A quantitative model for λ-calculus

1 Resource Monoids and Length Spaces

Let us recall the definition of resource monoids and length spaces from [1].

Definition 1. A *resource monoid* is a triple $\mathcal{M} = (|\mathcal{M}|, +, \leq)$ such that:

- $(|\mathcal{M}|, +)$ is a commutative monoid.
- $\leq$ is a pre-order on $|\mathcal{M}|$ compatible with $+$.

Remark: In [1], resource monoids also contain a fourth component $\mathcal{D}$ which allows to control the cost of a computation (e.g., time complexity). Here, we will only be interested in comparing the size of programs and their results, hence we drop this $\mathcal{D}$ component in all the subsequent definitions. Everything else remains the same.

Definition 2. A *length space* on a resource monoid $\mathcal{M}$ is a pair $\mathcal{A} = (|\mathcal{A}|, \Vdash)$ where $|\mathcal{A}|$ is a set and $\Vdash \subseteq |\mathcal{M}| \times |\mathcal{A}|$ satisfying the following properties:

- for all a, there is α such that $\alpha \Vdash a$.
- if $\alpha \Vdash a$ and $\alpha \leq \beta$, then $\beta \Vdash a$.

When $\alpha \Vdash a$, we call α a *majorizer* of a.

Remark: In [1], the relation $\Vdash$ also contains a third component e, called a *realizer* of a. It plays a role to measure the computation time of programs, and for cardinality issues when interpreting the second order universal quantification. We do not need these features, hence we also drop these realizers from our definitions.

Definition 3. Fix a resource monoid $\mathcal{M}$. A *morphism of length spaces* from $\mathcal{A}$ to $\mathcal{B}$ is a function $f : |\mathcal{A}| \to |\mathcal{B}|$ such that there is $\varphi \in |\mathcal{M}|$, and whenever $\alpha \Vdash_{\mathcal{A}} a$, there exists β such that:

- $\beta \Vdash_{\mathcal{B}} f(a)$.

- $\beta \leq \varphi + \alpha$.

When these properties hold, we call φ a majorizer of f, and we write $\varphi \Vdash_{\mathcal{A} \multimap \mathcal{B}} f$. Define $\mathcal{A} \multimap \mathcal{B} = (\mathrm{Hom}(\mathcal{A}, \mathcal{B}), \Vdash_{\mathcal{A} \multimap \mathcal{B}})$, where $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ is the set of morphisms from $\mathcal{A}$ to $\mathcal{B}$. It can be checked that this is again a length space.

Definition 4. Let $\mathcal{A}$ and $\mathcal{B}$ be two length spaces on the same resource monoid $\mathcal{M}$. We define $\mathcal{A} \otimes \mathcal{B} = (|\mathcal{A}| \times |\mathcal{B}|, \Vdash_{\mathcal{A} \otimes \mathcal{B}})$, where $\gamma \Vdash_{\mathcal{A} \otimes \mathcal{B}} (a, b)$ iff there are α, β such that:

- $\alpha \Vdash_{\mathcal{A}} a$.
- $\beta \Vdash_{\mathcal{B}} b$.
- $\alpha + \beta \leq \gamma$.

Again, we can check that this is a well-defined length space on $\mathcal{M}$.

Theorem 1. *Given any resource monoid $\mathcal{M}$, the category of length spaces on $\mathcal{M}$ is symmetric monoidal closed with respect to the tensor product and linear map defined above. This allows us to interpret multiplicative linear logic in the category of length spaces.*

Proof. The proof should be the same as in [1], where the parts dealing with the $\mathcal{D}$ component of resource monoids and the e component of length spaces are dropped.

For example, to define the composition, given morphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, majorized respectively by φ and ψ, let us show that $\mathrm{comp}(f, g) = x \mapsto g(f(x)) : \mathcal{A} \to \mathcal{C}$ is majorized by $\varphi + \psi$.

Let $\alpha \Vdash_{\mathcal{A}} a$, we need to find $\gamma \leq \varphi + \psi + \alpha$ such that $\gamma \Vdash_{\mathcal{C}} g(f(a))$. Since φ majorizes f, there is $\beta \leq \varphi + \alpha$ such that $\beta \Vdash_{\mathcal{B}} f(a)$, and thus since ψ majorizes g, there is $\gamma \leq \psi + \beta \leq \psi + \varphi + \alpha$ such that $\gamma \Vdash_{\mathcal{C}} g(f(a))$.

Another example, the morphism $\mathrm{eval} : \mathcal{A} \otimes (\mathcal{A} \multimap \mathcal{B}) \to \mathcal{B}$ such that $\mathrm{eval}(a, f) = f(a)$. Let us show that it is majorized by $0_{\mathcal{M}}$.

Let $\varepsilon \Vdash_{\mathcal{A} \otimes (\mathcal{A} \multimap \mathcal{B})} (a, f)$. By definition, there are α, φ such that $\alpha \Vdash_{\mathcal{A}} a$, $\varphi \Vdash_{\mathcal{A} \multimap \mathcal{B}} f$ and $\alpha + \varphi \leq \varepsilon$. Thus, since φ majorizes f, there is $\beta \leq \alpha + \varphi \leq \varepsilon$ such that $\beta \Vdash_{\mathcal{B}} f(a)$, which is what we wanted to prove.

The *unit object* is defined as $\mathcal{I} = (\{\star\}, \Vdash_{\mathcal{I}})$ where $\alpha \Vdash_{\mathcal{I}} \star$ for all α.

This is trivially a length space, and for all $\mathcal{A}$, we have a unique weakening map from $\mathcal{A}$ to $\mathcal{I}$ majorized by $0_{\mathcal{M}}$. $\square$

2 Interpreting simply typed λ-calculus

Consider the simply-typed λ-calculus equipped with one base type o, and two constants $0 : o$ and $S : o \to o$. We are interested in finding a semantic proof of the following property:

For each closed term f of type $o \to o$, there exists a constant $C \in \mathbb{N}$ such that for all $n : o$, $|\mathrm{nf}(f\ n)| \leq |\mathrm{nf}(n)| + C$, where $|\mathrm{nf}(t)|$ denotes the size of the normal form of t.

