Variations on inductive-recursive definitions

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Joint work with Neil Ghani, Conor McBride, Peter Hancock and Stephan Spahn
An inductive definition

```
data Rose (A : Set) : Set where
  leaf : Rose A
  node : A → List (Rose A) → Rose A
```

We can represent \( \text{Rose } A \) by a functor \( F_{\text{Rose}} : \text{Set} → \text{Set} \):

\[
F_{\text{Rose}}(X) = 1 + A \times \text{List } X
\]
An inductive definition

```haskell
data Rose (A : Set) : Set where
  leaf : Rose A
  node : A → List (Rose A) → Rose A
```

We can represent \( \text{Rose} \ A \) by a functor \( F_{\text{Rose}} : \text{Set} \to \text{Set} \):

\[
F_{\text{Rose}}(X) = 1 + A \times \text{List} \ X
\]

\( \text{Rose} \ A \) is the initial algebra of \( F_{\text{Rose}} \).
An inductive-recursive definition

A universe closed under $\mathbb{N}$ and $\Sigma$.

```plaintext
data U : Set
T : U → Set

data U where
  nat : U
  sig : (a : U) → (b : T a → U) → U

T nat = \mathbb{N}
T (sig a b) = \Sigma (T a) (T \circ b)
```
An inductive-recursive definition

A universe closed under $\mathbb{N}$ and $\Sigma$.

\[
\text{data } U : \text{Set} \\
T : U \rightarrow \text{Set}
\]

\[
\text{data } U \text{ where} \\
\quad \text{nat} : U \\
\quad \text{sig} : (a : U) \rightarrow (b : T \ a \rightarrow U) \rightarrow U
\]

\[
T \text{ nat} = \mathbb{N} \\
T \ (\text{sig} \ a \ b) = \Sigma (T \ a) (T \circ b)
\]

$U$ and $T$ defined simultaneously.
An inductive-recursive definition

A universe closed under $\mathbb{N}$ and $\Sigma$.

```haskell
data U : Set
T : U → Set

data U where
  nat : U
  sig : (a : U) → (b : T a → U) → U

T nat = N
T (sig a b) = Σ (T a) (T • b)
```

$U$ and $T$ defined simultaneously.

Also $(U, T)$ is the initial algebra of a functor.
Category of families of $Ds$

The category $\text{Fam } D$ for $D : \text{Set}_1$:

- objects pairs $(U, T)$ where

$$U : \text{Set}$$
$$T : U \to D$$

- morphisms $(U, T) \to (U', T')$ are $f : U \to U'$ s.t.

$$U \quad f \quad U'$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$T \quad \quad \quad \quad \quad T'$$
$$\quad \quad \quad \quad \quad D$$

commutes.

Note: $\text{Fam} : \text{Cat} \to \text{Cat}$ is a monad; $D$ considered as discrete category.
An endofunctor on \textit{Fam Set}

\begin{verbatim}
data U : Set where
  nat : U
  sig : (a : U) → (b : T a → U) → U

T : U → Set
T nat = \texttt{N}
T (sig a b) = Σ (T a) (T ∘ b)
\end{verbatim}

is represented by \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

\begin{verbatim}
F(X, Q) = (1, _ ↦ \texttt{N}) + (\Σ a : X)(Q a → X), (a, b) ↦ Σ (Q a)(Q ∘ b))
\end{verbatim}
An endofunctor on \( \text{Fam Set} \)

\[
\begin{align*}
data \ U : \text{Set} & \quad \text{where} \\
nat : U & \\
sig : (a : U) \rightarrow (b : T a \rightarrow U) \rightarrow U
\end{align*}
\]

\[
\begin{align*}
T : U & \rightarrow \text{Set} \\
T \text{ nat} & = N \\
T (\text{sig } a \ b) & = \Sigma (T a) (T \circ b)
\end{align*}
\]

is represented by \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

\[
F(X, Q) = (1, \_ \mapsto N) + ((\Sigma a : X)(Q a \rightarrow X), (a, b) \mapsto \Sigma (Q a) (Q \circ b))
\]

\((U, T)\) is the initial algebra of \( F \).
Representing inductive definitions

Not every functor defines a datatype. We want our inductive definitions to be strictly positive.
Representing inductive definitions

Not every functor defines a datatype. We want our inductive definitions to be strictly positive.

We can codify such definitions as follows (baby Dybjer-Setzer [1999, 2003, 2006]):

```
data ID : Set₁ where
  stop : ID
  side : (A : Set) → (c : A → ID) → ID
  ind : (A : Set) → (c : ID) → ID
```
Representing inductive definitions

Not every functor defines a datatype. We want our inductive definitions to be strictly positive.

