Type Theory

Lecture 3: Metatheory of Type Theory

Fredrik Nordvall Forsberg
University of Strathclyde, Glasgow

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https://fredriknf.com/splv2025/

Course plan

- **Monday:** Using type theory.
- **Tuesday:** Semantics of type theory.
- **Thursday:** ~~Implementation~~ Models, and metatheory.
  - Some concrete models, and what they are good for
  - Canonicity and normalisation

Slides and exercises: `https://fredriknf.com/splv2025/`

Reminder: categories with families

Definition A category with families (CwF) is given by:

- A category $\mathcal{C}$ with a terminal object.
- A presheaf $\mathsf{Ty} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$.
- A presheaf $\mathsf{Tm} : (\int_{\mathcal{C}} \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$.
- A context extension $\Gamma \cdot A \in \mathcal{C}$ for every $\Gamma \in \mathcal{C}$ and $A \in \mathsf{Ty}(\Gamma)$ satisfying a certain universal property.

The Set model

We can take $\mathcal{C} = \mathbf{Set}$, the category of sets and functions.

We define $\mathsf{Ty}(\Gamma) :\equiv \Gamma \to \mathsf{Set}$.

Type substitution for $f : \Delta \to \Gamma$: $A[f] :\equiv A \circ f$.

We define $\mathsf{Tm}(\Gamma, A) :\equiv (\Pi\gamma \in \Gamma).A(\gamma)$.

Term substitution for $f : \Delta \to \Gamma$ and $t \in \mathsf{Tm}(\Gamma, A)$: $t[f]_\delta :\equiv t_{f(\delta)}$.

Finally we define $\Gamma \cdot A :\equiv (\Sigma\gamma \in \Gamma).A(\gamma)$ with $\mathsf{p} :\equiv \mathsf{fst}$, $\mathsf{q} :\equiv \mathsf{snd}$.

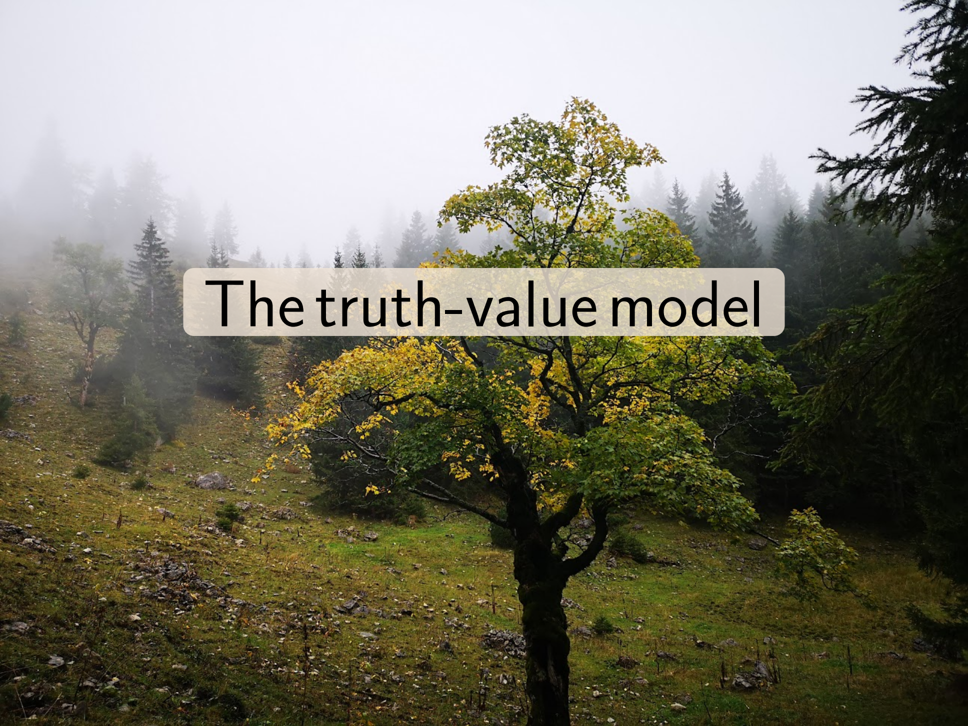
Some concrete models

Let us take a look at some concrete models and how they can be used for independence results:

- Smith's almost-trivial model (1988)
- Hofmann and Streicher's groupoid model (1994)
- A realizability model (see e.g. Beeson (1982))

Models such as the cubical sets model (Bezem, Coquand, and Huber 2013) can also inspire new syntax.

