(n)$ for $n \geq k$ and $x'(n)$ is an arbitrary fixed element in X for $n < k$, and analogously for y'. Then $[x'] = [x]$, $[y'] = [y]$, and $(x'(n))_n$, $(y'(n))_n$ are sequences in X. Since $x' \simeq y'$, $x'(n) - y'(n) \to 0$, so by the assumption, $f(x'(n)) - f(y'(n)) \to 0$. Hence $f(x'(\mu)) \simeq f(y'(\mu))$ for all infinite μ, and $f(x) \simeq f(y)$. □

Thus, if we had "f uniformly sequentially continuous $\implies f$ uniformly continuous", the converse of Theorem 5.7 would also hold. We will however show that this statement is equivalent to **BD**. First we need two lemmas from [Ish92]:

Lemma 5.10. *Let A be an inhabited pseudobounded subset of $\mathbb{N}$. Then there exists a complete subset $X \subseteq \mathbb{R}$ and a uniformly sequentially continuous function $f : X \to \{0, 1\}$ such that*

$$0 \in X \wedge f(0) = 0 \wedge \forall m\big(m \in A \Rightarrow 2^{-m} \in X \wedge f(2^{-m}) = 1\big).$$

Proof. Let $Z := \{0\} \cup \{2^{-m} : m \in A\}$, and construct $g : Z \to \{0, 1\}$ such that

$$\forall p \in Z\big((g(p) = 0 \Rightarrow p = 0) \wedge (g(p) = 1 \Rightarrow \exists m \in A(p = 2^{-m}))\big).$$

Then Z and g satisfies

$$0 \in Z \wedge g(0) = 0 \wedge \forall m\big(m \in A \Rightarrow 2^{-m} \in Z \wedge g(2^{-m}) = 1\big),$$

but Z probably isn't complete. Let X be the completion of Z, i.e. X consists of Cauchy sequences of elements of Z, with two sequences being equal if they tend to each other.

Note that for each $p, q \in Z$ and $n \in \mathbb{N}$,

($*$) if $g(p) \neq g(q)$ and $|p - q| \leq 2^{-n}$ then there is $m \in A$ with $m \geq n$:

without loss of generality, assume that $g(p) = 0$ and $g(q) = 1$. Then $p = 0$ and $q = 2^{-m}$ for some $m \in A$, and $2^{-m} = q = |0 - q| = |p - q| \leq 2^{-n}$, so $m \geq n$.

We show that g can be extended to X. For $(p_n)_{n=0}^{\infty} \in X$, construct a strictly increasing sequence $(N_n)_n$ in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, if $i, j \geq N_n$ then $|p_i - p_j| < 2^{-n}$ (this is possible since (p_n) is Cauchy). Then construct a binary sequence (α_n) such that

$$\begin{aligned} &\alpha_n = 0 \Rightarrow \forall i, j\big(N_n \leq i, j < N_{n+1} \Rightarrow g(p_i) = g(p_j)\big) \\ &\alpha_n = 1 \Rightarrow \exists i, j\big(N_n \leq i, j < N_{n+1} \wedge g(p_i) \neq g(p_j)\big) \end{aligned} \tag{3}$$

Let $a \in A$, and define a sequence (m_n) in A as follows: if $\alpha_n = 0$, let $m_n := a$. Otherwise, if $\alpha_n = 1$, let i, j be such that $N_n \leq i, j < N_{n+1}$ and $g(p_i) \neq g(p_j)$ (possible by (3)). Since $N_n \leq i, j$, $|p_i - p_j| \leq 2^{-n}$ and $(*)$ gives a $m \in A$ with $m \geq n$. Let $m_n := m$. Since (m_n) is a sequence in A, which is pseudobounded, there is a ν such that $\frac{m_n}{n} < 1$ for all $n \geq \nu$. For each n, we have $\alpha_n = 0 \vee \alpha_n = 1$. Assume $\alpha_n = 1$ for some $n \geq \nu$. Then $m_n \geq n$, so that $1 > \frac{m_n}{n} \geq \frac{n}{n} = 1$, i.e. $1 > 1$, a contradiction. Thus $\alpha_n = 0$ for all $n \geq \nu$. Hence the sequence $(g(p_n))_n$ is Cauchy, and we can define $f(x) = f((p_n)_n) := (g(p_n))_n$. It is clear that f extends g.