We can codify such definitions as follows (baby Dybjer-Setzer [1999, 2003, 2006]):

```haskell
data ID : Set₁ where
  stop : ID
  side : (A : Set) → (c : A → ID) → ID
  ind : (A : Set) → (c : ID) → ID
```

Each code gives rise to a functor:

$$\llbracket \_ \rrbracket : \text{ID} \to (\text{Set} \to \text{Set})$$

$$\llbracket \text{stop} \rrbracket X = 1$$

$$\llbracket \text{side } A \; c \rrbracket X = (\Sigma x : A)(\llbracket c \; x \rrbracket X)$$

$$\llbracket \text{ind } A \; c \rrbracket X = (A \to X) \times \llbracket c \rrbracket X$$
A code for List A

\[
\text{stop : ID} \\
\text{side : } (A : \text{Set}) \to (c : A \to \text{ID}) \to \text{ID} \\
\text{ind : } (A : \text{Set}) \to (c : \text{ID}) \to \text{ID}
\]

\[
[\text{stop}] X = 1 \\
[\text{side } A c] X = (\Sigma x : A)[c x] X \\
[\text{ind } A c] X = (A \to X) \times [c] X
\]
A code for List A

stop : ID
side : (A : Set) → (c : A → ID) → ID
ind : (A : Set) → (c : ID) → ID

\[ \begin{align*}
\text{stop} & : \text{ID} \\
\text{side} & : (A : \text{Set}) \rightarrow (c : A \rightarrow \text{ID}) \rightarrow \text{ID} \\
\text{ind} & : (A : \text{Set}) \rightarrow (c : \text{ID}) \rightarrow \text{ID}
\end{align*} \]

\[ \begin{align*}
\text{stop} & \colon X = 1 \\
\text{side} A c & \colon X = (\Sigma x : A) [c x] X \\
\text{ind} A c & \colon X = (A \rightarrow X) \times [c] X
\end{align*} \]

The datatype

\[ \textbf{data } \text{List } (A : \text{Set}) : \text{Set} \text{ where } \]
\[ [] : \text{List } A \\
_:_ : A \rightarrow \text{List } A \rightarrow \text{List } A \]

is represented by

\[ c_{\text{List}} = \text{side } \{ \text{'[]}, \text{'::} \} (\text{'[]} \mapsto \text{stop}; \text{'::} \mapsto \text{side } A (\_ \mapsto \text{ind } 1 \text{ stop})) \]
A code for List A

\[
\begin{align*}
\text{stop} &: \text{ID} \\
\text{side} &: (A : \text{Set}) \to (c : A \to \text{ID}) \to \text{ID} \\
\text{ind} &: (A : \text{Set}) \to (c : \text{ID}) \to \text{ID}
\end{align*}
\]

\[
\begin{align*}
[\text{stop}] X &= 1 \\
[\text{side} A c] X &= (\Sigma x : A)[c x] X \\
[\text{ind} A c] X &= (A \to X) \times [c] X
\end{align*}
\]

The datatype

\[
\textbf{data} \ \text{List} \ (A : \text{Set}) : \text{Set} \ \text{where} \\
[] &: \text{List} \ A \\
\_::\_ &: A \to \text{List} \ A \to \text{List} \ A
\]

is represented by

\[
\alpha_{\text{List}} = \text{side} \{', []', '::'\} ('[]' \mapsto \text{stop}; '::' \mapsto \text{side} A (\_ \mapsto \text{ind} 1 \ \text{stop}))
\]

Note: \(\text{side} \{\text{tag}_c, \text{tag}_d\} (\text{tag}_c \mapsto c; \text{tag}_d \mapsto d)\) for encoding coproducts of codes.
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors \( \text{Fam } D \rightarrow \text{Fam } E \):
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors \( \text{Fam } D \to \text{Fam } E \):

\[
\text{data } \mathbf{ID} : \text{Set}_1 \ \text{where}
\]
\[
\begin{align*}
\text{stop} & : \mathbf{ID} \\
\text{side} & : (A : \text{Set}) \to (c : A \to \mathbf{ID}) \to \mathbf{ID} \\
\text{ind} & : (A : \text{Set}) \to (c : \mathbf{ID}) \to \mathbf{ID}
\end{align*}
\]
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam } D \to \text{Fam } E$:

```
data DS (D E : Set₁) : Set₁ where
  stop : DS D E
  side : (A : Set) → (c : A → DS D E) → DS D E
  ind : (A : Set) → (c : DS D E) → DS D E
```
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam } D \to \text{Fam } E$:

\[
\text{data } \text{DS } (D \ E : \text{Set}_1) : \text{Set}_1 \ where \\
\iota : E \to DS \ D \ E \\
\text{side} : (A : \text{Set}) \to (c : A \to DS \ D \ E) \to DS \ D \ E \\
\text{ind} : (A : \text{Set}) \to (c : DS \ D \ E) \to DS \ D \ E
\]
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam} \; D \rightarrow \text{Fam} \; E$:

\[
\textbf{data} \; \text{DS} \; (D \; E : \; \text{Set}_1) : \; \text{Set}_1 \; \textbf{where}
\]
\[
\iota : \; E \rightarrow \text{DS} \; D \; E
\]
\[
\sigma : (A : \; \text{Set}) \rightarrow (c : \; A \rightarrow \text{DS} \; D \; E) \rightarrow \text{DS} \; D \; E
\]
\[
\text{ind} : (A : \; \text{Set}) \rightarrow (c : \; \text{DS} \; D \; E) \rightarrow \text{DS} \; D \; E
\]

