Different Notions of Ordinals in Homotopy Type Theory

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Joint work with Nicolai Kraus and Chuangjie Xu

HOTTEST seminar

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What are ordinals?

"Numbers" for ranking/ordering:

$$0, \quad 1, \quad 2, \quad \ldots, \quad \omega, \quad \omega + 1, \quad \ldots, \quad \omega \cdot 2, \quad \omega \cdot 2 + 1, \quad \ldots, \quad \omega \cdot 3, \quad \ldots$$

$$\omega^2, \quad \ldots, \quad \omega^2 \cdot 3 + \omega \cdot 7 + 13, \quad \ldots, \quad \omega^\omega, \quad \ldots, \quad \varepsilon_0 = \omega^{\omega^{\omega^{\cdots}}}, \quad \ldots, \quad \varepsilon_{17}, \quad \ldots$$

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Classically: sets with an order $<$, which is

- **transitive**: $(a<b)\to(b<c)\to(a<c)$
- **wellfounded**: every sequence $a_0>a_1>a_2>a_3>\ldots$ terminates
- and **trichotomous**: $(a<b)\vee(a=b)\vee(b<a)$

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- …or **extensional** (instead of trichotomous):

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Perhaps more importantly: what are they for?

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Useful: Definitional principle where ordinals are classified as 0, $\alpha + 1$ or a limit.

Termination of processes

- Programs terminating [Turing 1949]
- Consistency proof e.g. of Peano's axioms [Gentzen 1936]
- Termination of Goodstein sequences [Goodstein 1944], the Hydra game [Kirby&Paris 1982]:

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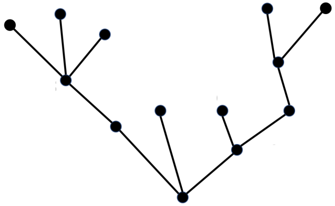

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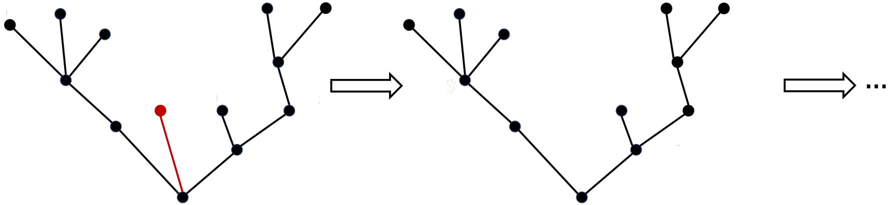

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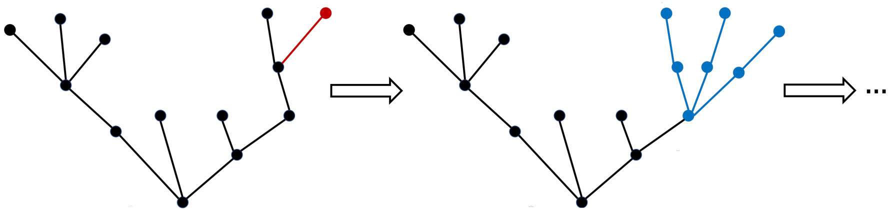

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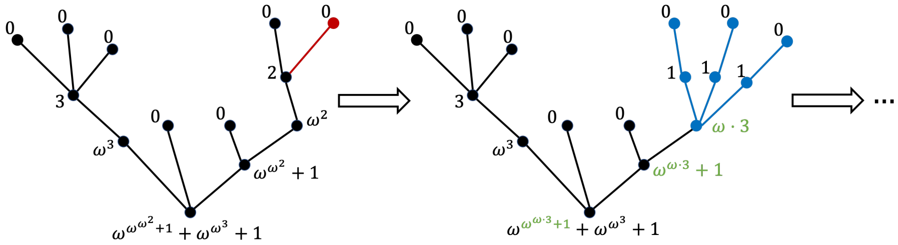

