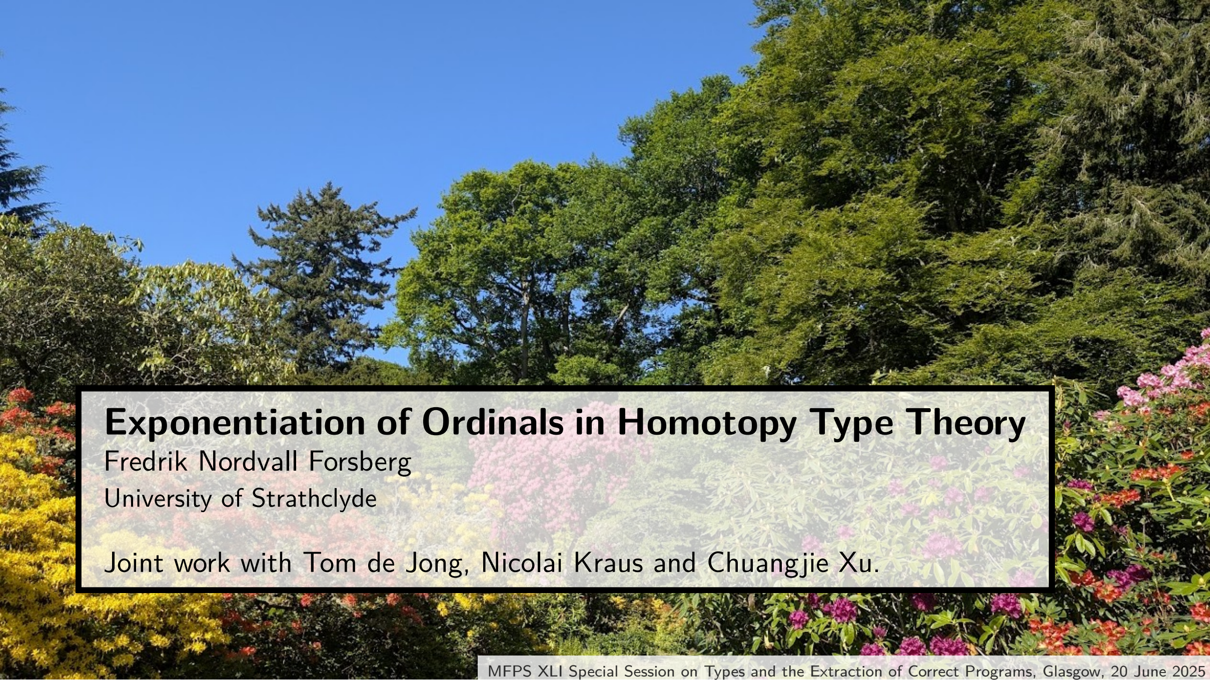

Exponentiation of Ordinals in Homotopy Type Theory

Fredrik Nordvall Forsberg
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Joint work with Tom de Jong, Nicolai Kraus and Chuangjie Xu.

MFPS XLI Special Session on Types and the Extraction of Correct Programs, Glasgow, 20 June 2025

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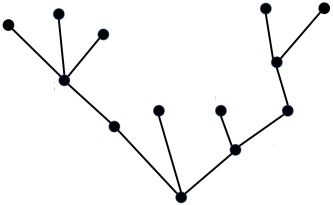

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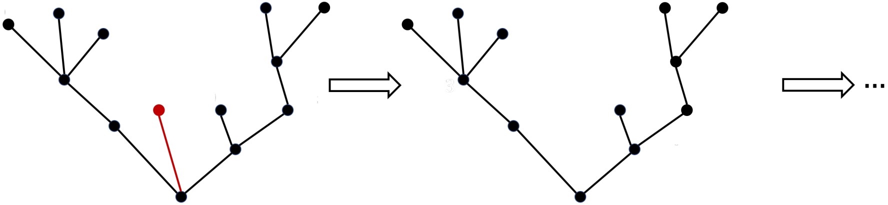