Note: $\text{Fam}_1 \sim \text{Set}$ and $\text{DS}_1 \sim \text{ID}$. 
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam } D \rightarrow \text{Fam } E$:

```haskell
data DS (D E : Set₁) : Set₁ where
  ι : E → DS D E
  σ : (A : Set) → (c : A → DS D E) → DS D E
  δ : (A : Set) → (c : (A → D) → DS D E) → DS D E
```

Note: $\text{Fam } 1 \sim = \text{Set}$ and $\text{DS } 1 1 \sim = \text{ID}$. 
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam } D \to \text{Fam } E$:

```
data DS (D E : Set₁) : Set₁ where
  ι : E → DS D E
  σ : (A : Set) → (c : A → DS D E) → DS D E
  δ : (A : Set) → (c : (A → D) → DS D E) → DS D E
```

```
[J_] : DS D E → Fam D → Fam E
```

```
[J_ ι e] (U, T) = (1, ⋆ ↦ e)
[J_ σ A f] (U, T) = (∑ a : A) ([f a] (U, T))
[J_ δ A F] (U, T) = (∑ g : A → U) ([F (T ◦ g)] (U, T))
```
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam } D \rightarrow \text{Fam } E$:

```
data DS (D E : Set) : Set where
  ι : E → DS D E
  σ : (A : Set) → (c : A → DS D E) → DS D E
  δ : (A : Set) → (c : (A → D) → DS D E) → DS D E
```

 coproducts in $\text{Fam } D$

```
⟦ ___ ⟧ : DS D E → Fam D → Fam E

⟦ ι e ⟧ (U, T) = (1, * ↦ e)

⟦ σ A f ⟧ (U, T) = (Σ a : A) (⟦ f a ⟧ (U, T))

⟦ δ A F ⟧ (U, T) = (Σ g : A ↠ U) (⟦ F (T ◦ g) ⟧ (U, T))
```
Representing inductive-recursive definitions

Dybjer-Setzer codes for functors $\text{Fam } D \rightarrow \text{Fam } E$:

**data** $\text{DS } (D \ E : \text{Set}_1) : \text{Set}_1$ **where**

\[
\begin{align*}
\iota &: E \rightarrow DS \ D \ E \\
\sigma &: (A : \text{Set}) \rightarrow (c : A \rightarrow DS \ D \ E) \rightarrow DS \ D \ E \\
\delta &: (A : \text{Set}) \rightarrow (c : (A \rightarrow D) \rightarrow DS \ D \ E) \rightarrow DS \ D \ E
\end{align*}
\]

\[
[\_] : DS \ D \ E \rightarrow \text{Fam } D \rightarrow \text{Fam } E
\]

\[
\begin{align*}
[\iota \ e] (U, T) &= (1, \star \mapsto e) \\
[\sigma \ A \ f] (U, T) &= (\Sigma a : A)([f \ a] (U, T)) \\
[\delta \ A \ F] (U, T) &= (\Sigma g : A \rightarrow U)([F (T \circ g)] (U, T))
\end{align*}
\]

Note: $\text{Fam } 1 \cong \text{Set}$ and $\text{DS } 1 1 \cong \text{ID}$. 
A code for a universe

The code

\[ c_{SN} = \sigma \{\text{nat, sig}\} \ (\text{nat} \mapsto \iota \ N; \ \text{sig} \mapsto \delta 1 \ (X \mapsto (\delta (X \star) (Y \mapsto \iota (\Sigma (X \star) Y)))))) \]

represents \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

\[
F(U, T) = (1, \star \mapsto \mathbb{N}) + ((\Sigma s : U)(Ts \rightarrow U), (s, p) \mapsto \Sigma (Ts)(T \circ p))
\]
A code for a universe

The code

\[ c_{\Sigma \mathbb{N}} = \sigma \{ \text{nat, sig} \} \ (\text{nat} \mapsto \iota \mathbb{N}; \]  
\[ \text{sig} \mapsto \delta 1 (X \mapsto (\delta (X *) (Y \mapsto \iota (\Sigma (X *) Y)))) \]

represents \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

\[ F(U, T) = \]
\[ (1, * \mapsto \mathbb{N}) + ((\Sigma s : U)(Ts \rightarrow U), (s, p) \mapsto \Sigma (Ts)(T \circ p)) \]
A code for a universe

The code

$$
c_{\Sigma N} = \sigma \{\text{nat}, \text{sig}\} (\text{nat} \mapsto \iota \mathbb{N};
\text{sig} \mapsto \delta 1 (X \mapsto (\delta (X \star) (Y \mapsto \iota (\Sigma (X \star) Y)))))))
$$

represents $F : \text{Fam Set} \rightarrow \text{Fam Set}$ where

$$
F(U, T) =
(1, \star \mapsto \mathbb{N}) + ((\Sigma s : 1 \rightarrow U) (T(s \star) \rightarrow U), (s, p) \mapsto \Sigma (T (s \star)) (T \circ p))
$$
A code for a universe