It remains to prove that f is well-defined. Let $x = (p_n)_n$ and $y = (q_n)_n$ with $x = y$. Since $f(z) = 0 \vee f(z) = 1$ for all z, we have $f(x) = f(y) \vee f(x) \neq f(y)$. Suppose that $f(x) \neq f(y)$. Then there is ν such that $g(p_n) \neq g(q_n)$ for all $n \geq \nu$. Since $x = y$, $|p_{2^{n+1}} - q_{2^{n+1}}| \leq 2^{-n}$ for all n, so $(*)$ gives a sequence (ℓ_n) in A with $\ell_n \geq n$ for all n with $2^{n+1} \geq \nu$. But then $\lim \frac{\ell_n}{n} \neq 0$, which is impossible since A is pseudobounded. Hence $f(x) = f(y)$ and f is well-defined.

Finally, we prove that f is uniformly sequentially continuous. Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences in X with $x_n - y_n \to 0$. Construct a strictly increasing sequence $(M_n)_{n=1}^{\infty}$ in $\mathbb{N}$ and a binary sequence $(\beta_n)_{n=0}^{\infty}$ as above so that $\forall k \geq M_n\big(|x_k - y_k| < 2^{-n}\big)$ and

$$\begin{aligned} &\beta_n = 0 \Rightarrow \forall k\big(M_n \leq k < M_{n+1} \Rightarrow f(x_k) = f(y_k)\big) \\ &\beta_n = 1 \Rightarrow \exists k\big(M_n \leq k < M_{n+1} \wedge f(x_k) \neq f(y_k)\big) \end{aligned}$$

holds for all n. Let $a \in A$ and define a sequence (s_n) in A as follows: if $\beta_n = 0$, set $s_n := a$. If $\beta_n = 1$, there is k with $M_n \leq k < M_{n+1}$ and $f(x_k) \neq f(y_k)$. Choose $p, q \in Z$ such that $|p - q| < 2^{-n}$ and $g(p) = f(x_k) \neq f(y_k) = g(q)$. Then $(*)$ gives a $m \in A$ with $m \geq n$. Set $s_n := m$. Since A is pseudobounded, there is N such that $\frac{s_n}{n} < 1$ for all $n \geq N$. Hence $\beta_n = 0$ for all $n \geq N$, and thus $f(x_k) = f(y_k)$ for all $k \geq M_N$, i.e. $f(x_n) - f(y_n) \to 0$. $\square$

Lemma 5.11. *Let $f : X \to \mathbb{R}$ be uniformly sequentially continuous. Then for each $\varepsilon > 0$ there exists an inhabited pseudobounded subset $A \subseteq \mathbb{N}$ such that*

$$\forall m > 0\big(\exists x, y \in X(|x - y| < 1/m \wedge |f(x) - f(y)| > \varepsilon) \Rightarrow m \in A\big).$$

Proof. For each $\varepsilon > 0$, let

$$A = \{0\} \cup \{m > 0 \colon \exists x, y \in X(|x - y| < 1/m \wedge |f(x) - f(y)| > \varepsilon)\},$$

and let $(s_n)_{n=1}^{\infty}$ be a sequence in A. For each $\delta > 0$, construct a binary sequence (λ_n) such that for each n,

$$\lambda_n = 0 \Rightarrow \frac{s_n}{n} < \delta$$
$$\lambda_n = 1 \Rightarrow \frac{s_n}{n} > \frac{\delta}{2}$$

