Quotient inductive-inductive types

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In Homotopy Type Theory [Awodey, Warren 2009; Voevodsky 2010]:

- Homotopical models suggest that equality can be given much more intricate *proof-relevant* structure.
- Equality type $\equiv_A$ provides access to this structure, and is morally part of A (cf. `cubicaltt` [Cohen, Coquand, Huber, Mörtberg 2015]).

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Applications:

- Synthetic homotopy theory:
  - Definition of the circle $\mathbb{S}^1$, with $\pi_1(\mathbb{S}^1) = \mathbb{Z}$,
  - Higher spheres $\mathbb{S}^n$,
  - The Hopf fibration, ...
- Quotidian applications:
  - Cauchy Reals $\mathbb{R}_c$,
  - the Partiality monad $(-)_\perp$,
  - Type Theory in Type Theory.

Quotient Inductive-Inductive Types

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$$A : \mathsf{Set} \qquad B : A \to \mathsf{Set}$$

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Quotient Inductive-Inductive Types (QIITs): QITs + IITs = QIITs.

All quotidian applications of HITs are QIITs.

Type theory in type theory as a QIIT

Simplified adaption after Altenkirch and Kaposi [2016]:

$$
\begin{aligned}
&\textbf{data}\ \mathsf{Con} : \mathsf{Set}\\
&\textbf{data}\ \mathsf{Ty} : \mathsf{Con} \to \mathsf{Set}
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon &: \mathsf{Con}\\
\mathsf{ext} &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}\\
\mathsf{U} &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma\\
\sigma &: (\Gamma : \mathsf{Con}) \to (\mathtt{A} : \mathsf{Ty}\,\Gamma) \to \mathsf{Ty}\,(\mathsf{ext}\,\Gamma\,A) \to \mathsf{Ty}\,\Gamma\\
\sigma_{eq} &: (\Gamma : \mathsf{Con}) \to (\mathtt{A} : \mathsf{Ty}\,\Gamma) \to (\mathtt{B} : \mathsf{Ty}\,(\mathsf{ext}\,\Gamma\,A))\\
&\qquad \to (\mathsf{ext}\,(\mathsf{ext}\,\Gamma\,\mathtt{A})\,\mathtt{B} \equiv_{\mathsf{Con}} \mathsf{ext}\,\Gamma\,(\sigma\,\Gamma\,\mathtt{A}\,\mathtt{B}))
\end{aligned}
$$

Challenging features

- Constructors for Con refer to Ty (and vice versa):

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$$\sigma : (\Gamma : \mathsf{Con}) \to (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Ty}\,(\mathsf{ext}\,\Gamma\,A) \to \mathsf{Ty}\,\Gamma$$

- "Path constructors" construct equalities, not elements:

$$\begin{aligned}\sigma_{eq} : (\Gamma : \mathsf{Con}) \to (A : \mathsf{Ty}\,\Gamma) \to (B : \mathsf{Ty}\,(\mathsf{ext}\,\Gamma\,A))\\ \to (\mathsf{ext}\,(\mathsf{ext}\,\Gamma\,A)\,B \equiv_{\mathsf{Con}} \mathsf{ext}\,\Gamma\,(\sigma\,\Gamma\,A\,B))\end{aligned}$$

This work: representing QIITs

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We will take an (internal) "semantics first" perspective, and represent QIITs as initial objects in a category of algebras;

This work: representing QIITs

How do we represent such definitions, in general?

How do we know that we have derived the right elimination rules?

We will take an (internal) "semantics first" perspective, and represent QIITs as initial objects in a category of algebras;

Then derive/show that initiality corresponds exactly to ordinary elimination rules. The key lemma used is that the category of algebras is complete.