The code

\[
c_{\Sigma \mathbb{N}} = \sigma \{ \text{nat}, \text{sig} \} (\text{nat} \mapsto \iota \mathbb{N};
\]

\[
\text{sig} \mapsto \delta 1 (X \mapsto (\delta (X \star) (Y \mapsto \iota (\Sigma (X \star) Y))))
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represents \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

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F(U, T) =
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A code for a universe

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represents \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

\[ F(U, T) = \]
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A code for a universe

The code

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c_{\Sigma N} = \sigma \{\text{nat, sig}\} (\text{nat} \mapsto \iota N; \\
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represents \( F : \text{Fam Set} \rightarrow \text{Fam Set} \) where

\[
F(U, T) = \\
(1, \star \mapsto N) + ((\Sigma s : 1 \rightarrow U) (T(s \star) \rightarrow U), (s, p) \mapsto \Sigma (T(s \star)) (T \circ p))
\]
Closure under composition?

DS codes represent functors; are they closed under composition?

That is, given $c : \text{DS} \ C \ D$ and $d : \text{DS} \ D \ E$, is there a code $d \cdot c : \text{DS} \ C \ E$ representing $[d] \circ [c] : \text{Fam} \ C \to \text{Fam} \ E$?
Closure under composition?

DS codes represent functors; are they closed under composition?

That is, given \( c : \text{DS} \ C \ D \) and \( d : \text{DS} \ D \ E \), is there a code \( d \bullet c : \text{DS} \ C \ E \) representing \([d] \circ [c] : \text{Fam} \ C \to \text{Fam} \ E\)?

Why care?
Closure under composition?

DS codes represent functors; are they closed under composition?

That is, given $c : \text{DS } C D$ and $d : \text{DS } D E$, is there a code $d \bullet c : \text{DS } C E$ representing $[d] \circ [c] : \text{Fam } C \to \text{Fam } E$?

Why care?

- Modularity: plug in $c$ later.
Closure under composition?

**DS** codes represent functors; are they closed under composition?

That is, given \( c : DS \ C \ D \) and \( d : DS \ D \ E \), is there a code \( d \bullet c : DS \ C \ E \) representing \([d] \circ [c] : Fam \ C \to Fam \ E\)?

**Why care?**

- Modularity: plug in \( c \) later.
- Solve \( F(G(X)) \cong X \), not just \( F(X) \cong X \). E.g. \( c_{\text{Rose}} = c_{\text{List}} \bullet c_{\text{List}} \).
Closure under composition?

**DS** codes represent functors; are they closed under composition?

That is, given \( c : DS \ C \ D \) and \( d : DS \ D \ E \), is there a code \( d \bullet c : DS \ C \ E \) representing \([d] \circ [c] : \text{Fam} \ C \to \text{Fam} \ E\)?

**Why care?**

- **Modularity**: plug in \( c \) later.
- **Solve** \( F(G(X)) \cong X \), not just \( F(X) \cong X \). E.g. \( c_{\text{Rose}} = c_{\text{List}} \bullet c_{\text{List}} \).
- **Longer term goal**: want **syntax-independent** characterisation of induction-recursion (cf polynomial functors [Gambino and Kock]) — will likely be closed under composition.
A proof attempt

Define $d \bullet c$ by induction on $d$:
A proof attempt

Define $d \bullet c$ by induction on $d$:

Since $[[\iota \ e]] ([[c]] (U, T)) = (1, \star \mapsto e)$,

$$(\iota \ e) \bullet c = \iota \ e$$

is easy.
A proof attempt

Define \( d \bullet c \) by induction on \( d \):

Since \( \llbracket \iota \ e \rrbracket (\llbracket c \rrbracket (U, T)) = (1, \star \mapsto e) \),

\[
(\iota \ e) \bullet c = \iota \ e
\]
is easy.

Similarly \( (\sigma \ A \ f) \bullet c = \sigma \ A (a \mapsto (f \ a) \bullet d) \) by the induction hypothesis.
A proof attempt

Define \( d \odot c \) by induction on \( d \):

Since \([\iota \ e] ([c] (U, T)) = (1, \ast \mapsto e)\),

\[
(\iota \ e) \odot c = \iota \ e
\]

is easy.

Similarly \((\sigma \ A \ f) \odot c = \sigma \ A (a \mapsto (f \ a) \odot d)\) by the induction hypothesis.