Let $z \in X$. Construct two sequences (x_n), (y_n) in X as follows: if $\lambda_n = 0$, let $x_n := y_n := z$. Otherwise, $\lambda_n = 1$ and $s_n > n\delta/2 > 0$, so by the definition of A, there is $x, y \in X$ such that $|x - y| < 1/s_n$ and $|f(x) - f(y)| > \varepsilon$. Let $x_n := x$ and $y_n := y$. Now $x_n - y_n \to 0$: if $\lambda_n = 0$, $|x_n - y_n| = |z - z| = 0 < \frac{2}{\delta n}$, and if $\lambda_n = 1$, $|x_n - y_n| < 1/s_n < \frac{2}{\delta n} \to 0$. Since f is uniformly sequentially continuous, there exists ν such that $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq \nu$. Suppose that $\lambda_n = 1$ for some $n \geq \nu$. Then $|f(x_n) - f(y_n)| > \varepsilon$ by the construction of (x_n), (y_n), which is a contradiction. Thus $\lambda_n = 0$ for all $n \geq \nu$ and $s_n/n < \delta$ for all $n \geq \nu$, i.e. A is pseudobounded. □

Theorem 5.12 (Hajime Ishihara [Ish92])**.** *The following is equivalent:*

(i) Every uniformly sequentially continuous function $f : X \to \mathbb{R}$ is uniformly continuous.

(ii) Every uniformly sequentially continuous function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$ is complete, is pointwise continuous.

(iii) **BD**.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) Let $A \subseteq \mathbb{N}$ be pseudobounded. By Lemma 5.10, there exists a complete subset $X \subseteq \mathbb{R}$ and a uniformly sequentially continuous function $f : X \to \{0, 1\}$ such that

$$0 \in X \wedge f(0) = 0 \wedge \forall m\big(m \in A \Rightarrow 2^{-m} \in X \wedge f(2^{-m}) = 1\big).$$

As f is pointwise continuous, by assumption, there exists N such that for all $x \in X$, if $|x - 0| < 2^{-N}$, then $|f(x) - f(0)| = |f(x)| < 1$. For every $m \in A$, we have $m \leq N \vee m > N$ since $<$ is decidable on $\mathbb{N}$. Suppose that $m > N$ for some $m \in A$. Then $2^{-m} \in X$ and $0 < 2^{-m} < 2^{-N}$, so $1 = f(2^{-m})$ but also $f(2^{-m}) < 1$, i.e. $1 < 1$, a contradiction. Hence $m \leq N$ for all $m \in A$ and A is bounded, so **BD** holds.

(iii) $\Rightarrow$ (i) Let $f : X \to \mathbb{R}$ be uniformly sequentially continuous. We want to show that f is uniformly continuous, so let $\varepsilon > 0$ be given. By Lemma 5.11, there is a pseudobounded $A \subseteq \mathbb{N}$ such that

$$\forall m > 0\big(\exists x, y \in X(|x - y| < 1/m \wedge |f(x) - f(y)| > \varepsilon) \Rightarrow m \in A\big).$$

By assumption A is bounded, i.e. there exists N such that $m < N$ for all $m \in A$. Let $x, y \in X$ with $|x - y| < 1/N$. Suppose that $|f(x) - f(y)| > \varepsilon$. Then $N \in A$, and hence $N < N$, a contradiction. Thus $|f(x) - f(y)| \leq \varepsilon$ and f is uniformly continuous. □

We get the following result for the converse of Theorem 5.7.

Theorem 5.13. *The following is equivalent:*

(i) $f : X \to \mathbb{R}$ *is uniformly continuous* $\Leftrightarrow \forall x, y \in X^+ \big(x \simeq y \Rightarrow f(x) \simeq f(y)\big)$.

(ii) **BD**.

Proof. "$\Rightarrow$" of (i) always holds by Theorem 5.7, and "$\Leftarrow$" of (i) is equivalent to (ii) by Theorem 5.12. □

Corollary 5.14.

$$f : X \to \mathbb{R} \text{ is uniformly continuous} \Leftrightarrow \forall x, y \in X^+ \big(x \simeq y \Rightarrow f(x) \simeq f(y)\big)$$

holds in ***CLASS***, ***INT*** *and* ***RUSS***, *but not in* ***BISH***.

Proof. By Theorems 3.29, 3.30, 3.31 and 3.28, **BD** holds in **CLASS**, **INT** and **RUSS** but not in **BISH**. □

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