Equivalently, if we interpret the λ-terms as set-theoretic functions in the usual way ($[\![o]\!] = \mathbb{N}$, $[\![0]\!] = 0$ and $[\![S]\!] = n \mapsto n+1$), this means that the denotation $[\![f]\!] : \mathbb{N} \to \mathbb{N}$ of f is bounded by $n \mapsto n + C$ for some constant C.

2.1 The resource monoid

In this section, we define a suitable resource monoid $\mathcal{M}$ that will allow us to interpret our simply typed λ-calculus in the category of length spaces on $\mathcal{M}$.

Definition 5. Let $\mathcal{M} = (|\mathcal{M}|, +_{\mathcal{M}}, \leq_{\mathcal{M}})$ be the following structure:

- $|\mathcal{M}|$ is the set of sequences of natural numbers with finite support (that is, ending with infinitely many 0's). We think of them as finite lists of integers: an element $\alpha = (a_n)_{n \in \mathbb{N}}$ of $|\mathcal{M}|$ such that $a_k \neq 0$ and $\forall n > k, a_n = 0$ is denoted as $\alpha = [a_0, \ldots, a_k]$. We note $|\alpha|$ the length of that list (here, $k+1$).
- $+_{\mathcal{M}}$ is defined componentwise: we take the max on the first component, and the sum on the others. For ease of notation, we denote $\max(a, b)$ as $a \vee b$.
$$(a_n) +_{\mathcal{M}} (b_n) = (c_n) \text{ where } \begin{cases} c_0 = a_0 \vee b_0 \\ c_n = a_n + b_n \ \text{ for } n > 0 \end{cases}$$
- $(a_n) \leq_{\mathcal{M}} (b_n)$ iff there exists $(d_n) \in \mathcal{M}$ such that:
$$\begin{cases} a_0 \leq b_0 + d_0 \\ a_n + d_{n-1} \leq b_n \cdot 2^{d_n} \ \text{ for } n > 0 \end{cases}$$
The intuition is that if a_0 is greater than b_0, we can accommodate for it by accumulating a *debt* d_0. Then we compare the second components, possibly accumulating more debt d_1, and so on. However, notice that the debt must eventually be paid back: since (b_n) ultimately becomes null, so does (d_n). Also note that the longer you wait to repay the debt, the smaller it gets: at each step, it decreases exponentially (but never reaches 0 unless $a_i < b_i$ at some point).

Examples

- $[7, 12, 1] \leq_{\mathcal{M}} [0, 2, 5]$. First we must have $7 \leq 0 + d_0$; thus, let us take a debt $d_0 = 7$. The condition on the second component is $12+7 \leq 2 \cdot 2^{d_1}$; take $d_1 = 4$. Finally, $1 + 4 \leq 5 \cdot 2^{d_2}$; we can take $d_2 = 0$ and the debt is paid off.
- if $a_n \leq b_n$ for all n, then $(a_n) \leq_{\mathcal{M}} (b_n)$: no debt is ever needed.
- If $\alpha \leq_{\mathcal{M}} \beta$, then $|\alpha| \leq |\beta|$. Indeed, let $\beta = [b_0, \ldots, b_k]$ (the case $\beta = 0_{\mathcal{M}}$ is trivial). For every index $i > k$, we have the condition $a_i + d_{i-1} \leq 0$ since $b_i = 0$. Thus, since both a_i and d_{i-1} are non-negative integers, we must have $a_i = d_{i-1} = 0$.

  In the following, we will often use this fact implicitly: whenever we have an assumption of the form $\alpha \leq_{\mathcal{M}} [b_0, \ldots, b_k]$, only the first $(k+1)$ inequalities matter. In particular, the last one will just be $a_k + d_{k-1} \leq b_k$, since $d_k = 0$ (the debt must have been paid off by then).
- The converse of the previous item is not true: $[18] \not\leq_{\mathcal{M}} [1, 1, 1, 1, 1]$.

Lemma 1. *$(|\mathcal{M}|, +_{\mathcal{M}})$ is a commutative monoid.*

Proof. The neutral element $0_{\mathcal{M}}$ is the empty list (i.e., the constant sequence whose components are all 0). Associativity and commutativity are clear since $+_{\mathcal{M}}$ is defined componentwise, and $+$ and max are both associative and commutative. □

Lemma 2. *$\leq_{\mathcal{M}}$ is a pre-order.*

Proof. Reflexivity is trivial: take $(d_n) = 0_{\mathcal{M}}$.

Transitivity: assume $(a_n) \leq_{\mathcal{M}} (b_n)$ with debt (d_n), and $(b_n) \leq_{\mathcal{M}} (c_n)$ with debt (d'_n). Let us check that $(a_n) \leq_{\mathcal{M}} (c_n)$ with debt $(d_n + d'_n)$.

- $a_0 \leq b_0 + d_0$ and $b_0 \leq c_0 + d'_0$, thus $a_0 \leq c_0 + d_0 + d'_0$.
- For all $n > 0$, we have $a_n + d_{n-1} \leq b_n \cdot 2^{d_n}$ and $b_n + d'_{n-1} \leq c_n \cdot 2^{d_n}$. Thus,

$$\begin{aligned} a_n + d_{n-1} + d'_{n-1} &\leq b_n \cdot 2^{d_n} + d'_{n-1} \\ &\leq (b_n + d'_{n-1}) \cdot 2^{d_n} \\ &\leq c_n \cdot 2^{d'_n} \cdot 2^{d_n} \\ &\leq c_n \cdot 2^{d_n + d'_n} \end{aligned}$$

□

Lemma 3. *$\leq_{\mathcal{M}}$ is compatible with $+_{\mathcal{M}}$.*

Proof. Assume $(a_n) \leq_{\mathcal{M}} (b_n)$ with debt (d_n). Check that $(a_n) +_{\mathcal{M}} (c_n) \leq_{\mathcal{M}} (b_n) +_{\mathcal{M}} (c_n)$ with debt (d_n).

- $a_0 \vee c_0 \leq (b_0 + d_0) \vee c_0 \leq (b_0 \vee c_0) + d_0$.
- For all $n > 0$, $(a_n + c_n) + d_{n-1} \leq b_n \cdot 2^{d_n} + c_n \leq (b_n + c_n) \cdot 2^{d_n}$.