The truth-value model



Peano's Fourth Axiom

Using a universe, one can prove that $0 \neq \mathsf{suc}\, n$ for any $n : \mathbb{N}$.

Is it possible to prove this without using a universe?

Smith (1988) showed that this is impossible, by constructing a model where every type has at most one inhabitant.

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and

$$\begin{aligned} \text{Tm}(\Gamma, A) &:= \{\star \mid \Gamma \leq A\} \\ t[\sigma] &:= t \end{aligned}$$

That is, there is a (unique) term of type A unless $\Gamma = \text{true}$ and $A = \text{false}$.

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We take $\Gamma \cdot A := \Gamma \wedge A$, for which we can define $\mathsf{p} : \Gamma \wedge A \leq \Gamma$ and $\mathsf{q} = \star \in \text{Tm}(\Gamma \wedge A, A)$.

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\mathsf{Unit} &:= \mathsf{true} \\
\mathsf{Nat} &:= \mathsf{true} \\
\Pi\, A\, B &:= A \supset B \qquad \text{(Boolean implication)} \\
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Whenever we are asked to interpret a term, we can use $\star$ by construction.

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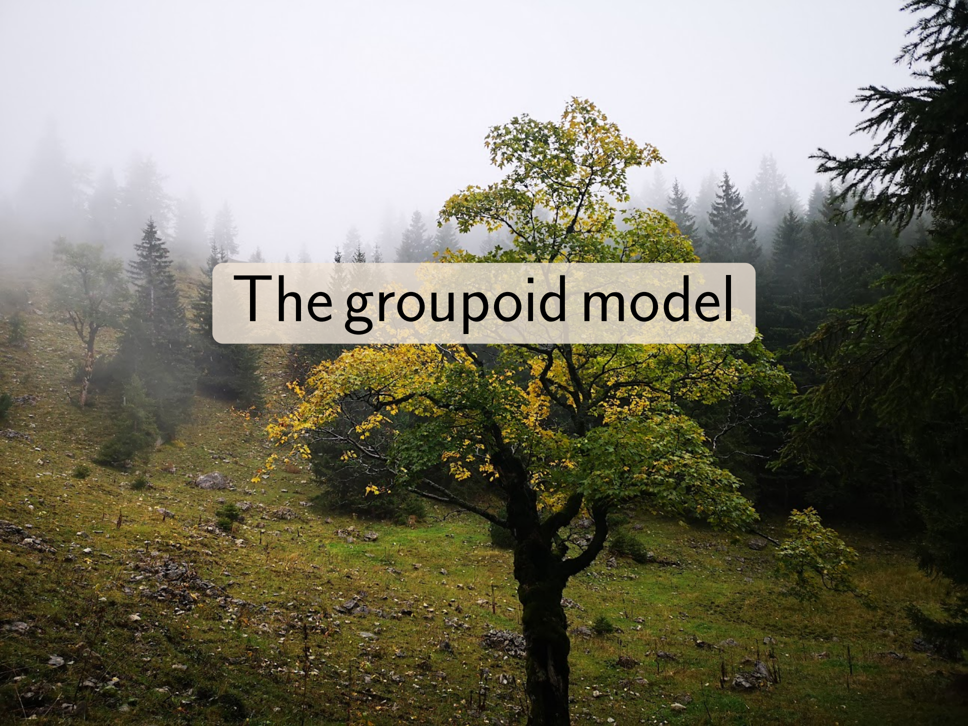
Note The model does not support universes, because they cannot afford to ignore all dependencies!

The groupoid model



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Also provable in a natural extension of type theory: Streicher's Axiom K (1993) or Coquand's dependent pattern matching (1992). (Mc Bride (1999) showed that in fact Axiom K and pattern matching are equivalent.)

Identities between identity proofs

Some equations are provable:

$$\begin{aligned}
\mathrm{trans}(p, \mathrm{refl}) &= p \\
\mathrm{trans}(\mathrm{refl}, q) &= q \\
\mathrm{trans}(\mathrm{trans}(p, q), r) &= \mathrm{trans}(p, \mathrm{trans}(q, r)) \\
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Hofmann's insight: we can turn this around and make a model out of groupoids!