High level view

A QIIT is specified by a sequence of constructors.

$$\boxed{\begin{array}{c}\text{cat. } \mathcal{A} \text{ of algebras}\\ \text{of previous constructors}\end{array}} \xRightarrow{\text{specify}} \boxed{\begin{array}{c}\text{new constructor}\\ \mathsf{c} \text{ on } \mathcal{A}\end{array}} \xRightarrow{\text{extend } \mathcal{A} \text{ with } \mathsf{c}} \boxed{\begin{array}{c}\text{new cat. of}\\ \text{algebras } \mathcal{A}'\end{array}}$$

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$$\boxed{\text{complete cat. } \mathcal{C}} \xRightarrow{\text{specify}} \boxed{\begin{array}{c}\text{new constructor} \\ \color{green}{\mathsf{c}} \text{ on } \mathcal{C}\end{array}} \xRightarrow{\text{extend } \mathcal{C} \text{ with } \color{green}{\mathsf{c}}} \boxed{\text{complete cat. } \mathcal{C}'}$$

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$$\mathsf{c} : (x : \mathsf{F}(\mathsf{X})) \to \mathsf{G}(\mathsf{X}, \mathsf{x})$$

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Of course, we need restrictions on these functors.

Well-behaved functors

$$\mathsf{c} : (x : \mathsf{F}(\mathsf{X})) \to \mathsf{G}(\mathsf{X}, \mathsf{x})$$

Argument functor $F : \mathcal{C} \Rightarrow \mathrm{Set}$ needs to be constrained (strictly positive etc) to prove existence, but can otherwise be arbitrary.

Target functor $G : \int^{\mathcal{C}} F \Rightarrow \mathrm{Set}$ definitely cannot be arbitrary.

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category of elements of F:
objects (X, x), where X in $\mathcal{C}$ and $x : \mathsf{F(X)}$,
morphisms $(X, x) \to (X', x')$ consists of
$f : X \to X'$ with $F(f)x \equiv x'$.

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Mathematically, G needs to be continuous, i.e. preserve limits.

Complication: $\int^{\mathcal{C}} F$ is often not complete, even if $\mathcal{C}$ is, so we need a more refined notion of continuity.

Relative continuity

Definition Let $\mathcal{C}$ be a category, $\mathcal{C}_0$ a complete category, and $U : \mathcal{C} \Rightarrow \mathcal{C}_0$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "U"'] & \mathrm{Set} \\ \mathcal{C}_0 \text{ (complete)} & \end{tikzcd}$$

- A cone in $\mathcal{C}$ is a *U-limit cone* if it is mapped to a limit cone by U.
- A functor $G : \mathcal{C} \Rightarrow \mathrm{Set}$ is *U-relatively continuous* if it maps U-limit cones to limit cones in Set.

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Example Let $U : \int^{\mathcal{C}} F \Rightarrow \mathcal{C}$ be the forgetful functor $U(X, x) = X$. If a functor $G : \int^{\mathcal{C}} F \Rightarrow \mathrm{Set}$ is U-relatively continuous, then e.g.

$$G(X \times Y, z) = G(X, z_0) \times G(Y, z_1)$$

where $z_i = F(\pi_i)z$.

Constructor specifications

Definition A constructor specification on a complete category $\mathcal{C}$ is given by

- A functor $F : \mathcal{C} \Rightarrow \mathrm{Set}$ (the *argument functor*).
- A U-relatively continuous functor $G : \int^{\mathcal{C}} F \Rightarrow \mathrm{Set}$ for the forgetful functor $U : \int^{\mathcal{C}} F \Rightarrow \mathcal{C}$ (the *target functor*).

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Example The constructor $\mathsf{ext} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Con}$ is specified on its base category by

$$\begin{aligned} F_{\mathsf{ext}}(C, T) &= (\Sigma x : C)(T(x)) \\ G_{\mathsf{ext}}(C, T, (x, A)) &= C \end{aligned}$$

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Example The constructor

$$\sigma : (\Gamma : \mathsf{Con}) \to (A : \mathsf{Ty}\,\Gamma) \to \mathsf{Ty}\,(\mathsf{ext}\,\Gamma\,A) \to \mathsf{Ty}\,\Gamma$$

is next specified by

$$
\begin{aligned}
F_{\sigma}(C, T, ext) &= (\Sigma x : C)(\Sigma a : T(x))(T(ext\,x\,a)) \\
G_{\sigma}(C, T, ext, (x, a, b)) &= T(x)
\end{aligned}
$$