□

Putting all of this together:

Lemma 4. *$\mathcal{M}$ is a resource monoid.*

Remark: $\leq_{\mathcal{M}}$ is actually an order. If $\alpha \leq_{\mathcal{M}} \beta$ and $\beta \leq_{\mathcal{M}} \alpha$, then $|\alpha| = |\beta|$. We can then show by induction on their size that $\alpha = \beta$. However, we do not need this property.

2.2 The collapse functions

We define a function $\text{collapse}_n : |\mathcal{M}| \to |\mathcal{M}|$ for every $n \in \mathbb{N}$. The idea is that, given an element $\alpha = [a_0, \ldots, a_k]$ of $\mathcal{M}$, we decrease its length by one by plugging the last component into the penultimate one (either by addition or by multiplying by a power of 2, depending on the case). We then repeat this process until the size of the list becomes $n + 1$ or smaller.

$$\begin{array}{lcll} \text{collapse}_n(\alpha) & = & \alpha & \text{if } |\alpha| \leq n+1 \\ \text{collapse}_0([a_0, a_1]) & = & [a_0 + a_1] & \\ \text{collapse}_n([a_0, \ldots, a_k]) & = & \text{collapse}_n([a_0, \ldots, a_{k-2}, a_{k-1} \cdot 2^{a_k}]) & \text{otherwise} \end{array}$$

Examples

- $\text{collapse}_3([0, 1, 2, 3, 4, 5, 6]) = [0, 1, 2, 3 \cdot 2^{4 \cdot 2^{5 \cdot 2^6}}]$
- $\text{collapse}_0([6, 3, 11, 4]) = [6 + 3 \cdot 2^{11 \cdot 2^4}]$

Lemma 5. *For all n, $|\text{collapse}_n(\alpha)| \leq n + 1$.*

Proof. By straightforward induction on the size of α. □

Lemma 6. *For all n, $\text{collapse}_n(\alpha) \leq_{\mathcal{M}} \alpha$.*

Proof. By induction on the size of α.

If $|\alpha| = 1$, then $\text{collapse}_n(\alpha) = \alpha \leq_{\mathcal{M}} \alpha$.

If $\alpha = [a_0, \ldots, a_k]$, let us look at the three cases of the definition

- if $|\alpha| \leq n+1$, then $\text{collapse}_n(\alpha) = \alpha \leq_{\mathcal{M}} \alpha$.
- if $n = 0$ and $k = 1$, then $[a_0 + a_1] \leq_{\mathcal{M}} [a_0, a_1]$ by taking the debt $d_0 = a_1$ and $d_i = 0$ for $i > 0$.
- otherwise, the induction hypothesis gives us:

  $$\text{collapse}_n([a_0, \ldots, a_{k-2}, a_{k-1} \cdot 2^{a_k}]) \leq_{\mathcal{M}} [a_0, \ldots, a_{k-2}, a_{k-1} \cdot 2^{a_k}]$$

  Moreover,

  $$[a_0, \ldots, a_{k-2}, a_{k-1} \cdot 2^{a_k}] \leq_{\mathcal{M}} [a_0, \ldots, a_k]$$

  by taking debt $d_{k-1} = a_k$ and $d_i = 0$ otherwise. By transitivity,

  $$\text{collapse}_n([a_0, \ldots, a_k]) \leq_{\mathcal{M}} [a_0, \ldots, a_k]$$

□

Lemma 7. *If $\alpha \leq_{\mathcal{M}} \beta$ and $|\alpha| \leq n+1$ then $\alpha \leq_{\mathcal{M}}$ collapse$_n(\beta)$.*

Proof. By induction on the size of β.

The case $|\beta| = 1$ works since $\text{collapse}_n(\beta) = \beta$.

Let $\beta = [b_0, \ldots, b_k]$ and (d_i) be the debt that proves $\alpha \leq_{\mathcal{M}} \beta$.

- Case $|\beta| \leq n+1$: trivial since $\text{collapse}_n(\beta) = \beta$.
- Case $n = 0$ and $k = 1$: we have $a_0 \leq b_0 + d_0$ and $a_1 + d_0 \leq b_1$. Since $|\alpha| \leq 1$, $a_1 = 0$ and thus $d_0 \leq b_1$. Hence $a_0 \leq b_0 + b_1$, so we have $\alpha \leq_{\mathcal{M}} [b_0 + b_1] = \text{collapse}_0(\beta)$ without debt.
- By induction hypothesis, we just need to prove that

  $$\alpha \leq_{\mathcal{M}} [b_0, \ldots, b_{k-2}, b_{k-1} \cdot 2^{b_k}].$$

  We use the debt (d'_i) such that $\begin{cases} d'_k = 0 \\ d'_i = d_i \text{ otherwise} \end{cases}$.

  The conditions at ranks 0 to $k-2$ stay the same. We need to check the condition at rank $k-1$, which is $a_{k-1} + d_{k-1} \leq b_{k-1} \cdot 2^{b_k}$.

  From the hypothesis $\alpha \leq_{\mathcal{M}} \beta$, we get $a_{k-1} + d_{k-1} \leq b_{k-1} \cdot 2^{d_k}$ and $a_k + d_k \leq b_k$. But $k > n$ and $|\alpha| \leq n+1$, so $a_k = 0$. Thus $d_k \leq b_k$ and we are done.

□

Corollary 1. *For all n, if $\alpha \leq_{\mathcal{M}} \beta$ then collapse$_n(\alpha) \leq_{\mathcal{M}}$ collapse$_n(\beta)$.*

Proof. By Lemma 6, $\text{collapse}_n(\alpha) \leq_{\mathcal{M}} \alpha \leq_{\mathcal{M}} \beta$. But $|\text{collapse}_n(\alpha)| \leq n+1$, so by Lemma 7, $\text{collapse}_n(\alpha) \leq_{\mathcal{M}} \text{collapse}_n(\beta)$. □

Corollary 2. *If $n \leq m$, then collapse$_n(\alpha) \leq_{\mathcal{M}}$ collapse$_m(\alpha)$.*

Proof. $\text{collapse}_n(\alpha) \leq_{\mathcal{M}} \alpha$ by Lemma 6. Moreover, $|\text{collapse}_n(\alpha)| \leq n+1 \leq m+1$, so we conclude by Lemma 7. $\square$

Lemma 8. *If $|\alpha| \leq n$ then $\text{collapse}_n(\alpha+\beta) = \alpha + \text{collapse}_n(\beta)$.*

Proof. By induction on the size of β.