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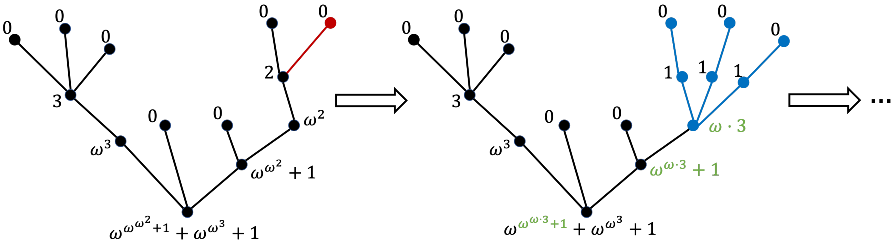

Useful: Arithmetic, and every decreasing sequence of ordinals hits 0.

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Three standard notions of "ordinals" in computer science:

- Cantor normal forms
- Brouwer trees
- Wellfounded, extensional, and transitive orders

How are they connected? Why can we call them "ordinals"?

Need features and concepts of HoTT to give "correct" formulations.

Cantor normal forms

Motivational classical theorem

Every ordinal α can be written uniquely

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n}$$

for some $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

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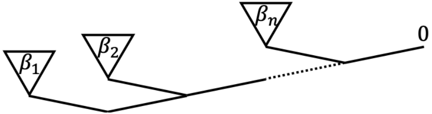

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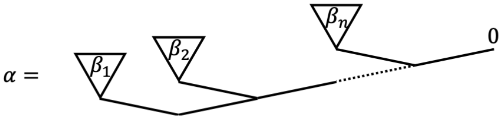

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Let $\mathcal{T}$ be the type of *unlabeled binary trees*:

$$\begin{aligned} 0 &: \mathcal{T} \\ \omega^{-} + - &: \mathcal{T} \to \mathcal{T} \to \mathcal{T} \end{aligned}$$

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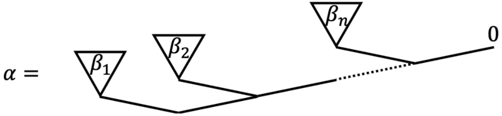

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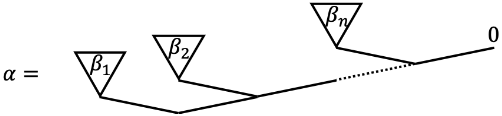

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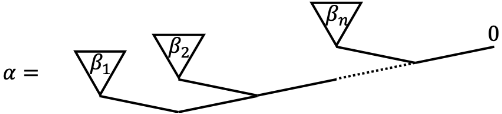

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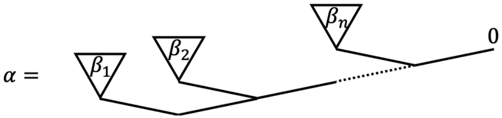

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We write Cnf $= (\Sigma\alpha : \mathcal{T})$ isCNF(α) for the type of Cantor Normal Forms.

Basic properties of Cantor Normal Forms

Equivalent implementations [Ghani, N.-F., Xu 2020]:

(i) inductive-inductively inlining the isCNF condition (no junk!)

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Theorem: Can classify each Cnf as zero, successor or limit, but cannot compute limits (implies WLPO).

Brouwer ordinal trees

Another definition: the usual inductive type $\mathcal{O}$ generated by

$$\mathsf{zero} : \mathcal{O} \qquad \mathsf{succ} : \mathcal{O} \to \mathcal{O} \qquad \mathsf{sup} : (\mathbb{N} \to \mathcal{O}) \to \mathcal{O}$$

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How to fix this without losing wellfoundedness, classification, and so on?