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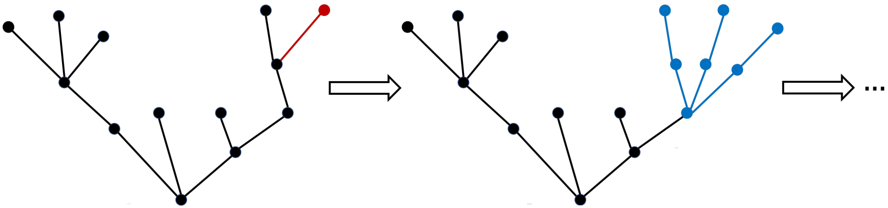

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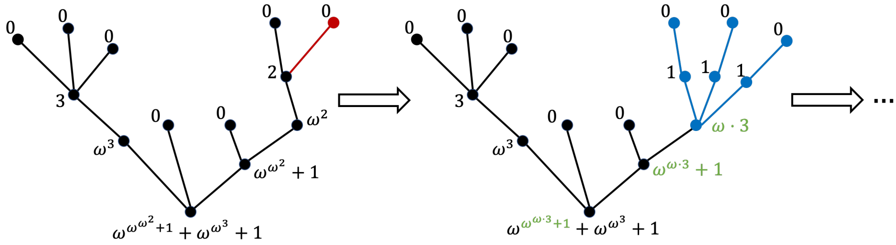

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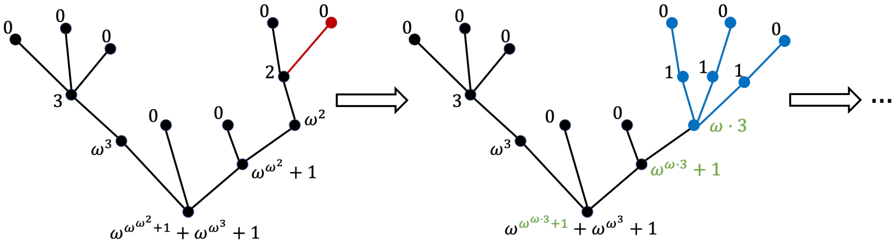

Proof of termination makes use of ordinal arithmetic, in particular exponentiation.

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- Many other more specialised (and well behaved) notions of ordinals [Martin-Löf 1970; Taylor 1996; Coquand, Lombardi and Neuwirth 2023, ...] , but here we focus on the most general notion.

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- Moreover, Ord is a poset with

$$\alpha \leq \beta :\equiv \Sigma(f : \alpha \to \beta).\, \forall(a : A).\, \alpha \downarrow a = \beta \downarrow f\, a.$$

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- In particular we have maps $[i, -] : F_i \leq \sup F_\bullet$ such that for any $y : \sup F_\bullet$ there exists $i : I$ and $x : F_i$ with

$$y = [i, x] \quad \text{and} \quad \sup F_\bullet \downarrow y = F_i \downarrow x.$$

Natural number arithmetic

$$\alpha + 0 = \alpha$$
$$\alpha + (\beta + 1) = (\alpha + \beta) + 1$$

$$\alpha \times 0 = 0$$
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Not a definition, constructively! But a good specification.

Addition and multiplication

- For addition and multiplication, there are well known explicit constructions:

$$\langle \alpha + \beta \rangle :\equiv \langle \alpha \rangle + \langle \beta \rangle$$

with $\mathsf{inl}\ a \prec \mathsf{inr}\ b$, and

$$\langle \alpha \times \beta \rangle :\equiv \langle \alpha \rangle \times \langle \beta \rangle$$

ordered reverse lexicographically:

$$(a, b) \prec (a', b') :\equiv (b \prec b') + ((b = b') \times (a \prec a')).$$

- Thm (,). (well known) The operations $\alpha + \beta$ and $\alpha \times \beta$ satisfy the specifications for addition and multiplication, respectively.

Exponentiation?

$$\alpha^0 = 1 \qquad 0^\beta = 0 \qquad \text{(if } \beta \neq 0\text{)}$$

$$\alpha^{\beta+1} = \alpha^\beta \times \alpha \qquad \alpha^{\sup_{i:I} \gamma_i} = \sup_{i:I} \alpha^{\gamma_i} \qquad \text{(if } I \text{ inhabited, and } \alpha \neq 0\text{)}$$

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  and P or $\neg P$ holds depending on if $f(\star) = \mathrm{inl}\ p$ or $f(\star) = \mathrm{inr}\ \star$ for $f : 1 \leq P+1$.