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Terms $\mathrm{Tm}(\Gamma, A)$ are "dependent functors":

$$M_0 \in (\Pi\gamma \in \Gamma).A(\gamma)$$
$$M_1 \in (\Pi f : \gamma \to \gamma').\big(A(f)(M_0(\gamma)) \to M_0(\gamma')\big)$$

s.t. $M_1(\mathrm{id}_\gamma) = \mathrm{id}_{M_0(\gamma)}$ and $M_1(f \circ g) = M_1(f) \circ A(f)(M_1(g))$.
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We define $\Gamma \cdot A \coloneqq \int_\Gamma A$, i.e., objects are pairs $(\gamma \in \Gamma, a \in A(\gamma))$ and $(f, g) : (\gamma, a) \to (\gamma', a')$ if $f : \gamma \to \gamma'$ and $g : A(f)(a) \to a'$.

Interpreting identity types

We interpret $\mathsf{Id}\, A\, a\, b$ as the discrete groupoid with objects $\mathrm{Hom}_A(a, b)$. On morphisms, we define $(\mathsf{Id}\, A\, f\, g)(r) :\equiv g \circ r \circ f^{-1}$.

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C is a functor, so it suffices to construct a morphism $(x, x, \mathrm{id}_x) \to (x, y, r)$. Such a morphism is given by

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f &: x \to x && \text{in } A \\
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So we can take $f = \mathrm{id}$, $g = r$, $h = \mathrm{id}$, and define $\mathrm{elim}_=(d, r) :\equiv C(\mathrm{id}, r, \mathrm{id})(d(x))$. (We also need to define actions on morphisms.)

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In particular, this means that $A =_U B$ in the model iff $A \cong B$.
$\rightsquigarrow$ Precursor to the Univalence Axiom.

Refuting UIP

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We would then have

$$u(B\mathbb{Z}, \star, \star, 0, 1) \in \mathsf{Id}\,(\mathsf{Id}\,B\mathbb{Z}\;\star\;\star)\,0\,1$$

in the model, but $\mathsf{Id}\,B\mathbb{Z}\;\star\;\star$ is a discrete groupoid, hence $\mathsf{Id}\,(\mathsf{Id}\,B\mathbb{Z}\;\star\;\star)\,0\,1 = \emptyset$ since $0 \neq 1$. Hence no such proof u can exist.

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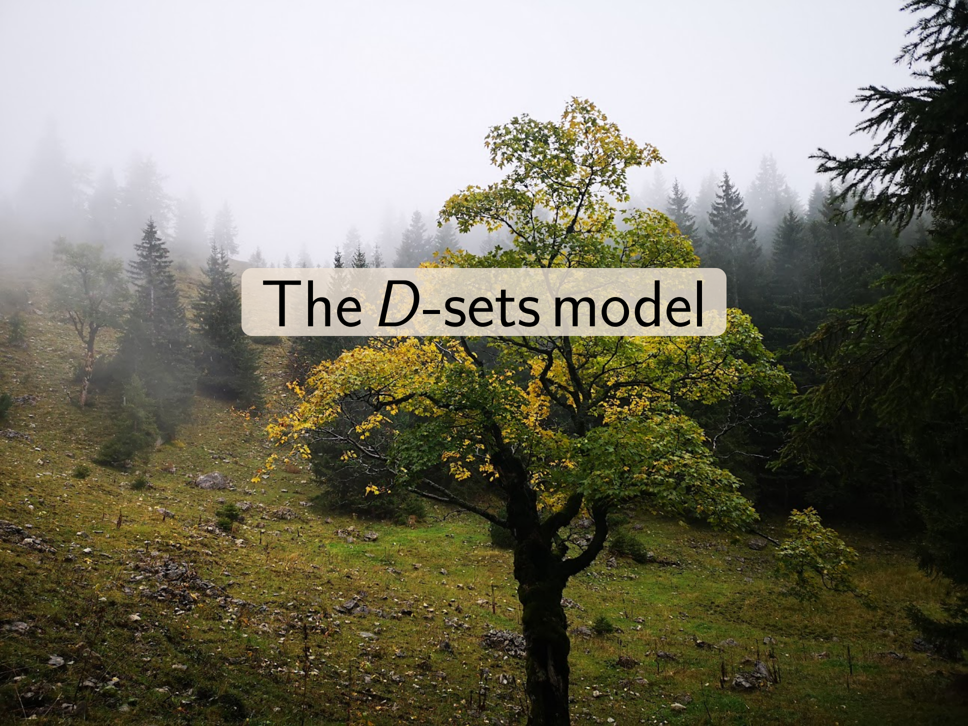
In the limit, we would rediscover Voevodsky's simplicial sets (aka ∞-groupoids) model of homotopy type theory (Kapulkin and Lumsdaine, 2021).