Brouwer trees quotient inductive-inductively

```
data Brw : Set where
  zero  : Brw
  succ  : Brw → Brw
  limit : (f : ℕ → Brw) {f↑ : increasing f} → Brw
  bisim : f ≈ g → limit f ≡ limit g

data _≤_ : Brw → Brw → Prop where
  ≤-zero      : zero ≤ x
  ≤-trans     : x ≤ y → y ≤ z → x ≤ z
  ≤-succ-mono : x ≤ y → succ x ≤ succ y
  ≤-cocone    : x ≤ f k → x ≤ limit f
  ≤-limiting  : (∀ k → f k ≤ x) → limit f ≤ x
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$x < y$ if succ $x \leq y$.

Characterising $\leq$ using encode-decode

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We use an encode-decode method to characterise $x \leq y$: define

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Technically involved: need to simultaneously prove transitivity, reflexivity of Code, and $(x \leq y) \to \mathsf{Code}\, x\, y$.

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Can prove expected properties such as:

- $n \cdot \omega \equiv \omega$;
- If $a < \omega^b$ then $a + \omega^b \equiv \omega^b$;
- $\epsilon_0 = \text{limit}\,(\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots)$ is a fixed point $\omega^{\epsilon_0} = \epsilon_0$;
- and so on.

Extensional wellfounded orders

The type Ord consists of pairs $(X : \text{Type}, \prec : X \to X \to \text{Prop})$ such that:

- $\prec$ is transitive
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The order on extensional wellfounded orders

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$X < Y$ is:

- a simulation $f : X \leq Y$
- such that there is $y : Y$ and f factors through $X \simeq Y_{/y}$.

$f : X < Y$ is a *bounded simulation*.

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Theorem: limits of increasing sequences of Ord can be calculated.

Theorem: “nothing” is decidable.

For example, deciding whether an Ord is a successor implies LEM.

Abstract setting

What do Cnf, Brw, Ord have to do with each other?

Why are they “types of ordinals”?

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Note: $(\leq \circ <) \subseteq (<)$ for Ord is equivalent to LEM (cf. Taylor).

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"Concrete" results:

- Cnf, Brw, Ord uniquely have zero and strong successor.
- Brw, Ord uniquely have limits; Cnf does not.
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"Abstract" result:

- is-zero$(a) \uplus$ is-str-suc$(a) \uplus$ is-limit(a) is a proposition.
- Corollary: "Classifiability" induction implies classification. (Conversely classification + wellfounded induction implies classifiability induction.)

Abstract arithmetic: addition

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$(A, <, \leq)$ has addition if there is a function $+ : A \to A \to A$ such that:

$$\text{is-zero}(a) \to c + a = c$$
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Concrete results: Cnf and Brw have unique addition. Ord has addition.

Addition for Cantor Normal Forms

Standard definition:

$$\mathbf{0} + b = b$$
$$a + \mathbf{0} = a$$
$$(\omega\hat{}\ a + c) + (\omega\hat{}\ b + d) \text{ with } {<}\text{-tri } a\ b$$
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Perhaps less standard: to prove correctness, need to define subtraction.

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Concrete results: Cnf and Brw have unique multiplication. Ord has multiplication.

Multiplication for Brouwer trees

Seemingly straightforward definition:

$$
\begin{aligned}
x \cdot \mathsf{zero} &= \mathsf{zero} \\
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$$\ldots \mid \text{yes}\ \textit{x≡0} = \text{zero}$$

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A has exponentation with base c if there is $\mathsf{exp}(c, -) : A \to A$ such that:

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Concrete results: Brw and Cnf and have unique exponentation (with base ω).

Connections between the notions

"decidable"	"partially decidable"	"undecidable"
Cnf	Brw	Ord

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Cnf $\xhookrightarrow{\mathsf{CtoB}}$ Brw — Ord

$(\omega^a + b) \mapsto \omega^{\mathsf{CtoB}(a)} + \mathsf{CtoB}(b)$

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Constructively, different definitions of ordinals are useful for different purposes.

We have considered three different notions, ranging from “decidable” to “undecidable” in general.

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Future work:

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- Can we make Brw being "partially decidable" precise using the notion of semi-decidability? [Veltri 2017, Escardó and Knapp 2017]

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More details:

- Connecting Constructive Notions of Ordinals in Homotopy Type Theory, MFCS 2021 (`arxiv:2104.02549`)
- Cubical Agda formalisation: `bitbucket.org/nicolaikraus/constructive-ordinals-in-hott/`

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Thank you!

References

In order of appearance

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