But what about \( \delta \)? (So far, we can compose with constant functors... )
Composing with $\delta$

$$[\delta A F]_0([c]_0 Z) = (\sum g : A \rightarrow [c]_0 Z)([F([c]_1(Z) \circ g)]_0([c] Z))$$
Composing with $\delta$

$$[\delta \ A \ F]_0([c]_0 Z) = (\Sigma g : A \to [c]_0 Z)([F([c]_1 (Z) \circ g)]_0([c] Z))$$

Progress could be made if we had

1. $A \longrightarrow c$

2. “Concatenation” of codes
Composing with $\delta$

$$[\delta A F]_0([c]_0Z) = (\Sigma g : A \to [c]_0Z)([F([c]_1(Z) \circ g)]_0([c]Z))$$

Progress could be made if we had

1. $A \rightarrow c$

2. “Concatenation” of codes

Spoiler alert: these are also necessary conditions.
“Concatenation” of codes

Item 2 is easy, because $\text{DS } D$ is a monad (Ghani and Hancock [2016]):

**Proposition.** There is an operation

$\_ \ggg \_ : \text{DS } C \ D \to (D \to \text{DS } C \ E) \to \text{DS } C \ E$

such that $\llbracket \_ \ggg g \rrbracket Z \cong \llbracket c \rrbracket Z \ggg \text{Fam} (e \mapsto \llbracket g \ e \rrbracket Z)$. 

□
"Concatenation" of codes

Item 2 is easy, because $DS \ D$ is a monad (Ghani and Hancock [2016]):

**Proposition.** There is an operation

$$\_ \ggg \_ : DS \ C \ D \rightarrow (D \rightarrow DS \ C \ E) \rightarrow DS \ C \ E$$

such that $[[c \ggg g]] Z \cong [[c]] Z \ggg_{Fam} (e \mapsto [[g \ e]] Z)$. 
Concretely,

$$[[c \ggg g]]_0 Z = \left( \Sigma x : [[c]]_0 Z \right) [[g ([[c]]_1 Z x)]]_0 Z$$

$$[[c \ggg g]]_1 Z (x, y) = [[g([[c]]_1 Z x)]]_1 Z y$$
Trying to define $S \rightarrow c$

This time $\iota$ and $\delta$ are easy, but:

$$S \rightarrow [\sigma \, A \, f]_0 Z = S \rightarrow (\Sigma a : A)([f \, a]_0 Z) \cong (\Sigma g : S \rightarrow A)((x : S) \rightarrow [f \, (g \, x)]_0 Z)$$
Trying to define $S \rightarrow c$

This time $\iota$ and $\delta$ are easy, but:

$$S \rightarrow [\sigma A f]_0 Z = S \rightarrow (\Sigma a : A)((f a)_0 Z)$$

$$\cong (\Sigma g : S \rightarrow A)((x : S) \rightarrow [f (g x)]_0 Z)$$

To continue inductively, we need to generalise to a dependent product

$$\pi : (S : \text{Set}) \rightarrow (S \rightarrow DS DE) \rightarrow DS DE$$
Trying to define $S \rightarrow c$

This time $\iota$ and $\delta$ are easy, but:

$$S \rightarrow [[\sigma \ A \ f]\_0 \ Z = S \rightarrow (\Sigma a : A)([[f \ a]\_0 \ Z)$$

$$\cong (\Sigma g : S \rightarrow A)((x : S) \rightarrow [[f \ (g \ x)]\_0 \ Z)$$

To continue inductively, we need to generalise to a dependent product

$$\pi : (S : \text{Set}) \rightarrow (S \rightarrow DS \ D \ E) \rightarrow DS \ D \ E$$

**But** we cannot define this because we have nothing to induct on anymore.
Powers from composition

In fact, any definition of composition would give us powers:

**Theorem.** A composition operator

\[
_\bullet_ : DS \, D \, E \rightarrow DS \, C \, D \rightarrow DS \, C \, E
\]

is definable if and only if a power operator

\[
\_ \rightarrow \_ : (S : Set) \rightarrow DS \, D \, E \rightarrow DS \, D \,(S \rightarrow E)
\]

is definable.
Powers from composition

In fact, any definition of composition would give us powers:

**Theorem.** A composition operator

\[
_\ ullet _\ : \ DS \ D \ E \ \rightarrow \ DS \ C \ D \ \rightarrow \ DS \ C \ E
\]

is definable if and only if a power operator

\[
_\ \rightarrow _\ : \ (S : \text{Set}) \ \rightarrow \ DS \ D \ E \ \rightarrow \ DS \ D \ (S \ \rightarrow \ E)
\]

is definable.

This (apparent) lack of powers thus suggests that \( DS \), as an axiomatisation of a class of functors, could perhaps be improved upon.
Variations on inductive-recursive definitions

This leads us to investigate alternative classes of functors axiomatising inductive-recursive definitions.

If one wants closure under composition, two natural options suggest themselves:

1. Restrict dependency so that $S \rightarrow c$ is definable $\leadsto$ uniform codes (Peter Hancock).

2. Add a $\pi$ combinator to the system $\leadsto$ polynomial codes (Conor McBride).
Variations on inductive-recursive definitions

This leads us to investigate alternative classes of functors axiomatising inductive-recursive definitions.

If one wants closure under composition, two natural options suggest themselves:

1. Restrict dependency so that \( S \rightarrow c \) is definable \( \rightsquigarrow \) uniform codes (Peter Hancock).

2. Add a \( \pi \) combinator to the system \( \rightsquigarrow \) polynomial codes (Conor McBride).

Take-home message: There are many axiomatisations of induction-recursion.
Uniform codes
Uniform codes

Originally due to Peter Hancock (2012).