All the cases are trivial except the last one: $\beta = [b_0, \ldots, b_k]$ with $k > n$. Since $|\alpha| \leq n$, $a_{k-1} = a_k = 0$. Thus, $\alpha+\beta = [a_0 \vee b_0, \ldots, a_{k-2}+b_{k-2}, b_{k-1}, b_k]$ Therefore,

$$\begin{aligned}
\text{collapse}_n(\alpha+\beta) &= \text{collapse}_n([a_0 \vee b_0, \ldots, a_{k-2}+b_{k-2}, b_{k-1} \cdot 2^{b_k}]) \\
&= \text{collapse}_n(\alpha + [b_0, \ldots, b_{k-2}, b_{k-1} \cdot 2^{b_k}]) \\
&= \alpha + \text{collapse}_n([b_0, \ldots, b_{k-2}, b_{k-1} \cdot 2^{b_k}]) \qquad \text{by IH} \\
&= \alpha + \text{collapse}_n(\beta)
\end{aligned}$$

$\square$

2.3 Interpretations of o, 0, S

We want to interpret our λ-calculus in the category of length spaces on $\mathcal{M}$, such that the underlying sets of the length spaces, and the underlying functions of the morphisms, are the usual set-theoretic semantics.

First, we need a length space whose underlying set is $\mathbb{N}$ in order to interpret the type o:

Definition 6. Define $\mathbf{N} = (\mathbb{N}, \Vdash_{\mathbf{N}})$, where $\alpha \Vdash_{\mathbf{N}} n$ iff $[n] \leq_{\mathcal{M}} \alpha$.

Lemma 9. *$\mathbf{N}$ is a length space on $\mathcal{M}$.*

Proof.

- For all n, $[n] \Vdash_{\mathbf{N}} n$.
- If $\alpha \Vdash_{\mathbf{N}} n$ and $\alpha \leq_{\mathcal{M}} \beta$, by transitivity $[n] \leq_{\mathcal{M}} \beta$ so $\beta \Vdash_{\mathbf{N}} n$.

$\square$

Lemma 10. *We have another characterization of $\Vdash_{\mathbf{N}}$: $\alpha \Vdash_{\mathbf{N}} n$ iff $[n] \leq_{\mathcal{M}} \text{collapse}_0(\alpha)$. If we write $\text{collapse}_0(\alpha) = [a_0]$, then $\alpha \Vdash_{\mathbf{N}} n$ iff $n \leq a_0$.*

Proof. If $[n] \leq_{\mathcal{M}} \alpha$, then by Lemma 7, $[n] \leq_{\mathcal{M}} \text{collapse}_0(\alpha)$. Conversely, if $[n] \leq_{\mathcal{M}} \text{collapse}_0(\alpha)$, then $[n] \leq_{\mathcal{M}} \alpha$ by Lemma 6 and transitivity. $\square$

Next, we want to interpret the function symbols 0 and S. Recall that the unit of the category of length spaces is $\mathcal{I} = (\{\star\}, \Vdash_{\mathcal{I}})$ where $\alpha \Vdash_{\mathcal{I}} \star$ for all α.

We interpret $0 : o$ as the morphism in $\mathcal{I} \multimap \mathbf{N}$ that sends $\star$ to $0 \in \mathbb{N}$. This morphism is majorized by any element of $\mathcal{M}$, for example, take $0_{\mathcal{M}}$.

$S : o \to o$ is a little bit less trivial: we want to interpret it as the successor function $n \mapsto n+1 : \mathbb{N} \to \mathbb{N}$. To prove that it is indeed a morphism in $\mathbf{N} \multimap \mathbf{N}$, we need to find a majorizer. Let us check that $[0,1] \Vdash_{\mathbf{N} \multimap \mathbf{N}} n \mapsto n+1$.

Suppose that $\alpha \Vdash_{\mathbf{N}} n$. We must find β such that $\beta \Vdash_{\mathbf{N}} n+1$ and $\beta \leq_{\mathcal{M}} \alpha + [0,1]$. Take $\beta = [n+1]$. Then $\beta \Vdash_{\mathbf{N}} n+1$ by reflexivity. Moreover, write $[a_0] = \mathrm{collapse}_0(\alpha)$. By Lemma 10, $a_0 \geq n$. Then, check that $\beta \leq_{\mathcal{M}} [a_0, 1]$ with debt $d_0 = 1$. Thus, $\beta \leq_{\mathcal{M}} \mathrm{collapse}_0(\alpha) + [0,1] \leq_{\mathcal{M}} \alpha + [0,1]$ by Lemma 6 and compatibility.

2.4 The duplication morphism

Theorem 1 tells us that we can interpret multiplicative linear logic with full weakening in the category of length spaces on $\mathcal{M}$. We have also given the interpretation of the base type and function symbols.

What remains to be done is to define a *duplication* morphism in $\mathcal{A} \multimap \mathcal{A} \otimes \mathcal{A}$ whose underlying function is $a \mapsto (a, a)$. Unfortunately, this is not possible in general: we would have to find an element $\varphi \in \mathcal{M}$ such that $\alpha + \alpha \leq_{\mathcal{M}} \varphi + \alpha$ for all α, and no such element exists (take $|\alpha| > |\varphi|$).

Thus, we restrict ourselves to the full subcategory $\mathcal{S}$ of the category of length spaces on $\mathcal{M}$, whose objects are the length spaces built from $\mathbf{N}$, $\mathcal{I}$, $\multimap$ and $\otimes$.

Definition 7. The *order* $\mathrm{Ord}(\tau)$ of a type τ is defined inductively as follows:

$$\begin{array}{lcl} \mathrm{Ord}(o) & = & 0 \\ \mathrm{Ord}(\tau_1 \to \tau_2) & = & \max(\mathrm{Ord}(\tau_1)+1, \mathrm{Ord}(\tau_2)) \\ \mathrm{Ord}(\tau_1 \times \tau_2) & = & \max(\mathrm{Ord}(\tau_1), \mathrm{Ord}(\tau_2)) \end{array}$$

When $\mathcal{A}$ is in $\mathcal{S}$, we also define $\mathrm{Ord}(\mathcal{A})$ in a similar way.