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- We show that our two constructions agree (whenever the base ordinal has a trichotomous least element).
- We use this equivalence together with **univalence (representation independence)** to prove algebraic laws and decidability properties.

A stronger specification

- Inspired by the classical definition (and the no-go theorem), we now wish to construct, for $\alpha \geq \mathbf{1}$, an operation $\alpha^{(-)}$ satisfying the specification:

$$\begin{aligned}
\alpha^{\mathbf{0}} &= \mathbf{1} \\
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- We recover $\alpha^{\mathbf{0}} = \mathbf{1} \vee \mathbf{0}$ and $\alpha^{\mathsf{sup}_{i:I} F_i} = \mathsf{sup}_{i:I}(\alpha^{F_i})$ for inhabited I, since $\alpha^{F_i} \geq \mathbf{1}$.

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- Def. (⚙) Define abstract exponentiation α^β by transfinite induction on β as

$$\alpha^\beta :\equiv \sup_{x:\mathbf{1}+\beta} \begin{cases} \mathsf{inl}\,\star \mapsto \mathbf{1} \\ \mathsf{inr}\,b \mapsto \alpha^{\beta\downarrow b} \times \alpha \end{cases}$$

Properties of abstract exponentiation

- Def. (repeated) Abstract exponentiation α^β is given by transfinite induction on β:

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- Lemma (⚙). Using the characterization of initial segments of suprema and products, we have for $a : \alpha$, $b : \beta$ and $e : \alpha^{\beta\downarrow b}$ that

$$\alpha^\beta \downarrow [\mathsf{inr}\ b, (e, a)] = \alpha^{\beta\downarrow b} \times (\alpha \downarrow a) + (\alpha^{\beta\downarrow b} \downarrow e).$$

Functions with finite support

- Sierpiński [1958] constructs, for α with a least element $\bot : \alpha$, the exponential α^{β} as

$$\Sigma(f : \beta \to \alpha).\, \mathrm{supp}(f) \text{ finite}$$

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where $\mathsf{supp}(f) :\equiv \Sigma(x : \beta).(f\,x > \perp)$.

- The order is defined by
$$f \prec g :\equiv f(b^*) \prec_\alpha g(b^*),$$
where b^* is the largest element x such that $f(x) \neq g(x)$ — such b^* exists by the finite support assumption.

- This is. . . problematic, constructively!

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- Lemma (). Excluded Middle holds iff $\alpha_{>\bot}$ is an ordinal for all α.

Concrete exponentiation

- Constructively well behaved version: represent a function with finite support as a **list of (output, input) pairs**, ordered decreasingly in the input-component to ensure uniqueness of the representation.

  The least element $\perp : \alpha$ should not be an output, so we consider

  $$\alpha_{>\perp} :\equiv \Sigma(a : \alpha).\, a > \perp$$

  and define concrete exponentiation as

  $$\exp(\alpha, \beta) :\equiv \Sigma(\ell : \mathsf{List}(\alpha_{>\perp} \times \beta)).\, \ell \text{ is decreasing in the } \beta\text{-component}.$$

- Lemma (⚙). Excluded Middle holds iff $\alpha_{>\perp}$ is an ordinal for all α.

- Grayson [1978, 1982] suggested a variation of this construction, unfortunately with a subtle mistake, similar to assuming that $\alpha_{>\perp}$ always is an ordinal.

Concrete exponentiation for bases with a trichotomous least element

- Lemma (⚙). The following are equivalent:
  (i) α has a trichotomous least element $\bot$, i.e., $\forall x : \alpha. (\bot < x) + (\bot = x)$.
  (ii) α has a least and trichotomous element $\bot$, i.e., $\forall x : \alpha. \bot \leq x$ and $\forall x : \alpha. (\bot < x) + (\bot = x) + (\bot > x)$.
  (iii) $\alpha = \mathbf{1} + \alpha'$ for some (necessarily unique) ordinal α'. If this happens, then $\alpha' = \alpha_{>\bot}$.