Discovered while trying to define composition for DS.
Uniformity by associating like in the 60s

In

\[ \sigma : (A : \text{Set}) \rightarrow (c : A \rightarrow \text{DSDE}) \rightarrow \text{DSDE} \]

nonuniformity comes from \( c \) depending on \( A \).
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Left-nested instead of right-nested (Pollack: Dependently Typed Records in Type Theory [2002]).
Uniform codes $\text{UF}$

Let $D, E : \text{Set}_1$. $\text{Uni } D : \text{Set}_1$ and $\text{Info} : \text{Uni } D \to \text{Set}_1$ are inductive-recursively given by

$$
\iota_{\text{UF}} : \text{Uni } D \\
\sigma_{\text{UF}} : (c : \text{Uni } D) \to (A : \text{Info } c \to \text{Set}) \to \text{Uni } D \\
\delta_{\text{UF}} : (c : \text{Uni } D) \to (A : \text{Info } c \to \text{Set}) \to \text{Uni } D
$$

$$
\text{Info } \iota_{\text{UF}} = 1 \\
\text{Info } (\sigma_{\text{UF}} c A) = (\Sigma \gamma : \text{Info } c)(A \gamma) \\
\text{Info } (\delta_{\text{UF}} c A) = (\Sigma \gamma : \text{Info } c)(A \gamma \to D)
$$

Large set of uniform codes $\text{UF } D E = (\Sigma c : \text{Uni } D)(\text{Info } c \to E)$. 
Decoding uniform codes

\[ [\_\_]_{\text{Uni}} : \text{Uni } D \rightarrow \text{Fam } D \rightarrow \text{Set} \]
\[ [\_\_]_{\text{Info}} : (c : \text{Uni } D) \rightarrow (Z : \text{Fam } D) \rightarrow [c]_{\text{Uni}} Z \rightarrow \text{Info } c \]
Decoding uniform codes

\[ \llbracket \_ \rrbracket_{\text{Uni}} : \text{Uni} \ D \rightarrow \text{Fam} \ D \rightarrow \text{Set} \]
\[ \llbracket \_ \rrbracket_{\text{Info}} : (c : \text{Uni} \ D) \rightarrow (Z : \text{Fam} \ D) \rightarrow \llbracket c \rrbracket_{\text{Uni}} Z \rightarrow \text{Info} \ c \]

\[
\llbracket \nu_{\text{UF}} \rrbracket_{\text{Uni}} (U, T) = 1
\]
\[
\llbracket \sigma_{\text{UF}} \ c \ A \rrbracket_{\text{Uni}} (U, T) = (\Sigma x : \llbracket c \rrbracket_{\text{Uni}} (U, T))(A(\llbracket c \rrbracket_{\text{Info}} (U, T) x))
\]
\[
\llbracket \delta_{\text{UF}} \ c \ A \rrbracket_{\text{Uni}} (U, T) = (\Sigma x : \llbracket c \rrbracket_{\text{Uni}} (U, T))(A(\llbracket c \rrbracket_{\text{Info}} (U, T) x) \rightarrow U)
\]

\[ \vdots \]

\[
\llbracket \delta_{\text{UF}} \ c \ S \rrbracket_{\text{Info}} (U, T) (x, g) = (\llbracket c \rrbracket_{\text{Info}} (U, T) x, T \circ g)
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\[ \vdots \]

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Finally for \((c, \alpha) : \text{UF } D E = (\Sigma c : \text{Uni } D)(\text{Info } c \to E)\)
\[ \llbracket (c, \alpha) \rrbracket = (\llbracket c \rrbracket_{\text{Uni}} _, \alpha \circ \llbracket c \rrbracket_{\text{Info}} _) : \text{Fam } D \to \text{Fam } E \]
A code for $W$-types

```haskell
data W (S : Set)(P : S → Set) : Set where
  sup : (s : S) → (P s → W S P) → W S P
```

$$\alpha_{W S P, UF} = \delta_{UF} (\sigma_{UF} \iota_{UF} (_ \mapsto S)) (((_ , s) \mapsto (P s)) : \text{Uni} 1)$$

$$\llbracket \alpha_{W S P, UF} \rrbracket_{\text{Uni}} (U, T) = (\Sigma(\star, s) : 1 \times S)(P(s) \rightarrow U)$$
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\[ a_{W S P, UF} = \delta_{UF} (\sigma_{UF} \iota_{UF} (_ \mapsto S)) (((_, s) \mapsto (P s)) : Uni 1) \]

\[ a_{W S P, DS} = \sigma S (s \mapsto \delta (P s) (_ \mapsto \iota \star)) : DS 1 1 \]

\[ \llbracket a_{W S P, UF} \rrbracket_{Uni} (U, T) = (\Sigma(*, s) : 1 \times S)(P(s) \rightarrow U) \]
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A code for \( W \text{-types} \)