The following lemma is the main motivation for the definition of the collapse function. Intuitively, it says that the function denoted by a term $t : \tau$ can always be majorized by an element of $\mathcal{M}$ of size at most $\mathrm{Ord}(\tau)+1$. Such a majorizer is obtained from any other majorizer by applying the function $\mathrm{collapse}_{\mathrm{Ord}(\tau)}$.

Lemma 11. *Let $\mathcal{A} \in \mathcal{S}$, and let $n = \mathit{Ord}(\mathcal{A})$. If $\alpha \Vdash_{\mathcal{A}} a$, then $\mathit{collapse}_n(\alpha) \Vdash_{\mathcal{A}} a$.*

Proof. By induction on the structure of $\mathcal{A}$.

- The case $\mathcal{A} = \mathcal{I}$ is trivial.
- The case $\mathcal{A} = \mathbf{N}$ follows from Lemma 10.
- Case $\mathcal{A} = \mathcal{B} \multimap \mathcal{C}$: assume $\varphi \Vdash_{\mathcal{B}\multimap\mathcal{C}} f$. By definition of the order, $\mathrm{Ord}(\mathcal{B}) \leq n-1$ and $\mathrm{Ord}(\mathcal{C}) \leq n$.

  Assume $\beta \Vdash_{\mathcal{B}} b$. By induction hypothesis, $\mathrm{collapse}_{\mathrm{Ord}(\mathcal{B})}(\beta) \Vdash_{\mathcal{B}} b$. By Corollary 2 and upward-closure of $\Vdash$, $\mathrm{collapse}_{n-1}(\beta) \Vdash_{\mathcal{B}} b$.

  Thus, since φ majorizes f, there is $\gamma \leq_{\mathcal{M}} \varphi + \mathrm{collapse}_{n-1}(\beta)$ such that $\gamma \Vdash_{\mathcal{C}} f(b)$. By the second induction hypothesis (and upward-closure), $\mathrm{collapse}_n(\gamma) \Vdash_{\mathcal{C}} f(b)$. Finally,

  $$\begin{aligned}
  \mathrm{collapse}_n(\gamma) &\leq_{\mathcal{M}} \mathrm{collapse}_n(\varphi + \mathrm{collapse}_{n-1}(\beta)) && \text{by monotonicity (1)}\\
  &\leq_{\mathcal{M}} \mathrm{collapse}_n(\varphi) + \mathrm{collapse}_{n-1}(\beta) && \text{by Lemma 8}\\
  &\leq_{\mathcal{M}} \mathrm{collapse}_n(\varphi) + \beta && \text{by Lemma 6}
  \end{aligned}$$

  Hence, $\mathrm{collapse}_n(\varphi) \Vdash_{\mathcal{B}\multimap\mathcal{C}} f$.
- Case $\mathcal{A} = \mathcal{B} \otimes \mathcal{C}$: assume $\alpha \Vdash_{\mathcal{B}\otimes\mathcal{C}} (b, c)$. There are β and γ such that $\beta \Vdash_{\mathcal{B}} b$, $\gamma \Vdash_{\mathcal{C}} c$ and $\beta + \gamma \leq_{\mathcal{M}} \alpha$.

  By induction hypothesis (and upward-closure of $\Vdash$), $\mathrm{collapse}_n(\beta) \Vdash_{\mathcal{B}} b$ and $\mathrm{collapse}_n(\gamma) \Vdash_{\mathcal{C}} c$.

  Moreover, $\mathrm{collapse}_n(\beta) + \mathrm{collapse}_n(\gamma) \leq_{\mathcal{M}} \beta + \gamma \leq_{\mathcal{M}} \alpha$. By Lemma 7, $\mathrm{collapse}_n(\beta) + \mathrm{collapse}_n(\gamma) \leq_{\mathcal{M}} \mathrm{collapse}_n(\alpha)$.

  Therefore, $\mathrm{collapse}_n(\alpha) \Vdash_{\mathcal{B}\otimes\mathcal{C}} (b, c)$.

$\square$

Lemma 12. *Let $\varphi = [1, \ldots, 1]$ where $|\varphi| = n + 1$. Then, for all α such that $|\alpha| \leq n$, $\alpha + \alpha \leq_{\mathcal{M}} \varphi + \alpha$.*

Proof. Write $\alpha = (a_i)$ and take the debt $d_i = 1$ for $i \leq n-1$ and $d_i = 0$ otherwise.

On the first component, $a_0 \vee a_0 \leq (a_0 \vee 1) + 1$ is verified.
For $1 \leq i \leq n-1$, $a_i + a_i + 1 \leq (a_i + 1) \cdot 2^1$ is verified.
The last condition is just $1 \leq 1$ since $a_n = 0$. $\square$

We can now define the duplication in $\mathcal{S}$:

Lemma 13. *For every object $\mathcal{A}$ of $\mathcal{S}$ with Ord($\mathcal{A}$) $= n$, the duplication function $a \mapsto (a, a)$ is majorized by $\varphi_n = [1, \ldots, 1]$ where $|\varphi_n| = n + 2$.*

Proof. Assume $\alpha \Vdash_{\mathcal{A}} a$, we must find $\beta \leq_{\mathcal{M}} \varphi_n + \alpha$ such that $\beta \Vdash_{\mathcal{A}\otimes\mathcal{A}} (a,a)$. Take $\beta = \text{collapse}_n(\alpha) + \text{collapse}_n(\alpha)$.

By Lemma 11, $\text{collapse}_n(\alpha) \Vdash_{\mathcal{A}} a$, so $\beta \Vdash_{\mathcal{A}\otimes\mathcal{A}} (a,a)$.

Remains to show that $\beta \leq_{\mathcal{M}} \varphi_n + \alpha$. Since $|\text{collapse}_n(\alpha)| \leq n+1$, by Lemma 12, $\beta \leq_{\mathcal{M}} \varphi_n + \text{collapse}_n(\alpha)$. Then we conclude by Lemma 6. $\square$

Finally:

Theorem 2. *The category $\mathcal{S}$ is cartesian closed. Thus, it is a model of the simply-typed λ-calculus.*

We can now prove the property mentioned in the beginning of this section.