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- Examples. $\omega = 1 + \omega$ and $17 = 1 + 16$ have trichotomous least elements.
- Thm (⚙). For α with a trichotomous least element, the lexicographic order on lists makes concrete exponentiation

$$\mathsf{exp}(\alpha, \beta) :\equiv \Sigma(\ell : \mathsf{List}(\alpha_{>\bot} \times \beta)).\, \ell \text{ is decreasing in the } \beta\text{-component}.$$

  an ordinal.
- Remark. In general, the lexicographic order on $\mathsf{List}(\alpha)$ is not wellfounded, but it is for decreasing lists.
- Thm (⚙). For α with a trich. least element, $\mathsf{exp}\ \alpha\ \beta$ satisfies the specification.

Properties of concrete exponentiation

- Thm (⚙). For ordinals α, β and γ with α having a trichotomous least element, we have $\exp(\alpha, \beta + \gamma) = \exp(\alpha, \beta) \times \exp(\alpha, \gamma)$.

  Other expected equations such as $\exp(\alpha, \beta \times \gamma) = \exp(\exp(\alpha, \beta), \gamma)$ proved too tedious to establish directly.

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- Thm (⚙). Concrete exponentiation **preserves decidability properties**, e.g. if α and β have decidable equality, then so does $\exp(\alpha, \beta)$.

  This is not at all obvious for abstract exponentiation.

Relating abstract and concrete exponentiation

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- For concrete exponentiation we can prove

$$\exp(\alpha, \beta) \downarrow ((a, b) :: \iota_b\, \ell) \;=\; \exp(\alpha, \beta \downarrow b) \times (\alpha \downarrow a) + \exp(\alpha, \beta \downarrow b) \downarrow \ell$$

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  where $\iota_b : \exp(\alpha, \beta \downarrow b) \hookrightarrow \exp(\alpha, \beta)$ is the obvious inclusion.

  Notice the **similarity to the above equation**!
- A proof by transfinite induction in Ord on β then shows:

  Thm (⚙). For α with a trichotomous least element we have $\exp(\alpha, \beta) = \alpha^\beta$.

Consequences

- Cor (⚙). Suppose that α has a trichotomous least element. If α and β have decidable equality, then so does α^{β}.
- Cor (⚙). Suppose α has a least element. If α and β are trichotomous, then so is α^{β}.
- Cor (⚙). For ordinals α, β and γ with α having a trichotomous least element, we have $\exp(\alpha, \beta \times \gamma) = \exp(\exp(\alpha, \beta), \gamma)$.

Constructive taboos

- Many properties of exponentiation can be proven constructively, e.g. monotonicity in the exponent or algebraic laws.

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- Lemma (⚙). For any proposition P we have $\mathbf{2}^P = \mathbf{1} + P$.

- Thm (⚙). The following are equivalent:
  (i) for all ordinals β, we have $\beta \leq \mathbf{2}^\beta$; $\beta = P + 1$
  (ii) for all ordinals β and $\alpha > \mathbf{1}$, we have $\beta \leq \alpha^\beta$;
  (iii) Excluded Middle.

Wrapping up

- We presented two constructively well behaved ordinal exponentiation functions for base ordinals with a least element, and showed them to be equivalent in case the base ordinal has a trichotomous least element.

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- Thanks to univalence we can transfer various results, such as algebraic laws and decidability properties, from one construction to the other.
- **Future work:** Ordinal subtraction, division and logarithms are also not constructively available in general — what can be done there?

Tom de Jong, Nicolai Kraus, Fredrik Nordvall Forsberg and Chuangjie Xu
Ordinal Exponentiation in Homotopy Type Theory
To appear at LICS 2025 (arXiv:2501.14542)

Fully formalised in Agda.
Building on Escardó's TypeTopology development. Click on ⚙ in paper and slides!
www.cs.bham.ac.uk/~mhe/TypeTopology/Ordinals.Exponentiation.Paper.html

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Thank you!

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