\[
\textbf{data} \; W \; (S : \text{Set})(P : S \to \text{Set}) : \text{Set} \; \text{where}
\]
\[
\text{sup} : (s : S) \to (P \; s \to W \; S \; P) \to W \; S \; P
\]

\[
\alpha_{W \; S \; P,UF} = \delta_{UF} \left( \sigma_{UF} \; \iota_{UF} \; (_ \mapsto \rightarrow S) \right) \; \left( (_\mapsto, s \mapsto (P \; s)) \right) : \text{Uni} \; 1
\]
\[
\alpha_{W \; S \; P,DS} = \sigma \; S \; (s \mapsto \delta \; (P \; s) \; (_ \mapsto \lambda \; *)) : \text{DS} \; 1 \; 1
\]

\[
\left\llbracket \alpha_{W \; S \; P,UF} \right\rrbracket_{\text{Uni}} (U, T) = (\Sigma(\star, s : 1 \times S)(P(s) \to U)
\]
\[
\left\llbracket \alpha_{W \; S \; P,DS} \right\rrbracket_0 (U, T) = (\Sigma s : S)(\Sigma f : (P(s) \to U))1
\]
Coproducts of uniform codes

A priori we do not longer have coproducts of codes — DS coproducts relied exactly on non-uniformity of $\sigma$. 

**Proposition.**

For every uniform code $c$, $J_c K Z \sim = J_\sigma UF_c (\_ \mapsto 1) K Z$ and $J_c K Z \sim = J_\delta UF_c (\_ \mapsto 0) K Z$.

By padding codes with such semantically redundant information, we can define $c + UF_d$.

E.g. $\sigma UF (\delta UF \iota UF A) B + UF \delta UF \iota UF A' = \sigma UF (\delta UF (\sigma UF \iota UF 2)) [A, A'] [B, 0]$. 

**Theorem.** $J_c + UF d K Z \sim = J_c K Z + J_d K Z$. 

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**Proposition.** For every uniform code $c$, $\llbracket c \rrbracket Z \cong \llbracket \sigma_{UF} c (\_ \mapsto 1) \rrbracket Z$ and $\llbracket c \rrbracket Z \cong \llbracket \delta_{UF} c (\_ \mapsto 0) \rrbracket Z$.

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E.g.

$$\sigma_{\text{UF}} (\delta_{\text{UF}} \nu_{\text{UF}} A) B +_{\text{UF}} \delta_{\text{UF}} \nu_{\text{UF}} A' = \sigma_{\text{UF}} (\delta_{\text{UF}} (\sigma_{\text{UF}} \nu_{\text{UF}} 2 [A, A']) [B, 0]$$
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E.g.

$$\sigma_{UF} (\delta_{UF} \nu_{UF} A) B +_{UF} \delta_{UF} \nu_{UF} A' = \sigma_{UF} (\delta_{UF} (\sigma_{UF} \nu_{UF} 2) [A, A']) [B, 0]$$

**Theorem.** $[c +_{UF} d] Z \cong [c] Z + [d] Z$. 

\[ \square \]
UF $\leftrightarrow$ DS

Since uniform codes are “backwards”, we can translate UF to DS the same way one reverses a list using an accumulator:

$$\text{accUFtoDS} : (c : \text{Uni } D) \rightarrow (\text{Info } c \rightarrow \text{DS } D E) \rightarrow \text{DS } D E$$
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defined by

\[
\text{accUFtoDS } \nu_{UF} F = F \star \\
\text{accUFtoDS } (\sigma_{UF} c A) F = \text{accUFtoDS } c (\gamma \mapsto \sigma (A \gamma) (a \mapsto F (\gamma, a))) \\
\text{accUFtoDS } (\delta_{UF} c A) F = \text{accUFtoDS } c (\gamma \mapsto \delta (A \gamma) (h \mapsto F (\gamma, h)))
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\]

\[
\text{accUFtoDS } (\delta_{UF} c A) F = \text{accUFtoDS } c (\gamma \mapsto \delta (A \gamma) (h \mapsto F (\gamma, h)))
\]

**Proposition.** \([\text{accUFtoDS } c (\iota \circ \alpha)] Z \cong [ (c, \alpha) ] Z.\]

Going the other way seems unlikely.
Consequences for soundness

This means that UF can piggyback on Dybjer and Setzer [1999]’s proof of existence of initial algebras.
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However the construction of \((\text{Uni}, \text{Info})\) itself is one instance of large induction-recursion, albeit a particularly simple instance. No additional assumptions are needed in the set-theoretical model.
UF is not a monad

We have gained uniformity, which makes powers definable.