Let $f : o \to o$, and $[\![f]\!] : \mathbb{N} \to \mathbb{N}$ its set-theoretic denotation. Then, there is a morphism $[\![f]\!]_{\mathcal{S}} : \mathbf{N} \multimap \mathbf{N}$ in $\mathcal{S}$ whose underlying function is $[\![f]\!]$. Let φ be a majorizer of $[\![f]\!]$. By Lemma 11, $\text{collapse}_1(\varphi) = [f_0, f_1]$ is also a majorizer. Then, for any n, since $[n] \Vdash_{\mathbf{N}} n$, there is $\beta \leq_{\mathcal{M}} [n] + [f_0, f_1]$ such that $\beta \Vdash_{\mathbf{N}} f(n)$. By upper-closure of $\Vdash$, $[n] + [f_0, f_1] \Vdash_{\mathbf{N}} f(n)$, and by Lemma 10, $f(n) \leq (n \vee f_0) + f_1 \leq n + (f_0 + f_1) = n + C_f$.

We also get other results: consider $f : o^k \to o$, then by a similar reasoning, we can prove that $f(n_1, \ldots, n_k) \leq \max(n_1, \ldots, n_k) + C_f$.

2.5 Bounding the constant

What does our model tell us about the constant C_f?

First, notice that in the previous paragraph, whenever $[\![f]\!]$ is majorized by $\varphi = [f_0, \ldots, f_n]$, then the constant C_f is such that $[C_f] = \text{collapse}_0(\varphi)$. Indeed, we first use collapse_1, then add the two components, which is by definition what collapse_0 does.

Therefore, C_f is a tower of exponentials involving the coefficients (f_i) which looks like this:

$$C_f = f_0 + f_1 \cdot 2^{f_2 \cdot 2^{\cdot^{\cdot^{\cdot^{f_n}}}}}$$

Thus, the intuition is that C_f strongly depends on the length of the majorizer, which determines the height of this tower of exponentials. In our model, the only thing that increases the length of a majorizer is when we use the duplication morphism: duplicating a variable of type τ requires a majorizer of length $\text{Ord}(\tau) + 1$.

We will show that C_f is bounded by a tower of exponentials whose height is related to the maximal order of a non-linear λ-abstraction that appears in f.

Definition 8. Let us define notations for these towers of exponentials:

- $\mathbf{2}^a_n = 2^{2^{\cdot^{\cdot^{\cdot^{a}}}}}$ of height n, i.e., $\begin{cases} \mathbf{2}^a_0 = a \\ \mathbf{2}^a_{n+1} = 2^{\mathbf{2}^a_n} \end{cases}$
- $\mathscr{C}([a_1, \ldots, a_n]) = a_1 \cdot 2^{a_2 \cdot 2^{\cdot^{\cdot^{\cdot^{a_n}}}}}$, i.e., $\begin{cases} \mathscr{C}([\,]) = 0 \\ \mathscr{C}([a_1, \ldots, a_n]) = a_1 \cdot 2^{\mathscr{C}([a_2, \ldots, a_n])} \end{cases}$

With this notation, we have $\mathrm{collapse}_0([a_0, \ldots, a_n]) = a_0 + \mathscr{C}([a_1, \ldots, a_n])$ (this is proved by an easy induction).

Lemma 14. *Let $\alpha = [a_0, \ldots, a_n] \in \mathcal{M}$, and write $a = \max(a_0, \ldots, a_n)$. Then $\mathit{collapse}_0(\alpha) \leq \mathbf{2}^a_n$.*

Proof. Since $a_i \leq a$ for all i, $[a_0, \ldots, a_n] \leq [a, \ldots, a]$ with $(n+1)$-many a's, and by Corollary 1, it is enough to prove that $\mathrm{collapse}_0([a, \ldots, a]) \leq \mathbf{2}^a_n$. To keep track of the number of a's, write $\mathscr{C}^a_n = \mathscr{C}([a, \ldots, a])$ with n-many a's. Thus we want to prove that $a + \mathscr{C}^a_n \leq \mathbf{2}^a_n$.

Proceed by induction on n.

- $n = 0$ is trivial: $\mathscr{C}^a_0 = 0$ and $\mathbf{2}^a_0 = a$.
- $\mathbf{2}^a_{n+1} = 2^{\mathbf{2}^a_n} \geq 2^{a + \mathscr{C}^a_n} = 2^a \cdot 2^{\mathscr{C}^a_n}$ by induction hypothesis.

  Moreover, since a is an integer, we have $2^a \geq a + a$.

  So $2^a \cdot 2^{\mathscr{C}^a_n} \geq (a + a) \cdot 2^{\mathscr{C}^a_n} \geq a + a \cdot 2^{\mathscr{C}^a_n} = a + \mathscr{C}^a_{n+1}$.

$\square$

Definition 9. We define the *rank* of a well-typed term t. The idea is that $\mathsf{rk}(t)$ is the maximal order of a contraction that occurs in the typing derivation of t. When t is closed, the rank is defined as follows:

$$\begin{array}{lcl} \mathsf{rk}(x) & = & 0 \\ \mathsf{rk}(t\ u) & = & \max(\mathsf{rk}(t), \mathsf{rk}(u)) \\ \mathsf{rk}(\lambda x^\tau.\ t) & = & \begin{cases} \mathsf{rk}(t) & \text{if } x \text{ appears at most once in } t \\ \max(\mathrm{Ord}(\tau), \mathsf{rk}(t)) & \text{otherwise} \end{cases} \end{array}$$

When t has free variables $x_1, \ldots, x_n$ and is typed in context $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$, then we define

$$\mathsf{rk}(\Gamma \vdash t : \tau) = \max(\{\mathrm{Ord}(\tau_i) \mid x_i \text{ appears at least twice in } t\} \cup \{\mathsf{rk}(t)\})$$

For $\alpha = [a_0, \ldots, a_n]$, we write $\max(\alpha) = \max(a_0, \ldots, a_n)$.
For $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$, we write $|\Gamma| = n$ and $\mathsf{var}(\Gamma) = \{x_1, \ldots, x_n\}$.

Lemma 15. *If $\Gamma \vdash t : \tau$ and $\mathsf{var}(\Gamma) \subseteq \mathsf{FV}(t)$, then there is a majorizer α of $[\![t]\!]$ such that $|\alpha| \leq \mathsf{rk}(\Gamma \vdash t : \tau) + 2$ and $\max(\alpha) + |\Gamma| \leq |t|$.*

Proof. We would like to reason by induction on the typing derivation of $\Gamma \vdash t : \tau$ (with explicit contraction and weakening), but first, we need to make sure that this derivation doesn't contain unnecessary contractions.