The D-sets model



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Definition A combinatory algebra is a set D with a binary operation $\$: D \times D \to D$ together with elements $K, S \in D$ such that

$$K \$ x \$ y = x \qquad\qquad S \$ x \$ y \$ z = (x \$ z) \$ (y \$ z)$$

(Can also work with *partial* combinatory algebras, i.e. $\$$ partial.)

Examples $D =$ untyped lambda terms, $D =$ an enumeration of Turing machines as natural numbers.

Functional completeness

D is functionally complete: for each term $t(x_1, \ldots, x_n) \in D$ there is $f \in D$ such that $f \mathbin{\$} a_1 \mathbin{\$} \ldots \mathbin{\$} a_n = t(a_1, \ldots, a_n)$.

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Hence we can do the usual Church encoding tricks and define pairing and projections: There are $\pi_1, \pi_2, <a, b> \in D$ such that

$$\pi_1 \mathbin{\$} a \mathbin{\$} b = a$$
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Hence $<a, b> \$ \pi_1 = a$ and $<a, b> \$ \pi_2 = b$.

Similarly we can define Church numerals c_n for natural numbers.

D-sets and their morphisms

Definition A D-set (or assembly) is a pair $(X, \Vdash_X)$, where X is a set and $\Vdash_X \subseteq D \times X$, such that for each $x \in X$, there exists $a \in D$ such that $a \Vdash_X x$.

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A morphism $(X, \Vdash_X) \to (Y, \Vdash_Y)$ is a function $X \to Y$ such that there exists $d \in D$ such that if $a \Vdash_X x$ then $d \, \$ \, a \Vdash_Y f(x)$.

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Terminology if $a \Vdash_X x$, we call a a realizer of x. We say that d above tracks f.

D-sets and their morphisms

Definition A D-set (or assembly) is a pair $(X, \Vdash_X)$, where X is a set and $\Vdash_X \subseteq D \times X$, such that for each $x \in X$, there exists $a \in D$ such that $a \Vdash_X x$.

A morphism $(X, \Vdash_X) \to (Y, \Vdash_Y)$ is a function $X \to Y$ such that there exists $d \in D$ such that if $a \Vdash_X x$ then $d \,\$\, a \Vdash_Y f(x)$.

Terminology if $a \Vdash_X x$, we call a a realizer of x. We say that d above tracks f.

There is an identity morphism, and D-set morphisms compose (easy by functional completeness).

The category of D-sets

The category of D-sets has lots of nice structure:

- Products $(X, \Vdash_X) \times (Y, \Vdash_Y) = (X \times Y, \Vdash)$ where $d \Vdash (x, y)$ iff $d = <a, b>$ such that $a \Vdash_X x$ and $b \Vdash_Y y$.
- Exponentials $(X, \Vdash_X) \Rightarrow (Y, \Vdash_Y)$ with underlying sets D-sets morphisms, and $d \Vdash f$ iff d tracks f.
- A natural numbers objects $(\mathbb{N}, \Vdash_{\mathbb{N}})$ where $d \Vdash_{\mathbb{N}} n$ iff $d = c_n$.
- Coproducts $(X_0, \Vdash_{X_0}) + (X_1, \Vdash_{X_1}) = (X_0 + X_1, \Vdash)$ where $d \Vdash \mathrm{in}_i\, x$ iff $d = <c_i, a>$ such that $a \Vdash_{X_i} x$.

D-sets as a CwF

We build a category with families structure on the category of D-sets.

We take

$$\begin{aligned} \mathrm{Ty}((X, \Vdash_X)) &:= X \to D\text{-Set} \\ \mathrm{Tm}((X, \Vdash_X), Y) &:= \{b : (\Pi x \in X).Y(x) \mid \exists d \in D.d \text{ tracks } b\} \end{aligned}$$

and define $(X, \Vdash_X) \cdot Y := ((\Sigma x \in X).Y(x), \Vdash)$ where $d \Vdash (x, y)$ iff $d = <a, b>$ such that $a \Vdash_X x$ and $b \Vdash_{Y(x)} y$.