Unfortunately, the uniformity also means that we no longer have a monad.
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Bind should graft trees, but grafting a collection of uniform trees might not result in a uniform tree.
Towards composition: combined bind and powers

Is all lost? No. We can still define the instance of bind that we need, combined with a power operation. (Note: only the set depends on Info c.)

\[
\triangleright\triangleright[- \rightarrow -] : (c : \text{Uni } D) \rightarrow (\text{Info } c \rightarrow \text{Set}) \rightarrow \text{Uni } D \rightarrow \text{Uni } D
\]
Towards composition: combined bind and powers

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\[
- \ggg[- \rightarrow -] : (c : \text{Uni } D) \rightarrow (\text{Info } c \rightarrow \text{Set}) \rightarrow \text{Uni } D \rightarrow \text{UniD}
\]

As usual, we need to define this simultaneously with its meaning on \(\text{Info}\):

\[
(\text{Info } c \ggg[E \rightarrow d])_{\text{Info}} : \text{Info} (c \ggg[E \rightarrow d]) \rightarrow (\sum x : \text{Info } c)(E x \rightarrow \text{Info } d)
\]
Towards composition: combined bind and powers

Is all lost? No. We can still define the instance of bind that we need, combined with a power operation. (Note: only the set depends on $\text{Info } c$.)

$$\xRightarrow{[\_ \to \_]} : (c : \text{Uni } D) \to (\text{Info } c \to \text{Set}) \to \text{Uni } D \to \text{Uni } D$$

As usual, we need to define this simultaneously with its meaning on $\text{Info}$:

$$(c \xRightarrow{[E \to d]}_{\text{Info}} : \text{Info } (c \xRightarrow{[E \to d]}) \to (\Sigma x : \text{Info } c)(E x \to \text{Info } d)$$

**Proposition.** There is an equivalence

$$[c \xRightarrow{[E \to d]}, (d \xRightarrow{[E \to d]}_{\text{Info}}] \cong (([c, \text{id }]) \xRightarrow{\text{Fam } (e \mapsto ((E e) \to \text{Fam } [d, \text{id }]))}$$
Composition for UF

_ •_\text{Uni} _ : \text{Uni } D \to \text{UF } C D \to \text{Uni } C

(\_ •\text{Info } _) : (c : \text{Uni } D) \to (R : \text{UF } C D) \to \text{Info } (c •\text{Uni } R) \to \text{Info } c
Composition for UF

\[ \_ \bullet_{\text{Uni}} \_ : \text{Uni} \, D \to \text{UF} \, C \, D \to \text{Uni} \, C \]

\[ \_ \bullet_{\text{Info}} \_ : (c : \text{Uni} \, D) \to (R : \text{UF} \, C \, D) \to \text{Info} \, (c \bullet_{\text{Uni}} R) \to \text{Info} \, c \]

\[ \nu_{\text{UF}} \bullet_{\text{Uni}} R = \nu_{\text{UF}} \]

\[ (\sigma_{\text{UF}} \, c \, A) \bullet_{\text{Uni}} R = \sigma_{\text{UF}} \, (c \bullet_{\text{Uni}} R) \, (A \circ (c \bullet_{\text{Info}} R)) \]

\[ (\delta_{\text{UF}} \, c \, A) \bullet_{\text{Uni}} (d, \beta) = (c \bullet_{\text{Uni}} (d, \beta)) \gg [(A \circ (c \bullet_{\text{Info}} (d, \beta))) \to d] \]
Composition for UF

\[ _\bullet_{\text{Uni}} : \text{Uni } D \to \text{UF } C \ D \to \text{Uni } C \]
\[ (_\bullet_{\text{Info}}) : (c : \text{Uni } D) \to (R : \text{UF } C \ D) \to \text{Info } (c \bullet_{\text{Uni}} R) \to \text{Info } c \]

\[ \nu_{\text{UF}} \bullet_{\text{Uni}} R = \nu_{\text{UF}} \]
\[ (\sigma_{\text{UF}} \ c \ A) \bullet_{\text{Uni}} R = \sigma_{\text{UF}} (c \bullet_{\text{Uni}} R) \ (A \circ (c \bullet_{\text{Info}} R)) \]
\[ (\delta_{\text{UF}} \ c \ A) \bullet_{\text{Uni}} (d, \beta) = (c \bullet_{\text{Uni}} (d, \beta)) \implies [(A \circ (c \bullet_{\text{Info}} (d, \beta))) \rightarrow d] \]

**Theorem.**

\[ \llbracket (c, \alpha) \bullet d \rrbracket Z = \llbracket c \bullet_{\text{Uni}} d, \alpha \circ (c \bullet_{\text{Info}} d) \rrbracket Z \cong \llbracket (c, \alpha) \rrbracket (\llbracket d \rrbracket Z). \]

How suitable are uniform codes?

Uniform codes (most likely) capture a smaller class of functors compared to DS.

However all inductive-recursive definitions “in the wild” are already uniform (because coproducts definable).
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However all inductive-recursive definitions “in the wild” are already uniform (because coproducts definable).

Conjecture: UF and DS have the same proof-theoretical strength.
Summary

Uniform codes **UF** and polynomial codes **PN** as new, alternative axiomatisations of inductive-recursive definitions.

\[
UF \leftrightarrow DS \leftrightarrow PN
\]

Both **UF** and **PN** are closed under composition; **DS** probably is not.

Existence of initial algebras for **UF** unproblematic. For **PN**, need to adjust the **DS** model slightly (but not much).

Are there other, even more well-behaved axiomatisations?

Thank you!