Claim 1: *For every derivation of $\Gamma, x : \sigma \vdash t : \tau$ with $x \notin \mathsf{FV}(t)$, there is a derivation of $\Gamma \vdash t : \tau$ whose height is smaller or equal.*

Proof: by induction on the derivation.

Claim 2: If $\Gamma \vdash t : \tau$ *with* $\mathsf{var}(\Gamma) \subseteq \mathsf{FV}(t)$, *then there is a derivation of $\Gamma \vdash t : \tau$ where the weakening rule only occurs right after a lambda rule that introduced the weakened variable.*

Proof: by induction on the height of the derivation.

- Axiom: $x : \tau \vdash x : \tau$.

  This derivation already satisfies the property.

- App: $\Gamma, \Delta \vdash t\ u : \tau$ comes from $\Gamma \vdash t : \sigma \to \tau$ and $\Delta \vdash u : \sigma$.

  Since $\mathsf{var}(\Gamma, \Delta) \subseteq \mathsf{FV}(t\ u)$ by assumption and $\Gamma \cap \Delta = \varnothing$, we have $\mathsf{var}(\Gamma) \subseteq \mathsf{FV}(t)$ and $\mathsf{var}(\Delta) \subseteq \mathsf{FV}(u)$, so we can use the induction hypothesis on both premises. Then by applying the App rule we get a derivation that satisfies the property.

- Lam: $\Gamma \vdash \lambda x.\ t : \sigma \to \tau$ comes from $\Gamma, x : \sigma \vdash t : \tau$.

  Either $x \in FV(t)$ or $x \notin \mathsf{FV}(t)$. In the first case, we can use our induction hypothesis and we are done. In the second case, Claim 1 gives us a derivation of $\Gamma \vdash t : \tau$ whose height is smaller, so we can use the induction hypothesis. We thus get a derivation of $\Gamma \vdash t : \tau$. Using weakening on x and the Lam rule, we get a derivation of $\Gamma \vdash \lambda x.\ t : \sigma \to \tau$ that satisfies the property.

- Contr: $\Gamma, x : \sigma \vdash t[x/y][x/z] : \tau$ comes from $\Gamma, y : \sigma, z : \sigma \vdash t : \tau$.

  At least one of the variables y and z is in $\mathsf{FV}(t)$, otherwise x wouldn't either. If they both are in $\mathsf{FV}(t)$, we use the induction hypothesis and we are done. Otherwise, if $y \notin \mathsf{FV}(t)$, then Claim 1 gives us a smaller derivation of $\Gamma, z : \sigma \vdash t : \tau$, we can then use the induction hypothesis, and by renaming z into x, we are done. Same reasoning if $z \notin \mathsf{FV}(t)$.

- Weak: $\Gamma, x : \sigma \vdash t : \tau$ comes from $\Gamma \vdash t : \tau$.

  This case is not possible since by assumption $x \in \mathsf{FV}(t)$.

We can now prove Lemma 15, with the extra assumption that the derivation doesn't contain weakening rules, except right after a lambda rule. By induction on the typing derivation of $\Gamma \vdash t : \tau$.

- Axiom: $x : \tau \vdash x : \tau$.

  The morphism $\mathrm{Id} : [\![\tau]\!] \to [\![\tau]\!]$ is majorized by $0_{\mathcal{M}}$, which satisfies the property.

- App: $\Gamma, \Delta \vdash t\ u : \tau$ comes from $\Gamma \vdash t : \sigma \to \tau$ and $\Delta \vdash u : \sigma$.

  By induction hypothesis, we get two majorizers α and β of $[\![t]\!]$ and $[\![u]\!]$ which satisfy the property. Then, the morphism $[\![t\ u]\!]$ is obtained by composing $[\![t]\!] \otimes [\![u]\!]$ with the evaluation morphism. Since eval is majorized by $0_{\mathcal{M}}$, $[\![t\ u]\!]$ is majorized by $\alpha + \beta$.

  Then,

$$\begin{aligned} |\alpha+\beta| &\leq \max(|\alpha|, |\beta|) \\ &\leq \max(\mathsf{rk}(\Gamma \vdash t : \sigma \to \tau) + 2, \mathsf{rk}(\Delta \vdash u : \sigma) + 2) \\ &= \max(\mathsf{rk}(\Gamma \vdash t : \sigma \to \tau), \mathsf{rk}(\Delta \vdash u : \sigma)) + 2 \\ &= \mathsf{rk}(\Gamma, \Delta \vdash t\ u : \tau) + 2 \end{aligned}$$

  and

$$\begin{aligned} \max(\alpha+\beta) + |\Gamma, \Delta| &\leq \max(\alpha) + |\Gamma| + \max(\beta) + |\Delta| \\ &\leq |t| + |u| \\ &\leq |t\ u| \end{aligned}$$

- Lam: $\Gamma \vdash \lambda x.\ t : \sigma \to \tau$ comes from $\Gamma, x : \sigma \vdash t : \tau$.

  If $x \in \mathsf{FV}(t)$, then by induction hypothesis, we have a majorizer α for $[\![t]\!] : \Gamma \otimes [\![\sigma]\!] \to \tau$. We obtain a morphism in $\Gamma \to \sigma \multimap \tau$ by currying, which is majorized by $0_{\mathcal{M}}$. Thus, $[\![\lambda x.\ \tau]\!]$ is majorized by α.

  Then, $\quad |\alpha| \leq \mathsf{rk}(\Gamma, x : \sigma \vdash t : \tau) + 2 = \mathsf{rk}(\Gamma \vdash \lambda x.\ t : \sigma \to \tau) + 2,$

  and $\quad \max(\alpha) + |\Gamma| \leq \max(\alpha) + |\Gamma, x : \sigma| - 1 \leq |t| - 1 \leq |\lambda x.\ t|.$

  If $x \notin \mathsf{FV}(t)$, the rule is immediately followed by a weakening whose premise is $\Gamma \vdash t : \tau$. The induction hypothesis on this premise gives us a majorizer α for $[\![t]\!] : \Gamma \to \tau$. By weakening and currying, we get a morphism in $\Gamma \to \sigma \multimap \tau$ majorized by α

The condition on the size still holds, and the one on the max becomes: $\max(\alpha) + |\Gamma| \leq |t| \leq |\lambda x.\ t|$.