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Using the categorical structure in D-Set, we interpret (dependent) functions and pairs, disjoint unions, natural numbers, etc.

An impredicative universe

There is an interesting subcategory of so-called modest D-sets:

Definition A D-set $(X, \Vdash_X)$ is modest if $d \Vdash_X x$ and $d \Vdash_X y$ implies $x = y$. (A family $Y : X \to D$-Set is called modest if each Y_x is modest.)

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Modest sets are isomorphic to partial equivalence relations on D, hence "all small". Thus: if $B \in \mathrm{Ty}(\Gamma \cdot A)$ is modest then $\Pi\, A\, B \in \mathrm{Ty}(\Gamma)$ is modest, for all $A \in \mathrm{Ty}(\Gamma)$.

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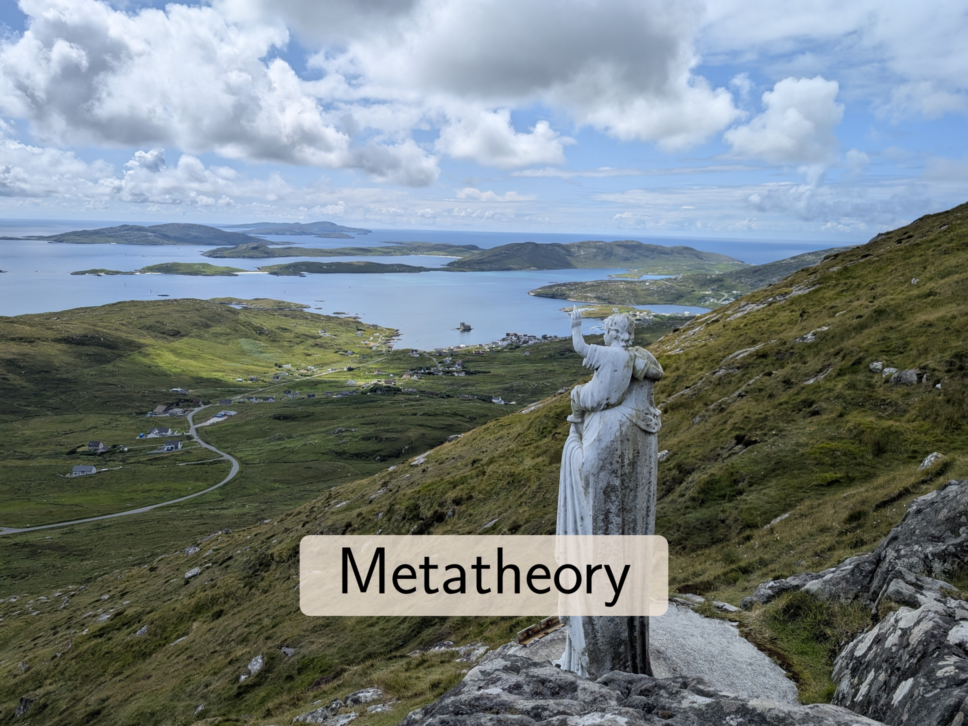
Modest sets form a universe closed under impredicative quantification, containing the natural numbers. Such a universe contradicts classical logic.


Metatheory

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- **Consistency:** There is no proof of $\mathbf{0}$ in the empty context.
- **Canonicity:** Every closed term of type $\mathbb{N}$ is equal to a numeral $\mathsf{suc}^n\,0$.
- **Normalisation:** Every term is equal to a term in *normal form*.
- (**Strong normalisation:** Every term reduces to a term in normal form, no matter the reduction strategy.)

Consistency

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If you propose an extension to a type theory, you want to know/show that it is still consistent. But there is not much you can do with a proof of consistency.

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How do we prove it? Unfortunately, a naive induction on typing judgements does not work.

A "proof-relevant" logical relation (Coquand 2019)

To each (closed) type A we associate a family of *sets* $A' : A \to \mathsf{Set}$ of "proofs of canonicity".

To each closed term $t : A$, we associate an element $t' \in A'(t)$.

