A Resource Monoid for Simply Typed λ-calculus

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Abstract

We use a simplified version of the framework of resource monoids, introduced by Dal Lago and Hofmann [5, 8], to interpret simply typed λ-calculus with constants zero and successor. We then use this model to prove a simple quantitative result about bounding the size of the normal form of λ-terms. To our knowledge, our proof is the first semantic proof of this result. Our use of resource monoids differs from the other instances found in the literature, in that it measures the size of λ-terms rather than time complexity.

1 Introduction

Resource monoids were introduced by Dal Lago and Hofmann as part of a semantic framework to prove soundness theorems in the field of Implicit Computational Complexity [8]. The aim of implicit complexity is to characterize complexity classes as classes of functions definable in a certain logical systems, often based on linear logic. In this context, we are usually interested in proving results of the form: “every definable function lies in a given complexity class”. By instantiating their general framework in various different ways, Dal Lago and Hofmann proved such soundness results for various programming languages: Elementary Affine Logic, LFPL and Soft Affine Logic in [8]; Light Affine Logic in [7]; and Bounded Affine Logic in [6]. The same technique was used by Brunel and Terui [2] to prove the polytime soundness of the system DIAL_{lin}. A modification of the resource monoid framework dubbed quantitative realizability was introduced in [1].

The main idea of Dal Lago and Hofmann’s framework is to consider a modification of realizability, where bounded-time algorithms are used as realizers. The bounds are expressed abstractly as elements of an algebraic structure called a resource monoid. This allows for more flexibility in the framework. Given a particular resource monoid $\mathcal{M}$, one can define the notion of length spaces on $\mathcal{M}$, that is, sets of data that can be bounded

*This work was carried out in 2016 while I, the second author, was doing a Masters Internship with Martin. While he could not take part in writing this article, he is undoubtedly an author of the work presented here.
by elements of $\mathcal{M}$ in a coherent way. As it turns out, the category of length spaces on any resource monoid $\mathcal{M}$ is symmetric monoidal closed. Thus, one can interpret second-order multiplicative affine logic in this category, independently of the choice of $\mathcal{M}$. This is the common base of this semantic framework. To be able to model logical systems with additional features, one must choose an appropriate resource monoid $\mathcal{M}$, so that the associated category of length spaces has the required structure.

To our knowledge, all instances of resource monoids previously found in the literature were used to measure the time complexity of programs. Moreover, they all deal with systems based on linear logic, where duplication of data is forbidden or restricted. In this paper, we present a new use-case of resource monoids. Our goal is to prove upper bounds on the size of normal forms in simply typed $\lambda$-calculus. More precisely, we give a semantic proof of the following well-known result due to Schwichtenberg [13]:

**Claim:** For any simply typed $\lambda$-term $t$, the size of its normal form $\text{nf}(t)$ is bounded by some tower of exponentials which depends on the size of $t$ and the order of its non-linear $\lambda$-abstractions.

To achieve this, we define a particular resource monoid $\mathcal{M}$ whose elements can be used to bound the potential of a $\lambda$-term to produce large normal forms. The main difficulty to be able to interpret simply typed $\lambda$-calculus is then to prove that the category of length spaces over $\mathcal{M}$ actually has duplication morphisms, allowing us to model the contraction rule.

This question is closely related to the notion of $\lambda$-definability of functions between natural numbers. A function $f : \mathbb{N} \to \mathbb{N}$ is $\lambda$-definable if there is a $\lambda$-term $t : \text{Nat}_o \to \text{Nat}_o$ such that for all $n \in \mathbb{N}$, $t \overline{n} =_{\beta} f(n)$. Here, $\text{Nat}_o = (o \to o) \to o \to o$ stands for the type of Church numerals over some base type $o$, and $\overline{n} \in \text{Nat}_o$ is the Church encoding of $n \in \mathbb{N}$. The famous result by Schwichtenberg claims that those functions are exactly the extended polynomials [12]. This is not the end of the question however: one can greatly augment the expressivity by considering terms $t : \text{Nat}_\tau \to \text{Nat}_o$, where $\tau$ is an arbitrary type. For this notion of $\lambda$-definability, the question of precisely characterizing the class of definable functions remains open [9]. Recent discoveries of links between linear logic and automata theory have sparked new interest in the notion of $\lambda$-definability [11, 10].