- CONTR: $\Gamma, x : \sigma \vdash t[x/y][x/z] : \tau$ comes from $\Gamma, y : \sigma, z : \sigma \vdash t : \tau$.

  Both y and z are necessarily in $\mathsf{FV}(t)$: otherwise, it would need to be weakened later, which is not possible because we assumed that all weakenings delete a variable which was introduced by an immediately preceding lambda rule.

  We use the induction hypothesis, and we get a majorizer α of $[\![t]\!]$. We compose $[\![t]\!]$ with the duplication morphism $\Gamma \otimes [\![\sigma]\!] \to \Gamma \otimes [\![\sigma]\!] \otimes [\![\sigma]\!]$. Thus, we get a morphism $[\![t[x/y][x/z]]\!]$ that is majorized by $\alpha + \varphi_n$, where $n = \mathrm{Ord}(\sigma)$ and $\varphi_n = [1, \ldots, 1]$ with $(n+2)$-many 1's.

  Then,

$$\begin{aligned} |\alpha + \varphi_n| &\leq \max(|\alpha|, |\varphi_n|) \\ &\leq \max(\mathsf{rk}(\Gamma, y : \sigma, z : \sigma \vdash t : \tau) + 2, n + 2) \\ &= \max(\mathsf{rk}(\Gamma, y : \sigma, z : \sigma \vdash t : \tau), \mathrm{Ord}(\sigma)) + 2 \\ &\leq \mathsf{rk}(\Gamma, x : \sigma \vdash t[x/y][x/z] : \tau) + 2 \end{aligned}$$

  and

$$\begin{aligned} \max(\alpha + \varphi_n) + |\Gamma, x : \sigma| &\leq \max(\alpha) + 1 + |\Gamma, y : \sigma, z : \sigma| - 1 \\ &\leq |t| \end{aligned}$$

- WEAK: $\Gamma, x : \sigma \vdash t : \tau$ comes from $\Gamma \vdash t : \tau$.

  This case is not possible since weakenings only occur after lambdas: it cannot be at the root.

- ZERO and SUCC: $\vdash 0 : o$ and $\vdash S : o \to o$.

  0 is majorized by $0_{\mathcal{M}}$, which trivially satisfies the desired properties.

  S is majorized by $[0, 1]$, whose size is $|[0, 1]| = 2 \leq \mathsf{rk}(S) + 2 = 2$, and $\max([0, 1]) = 1 \leq |S| = 1$.

$\square$

Finally, applying Lemma 15 to a closed term $\vdash f : o \to o$, we get a majorizer φ of $[\![f]\!]$ with $|\varphi| \leq \mathsf{rk}(f) + 2$ and $\max(\varphi) \leq |f|$.

By Lemma 14, we get a bound on the constant C_f:

$$C_f = \text{collapse}_0(\varphi) \leq \mathbf{2}^{|f|}_{\mathsf{rk}(f)+1}$$

3 Examples

$\mathbf{f} := \lambda \mathbf{x}.\ \mathbf{S^k\ x}$. The successor function is majorized by $[0, 1]$, and each application adds the majorizers together.

Thus, f is majorized by $[0, k]$, which gives $C_f = k$. Indeed, the denotation of this term is $[\![f]\!](n) = n + k$.

Our bound gives $C_f \leq \mathbf{2}^{|f|}_{\mathsf{rk}(f)+1} = \mathbf{2}^{k+1}_1 = 2^{k+1}$.

$\bar{\mathbf{2}} := \lambda \mathbf{f^{o \to o}}.\lambda \mathbf{x^o}.\ \mathbf{f\ (f\ x)}$. Because of the contraction on the variable f of type $o \to o$, this term is majorized by $[1, 1, 1]$.

More generally, $\bar{2} = \lambda f^{\tau \to \tau}.\lambda x^{\tau}.\ f\ (f\ x)$ where $\text{Ord}(\tau) = n$ is majorized by $\varphi_{n+1} = [1, \ldots, 1]$ with $|\varphi_{n+1}| = n + 3$.

$\mathbf{g} := \bar{\mathbf{2}}\ \mathbf{S}$ is thus majorized by $[1, 1, 1] + [0, 1] = [1, 2, 1]$, which gives the constant $C_g = 5$.

Its denotation is $[\![g]\!](n) = n + 2$.

Our bound gives $C_g \leq \mathbf{2}^{|g|}_{\mathsf{rk} g+1} = \mathbf{2}^4_2 = 2^{2^4} = 65536$.

$\mathbf{h} := \bar{\mathbf{2}}\ (\bar{\mathbf{2}}\ (\ldots\ (\bar{\mathbf{2}}\ \mathbf{S})))$, where $\bar{2}$ appears k times.

Its denotation is $[\![h]\!](n) = n + 2^k$.

It is majorized by $[1, k+1, k]$, thus $C_h = 1 + (k+1) \cdot 2^k$.

Our bound gives $C_h \leq 2^{2^{3k+1}}$.

$\mathbf{t} := (((\bar{\mathbf{2}}\ \bar{\mathbf{2}})\ \ldots)\ \bar{\mathbf{2}})\ \mathbf{S}$, where $\bar{2}$ appears k times.

Its denotation is $[\![t]\!](n) = n + \mathbf{2}^1_k$.

The $\bar{2}$ on the right is majorized by $[1, 1, 1]$, then each $\bar{2}$ has one more 1 than its neighbour, until the $\bar{2}$ on the left whose majorizer has size $k + 2$. Also, S is majorized by $[0, 1]$.

Thus, t is majorized by $[1, k+1, k, k-1, k-2, \ldots, 3, 2, 1]$ of size $k + 2$.

This gives us $C_t = 1 + (k+1) \cdot 2^{k \cdot 2^{\cdot^{\cdot^{\cdot^{2 \cdot 2^1}}}}}$ (the height is k).

Our bound gives $C_t \leq \mathbf{2}^{3k+1}_{k+1} = 2^{2^{\cdot^{\cdot^{\cdot^{3k+1}}}}}$ (the height is $k + 1$).

References

[1] Ugo Dal Lago and Martin Hofmann. Quantitative models and implicit complexity. In *Proc. Foundations of Software Technology and Theoretical Computer Science*, pages 189–200, 2005.