Category of algebras

Each constructor specification (F, G) on $\mathcal{C}$ gives rise to a category of algebras $\mathcal{C}.(F, G)$, with

objects pairs (X, f), where $X : \mathcal{C}$ and

$$f : (x : \mathsf{F(X)}) \to \mathsf{G(X,\ x)}$$

morphisms $(X, f) \to (Y, g)$ consisting of $\alpha : X \to Y$ making the obvious "dependent diagram" commute:

$$\begin{tikzcd}
(x : F(X)) \arrow[r, "f"] \arrow[d, "F(\alpha)"'] & G(X, x) \arrow[d, "G(\alpha{,}\mathrm{refl})"] \\
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The intended meaning of a QIIT is the initial object in such a category.

Algebras for type theory in type theory

The category of algebras $\mathcal{C}.(F_{\text{ext}}, G_{\text{ext}}).(F_\sigma, G_\sigma)$ for the ext and σ specification has objects (C, T, c_0, c_1), where

$$\begin{aligned}
C &: \mathsf{Set} \\
T &: C \to \mathsf{Set} \\
c_0 &: (y : F_{\text{ext}}(C, T)) \to G_{\text{ext}}(C, T, y) \\
c_1 &: (y : F_\sigma(C, T, c_0)) \to G_\sigma(C, T, c_0, y)
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Categories of algebras are complete

Theorem Let (F, G) be a constructor specification on a complete category $\mathcal{C}$. Then the category of algebras $C.(F, G)$ is also complete. $\square$

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1. Preconditions satisfied for adding another constructor to the category of algebras.
2. Allows using limits when reasoning about algebras, as is needed for the elimination rules.
3. Partial progress towards existence of initial algebras (solution set condition missing).

Point and path constructors

This works for any relatively continuous target functor.

QIITs are given by point and path constructors; we show that they can be specified using relatively continuous target functors.

Point constructors

Point constructors have target functors picking out a *base sort*.

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Example Point constructors of $\mathsf{Con} : \mathsf{Set}$ have target functor

$$U(C, T, \vec{X}) = C.$$

Example Point constructors of $\mathsf{Ty} : \mathsf{Con} \to \mathsf{Set}$ have target functor

$$U'(C, T, \vec{X}, \Gamma) = T(\Gamma)$$

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($U''(C, T, \vec{X}) = T(t(\vec{X}))$ can be obtained by composing U' with a suitable functor — this preserves relative continuity).

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Theorem All such base sort functors

$$\mathcal{C}.(F_1, G_1).\ldots.(F_k, G_k) \to \mathcal{C}$$

are relatively continuous. $\square$

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have target functor

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Theorem Suppose $G : \int^{\mathcal{C}} F \Rightarrow \mathsf{Set}$ is relatively continuous. For natural transformations $l, r : 1 \to G$, the functor $\mathsf{Eq}_G(l, r) : \int^{\mathcal{C}} F \Rightarrow \mathsf{Set}$ defined by

$$\mathsf{Eq}_G(l, r)(X, x) = ((l_{(X,x)} =_{G(X,x)} r_{(X,x)}))$$
$$\mathsf{Eq}_G(l, r)(f, p) = (p \cdot nat_l) \cdot (\mathsf{ap}\ G(f, \mathsf{refl})\ -) \cdot (p \cdot nat_r)^{-1}$$

is relatively continuous. $\square$

Initiality and induction

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Theorem

$$\text{initiality} \;\Leftrightarrow \text{induction}$$

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$$\text{initiality} \Leftrightarrow \text{section induction} \Leftrightarrow \text{induction}$$

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Dijkstra thesis [2017]
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initiality $\simeq$ section induction $\simeq$ induction

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□

Summary

QIITs represented by sequence of constructor specifications.

Constructor specification given by argument and target functors.

Each QIIT representation gives rise to a category of algebras; we are interested in its initial object.

An algebra is initial exactly when it satisfies the usual induction principle.

Same method should work also for higher inductive types, but we want to make sure that all categorical concepts still make sense.

Summary

QIITs repr

Constructo

Each QIIT … are
interested

An algebra … principle.

Same met … want to
make sure

Thank you!