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$$\begin{aligned} \mathbb{N}'(t) &\coloneqq \{n \mid t \equiv \mathrm{suc}^n\, 0\} \\ \big((\Pi x : A).B\big)'(t) &\coloneqq (\Pi a : A)(\Pi a' : A'(a)).B'(a, a')\,(t\, a) \end{aligned}$$

$$\begin{aligned} (t\, a)' &\coloneqq t'\, a\, a' \\ \big((\lambda x : A).t\big)' &\coloneqq (\lambda a : A)(\lambda a' : A'(a)).t'\, a\, a' \\ (\mathrm{suc}\, n)' &\coloneqq n' + 1 \\ &\vdots \end{aligned}$$

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By induction on derivations, we can show that if $\vdash a : A$ then $a' \in A'(t)$ and if $\vdash a \equiv b : A$ then $a' = b'$. (Need to generalise statement to closing substitutions.) In particular if $\vdash t : \mathbb{N}$ then $t \equiv \mathsf{suc}^n\, 0$ for some n.

A more structured approach?

We can organise the argument as follows:

For each model $\mathcal{M} = (\mathcal{C}, \mathsf{Ty}, \mathsf{Tm})$, we build a new "canonicity" model $\mathcal{M}^* = (\mathcal{C}^*, \mathsf{Ty}^*, \mathsf{Tm}^*)$ together with a model morphism $\mathcal{M}^* \to \mathcal{M}$.

This way, it is easier to not accidentally forget a clause.

The “canonicity” model

The objects of $\mathcal{C}^*$ are pairs (Γ, Γ') where $\Gamma \in \mathcal{C}$ and $\Gamma' : \mathrm{Hom}_{\mathcal{C}}(1, \Gamma) \to \mathrm{Set}$, with $1^* = (1, \lambda\_.\mathbf{1})$.

Morphisms are pairs (σ, σ') where

$$\begin{aligned} \sigma &: \Delta \to \Gamma \\ \sigma' &: (\Pi\tau : 1 \to \Delta).\big(\Delta'(\tau) \to \Gamma'(\sigma \circ \tau)\big) \end{aligned}$$

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We define $\mathrm{Ty}^*(\Gamma, \Gamma')$ to be the set of pairs (A, A') where

$$\begin{aligned} A &\in \mathrm{Ty}(\Gamma) \\ A' &\in (\Pi \sigma : 1 \to \Gamma)\big(\Gamma'(\sigma) \to \mathrm{Tm}(1, A[\sigma]) \to \mathrm{Set}\big) \end{aligned}$$

Similarly $\mathrm{Tm}^*((\Gamma, \Gamma'), (A, A'))$ consists of (t, t') such that

$$\begin{aligned} t &\in \mathrm{Tm}(\Gamma, A) \\ t' &\in (\Pi \sigma : 1 \to \Gamma)(\Pi \sigma' \in \Gamma'(\sigma)).A' \, \sigma \, \sigma' \, (t[\sigma]) \end{aligned}$$

Canonicity from $\mathcal{M}^*$

If $\mathcal{M}$ has natural numbers $\mathsf{Nat} \in \mathsf{Ty}(\Gamma)$, we can define $(\mathsf{Nat}, \mathsf{Nat}') \in \mathsf{Ty}^*(\Gamma, \Gamma')$ where

$$\mathsf{Nat}'\,\sigma\,\sigma'\,t :\equiv \{n \mid t \equiv \mathsf{suc}^n\,0\}$$

and similarly for other type and term constructors. The model morphism $\pi : \mathcal{M}^* \to \mathcal{M}$ is given by first projection.

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Theorem In the syntax, every closed term of type $\mathbb{N}$ is (judgementally) equal to a numeral $\mathsf{suc}^n\,0$.

Proof: The syntax forms an initial model $\mathcal{M}_0$. We thus have a map $i : \mathcal{M}_0 \to \mathcal{M}_0^*$, and $\pi \circ i = \mathsf{id}_{\mathcal{M}_0}$ by initiality. For closed terms $t : \mathbb{N}$ we thus have $t' \in \mathbb{N}' \star \star\, t$ so $t \equiv \mathsf{suc}^n\,0$ for some $n \in \mathbb{N}$. □

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Even more abstractly, this model construction is an instance of gluing for CwFs (Kaposi, Huber, and Sattler 2019) .

Summary

We have seen four models of type theory in the CwF framework:

1. Truth-value model demonstrating the independence of $0 = \mathsf{suc}\, n$ without universes.
2. Groupoid model demonstrating the independence of UIP, and suggesting the "universe extensionality axiom"
3. D-sets model enabling the extraction of computable data, and demonstrating the independence of classical logic.
4. Canonicity model allowing us to derive canonicity.

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