**Plan of the paper.** The preliminary Section 2 starts by formulating the precise statements that we will prove in the paper, and defines a simplified version of the notions of resource monoids and length spaces. Then in Section 3, we define our resource monoid $\mathcal{M}$ of interest, and prove some useful properties. Section 4 defines our interpretation of simply types $\lambda$-calculus in the category of length spaces over $\mathcal{M}$. Finally, Section 5 uses this model to give a semantic proof of Schwichtenberg’s theorem.
2 Preliminaries

2.1 A simple result about $\lambda$-calculus

We start by formulating a simple question about the computational power of simply typed $\lambda$-calculus. The result itself is well known; the point of this paper will be to provide a semantic proof of it, using the framework of resource monoids.

Simply typed $\lambda$-calculus with constants. We consider simply typed $\lambda$-calculus equipped with one base type $o$, and two constants $0 : o$ and $S : o \to o$. Thus, the types are given by the following grammar:

$$\sigma, \tau ::= o \mid \sigma \to \tau \mid \sigma \times \tau$$

The terms are given by the grammar below, where $x$ belongs to some infinite set of variables.

$$t, u ::= 0 \mid S \mid x \mid t \ u \mid \lambda x. t$$

The typing rules are given below, with explicit contraction and weakening.

\[
\begin{align*}
\text{Var} & \quad \dfrac{}{\Gamma, x : \sigma \vdash x : \sigma} \\
\text{Lam} & \quad \dfrac{}{\Gamma \vdash \lambda x. t : \sigma \to \tau} \\
\text{App} & \quad \dfrac{}{\Gamma, \Delta \vdash t \ u : \tau} \\
\text{ZERO} & \quad \dfrac{}{\vdash 0 : o} \\
\text{Succ} & \quad \dfrac{}{\vdash S : o \to o} \\
\text{Weak} & \quad \dfrac{}{\Gamma, x : \sigma \vdash t : \tau} \\
\text{Contr} & \quad \dfrac{}{\Gamma, z : \sigma \vdash t[x, y \leftarrow z] : \tau}
\end{align*}
\]

A simply typed busy beaver. Under the standard set-theoretic interpretation of $\lambda$-terms (with $[o] = \mathbb{N}$, $[0] = 0$ and $[S] = n \mapsto n + 1$), closed terms of type $o$ correspond to natural numbers. A natural question to ask is how compactly we can write large natural numbers. More precisely, given a closed term $t : o$ of size $|t|$, can we find a bound on its denotation $[t] \in \mathbb{N}$?

For instance, if we write $2 := \lambda f. \lambda x. f (f \ x)$, for the church encoding of the number two, we can compare the following $\lambda$-terms of type $o$:

(a) $t = S (S (S (\ldots (S 0))))$ with $n$ occurrences of $S$.
(b) $u = 2 (2 (2 (\ldots (2 S)))) 0$ with $n$ occurrences of $\bar{2}$.
(c) $v = ((2 \bar{2}) \bar{2}) \ldots \bar{2} S 0$ with $n$ occurrences of $\bar{2}$.

Their denotations are respectively $[t] = n$, $[u] = 2^n$, and $[v] = 2^{2^{\ldots^{2^n}}}$, a tower of exponentials of height $n$. Our goal is to show that one cannot do better than these towers of exponentials. More precisely, we will prove the following bound:
Claim 1. For every closed term \( t : o \), we have \([t] \leq 2^{2^{[t]}}\), where the height of the tower of exponentials is one plus the maximal order of a non-linear \( \lambda\)-abstraction occurring in \( t \).

A related question is to ask which set-theoretic functions arise as the denotation of some closed term of type \( o \to o \). Intuitively, all these terms can do is add some fixed number of \( S\)'s to their argument. Thus:

Claim 2. For every closed term \( f : o \to o \), there exists a constant \( C \in \mathbb{N} \) such that its denotation \([f] : \mathbb{N} \to \mathbb{N}\) is bounded by \( n \mapsto n + C \).

A mild generalization of this result is the case of functions with several arguments. Intuitively, only one of these arguments can be used, and the other ones are discarded:

Claim 3. For every closed term \( f : o^k \to o \), there exists a constant \( C \in \mathbb{N} \) such that its denotation \([f] : \mathbb{N}^k \to \mathbb{N}\) is bounded by \((n_1, \ldots, n_k) \mapsto \max(n_1, \ldots, n_k) + C \).

Once again, we stress that these three results are folklore, and not very difficult to prove by syntactic means. The originality of our proofs reside in their semantic nature: we construct a model for \( \lambda \)-calculus where fast-growing functions do not exist.

Remark 1. These claims should not be confused with other similar results about the so-called \( \lambda \)-definability of functions \( \mathbb{N} \to \mathbb{N} \). The well-known theorem due to Schwichtenberg \([12, 13]\) says that the functions \( \mathbb{N} \to \mathbb{N} \) definable in the simply typed \( \lambda \)-calculus are exactly the extended polynomials. In this context, the notion of \( \lambda \)-definability refers to terms of type \( \text{Nat}_o \to \text{Nat}_o \), where \( \text{Nat}_o \) denotes the type \((o \to o) \to (o \to o)\) of Church integers.

Another, more expressive, notion of \( \lambda \)-definability is obtained by considering the terms of type \( \text{Nat}_\tau \to \text{Nat}_o \), where \( \text{Nat}_\tau = (\tau \to \tau) \to (\tau \to \tau) \) is the type of Church integers on an arbitrary base type \( \tau \). In this case, fast-growing functions up to towers of exponentials can be defined, but other very simple functions such as subtraction cannot be expressed \([3]\). Some partial descriptions of this class of functions can be found in the literature \([4]\), but the question of precisely characterizing such functions remains open.

2.2 Resource monoids and length spaces

In this paper, we use a simplified version of the notion of resource monoid, introduced by Dal Lago and Hofmann in \([5]\). The elements of a resource monoid can be used to bound various quantitative properties of programs, such as the size of data, runtimes, etc. To allow more flexibility, these bounds need not be numbers: they can be elements of any commutative monoid, equipped with a suitable pre-order.
Definition 1. A resource monoid\(^1\) is a triple \(M = (|M|, +, \leq)\) such that:
- \((|M|, +)\) is a commutative monoid, and
- \(\leq\) is a pre-order on \(|M|\) compatible with +, that is, \(\alpha \leq \beta\) implies \(\alpha + \gamma \leq \beta + \gamma\).

As shown in [8], any resource monoid \(M\) gives rise to a notion of length spaces on \(M\), which form a symmetric monoidal closed category. Thus, this yields a model of multiplicative linear logic. In order to interpret programming languages with more features, one needs to choose a particular resource monoid of interest, whose associated category of length spaces has the desired additional structure. This is the approach taken in [8, 7, 6] to provide models of various programming languages: Elementary Affine Logic, Light Affine Logic, Soft Affine Logic, Bounded Affine Logic and LFPL. In the remainder of the section, we assume given an arbitrary resource monoid \(M\).

Definition 2. A length space\(^2\) on a resource monoid \(M\) is a pair \(A = (|A|, \vdash)\) where
- \(|A|\) is a set, and \(\vdash \subseteq |M| \times |A|\) satisfies the following properties:
  - for all \(a\), there exists \(\alpha\) such that \(\alpha \vdash a\), and
  - if \(\alpha \vdash a\) and \(\alpha \leq \beta\), then \(\beta \vdash a\).

When \(\alpha \vdash a\), we say that \(\alpha\) is a majorizer of \(a\), or that \(\alpha\) majorizes \(a\). Intuitively, \(a\) is the denotation of a program, and \(\alpha\) represents an upper bound on some quantitative property of \(a\). When we need to distinguish between several length spaces, we will write the majorizer relation \(\vdash\), with the name of the length space as a subscript.

Definition 3. Fix a resource monoid \(M\). A morphism of length spaces from \(A\) to \(B\) is a function \(f : |A| \to |B|\) satisfying the following property:
- there exists \(\varphi \in |M|\), such that \(\alpha \vdash a\) implies \(\varphi + \alpha \vdash_B f(a)\).

When this property holds, we call \(\varphi\) a majorizer of \(f\), and we denote it by \(\varphi \vdash_{A \to B} f\).

As the above notation suggests, given two length spaces \(A\) and \(B\), we can define a new length space \(A \to B := (\text{Hom}(A, B), \vdash_{A \to B})\), where \(\text{Hom}(A, B)\) is the set of morphisms from \(A\) to \(B\). It is easily checked that \(A \to B\) is indeed a length space. Similarly, we can define the tensor product of two length spaces, \(A \otimes B := (|A| \times |B|, \vdash_{A \otimes B})\), where the relation \(\vdash_{A \otimes B}\) is defined by \(\gamma \vdash_{A \otimes B} (a, b)\) iff there exist \(\alpha, \beta \in |M|\) such that:
- \(\alpha \vdash_A a\), and
- \(\beta \vdash_B b\), and

\(^1\)In [8], resource monoids also contain a fourth component \(D\) which can be interpreted as the difference \(D(\alpha, \beta)\) between two elements \(\alpha \leq \beta\) of the monoid. This allows to measure the cost of a computation (e.g., time complexity) as a trade-off between the size of the data and the time taken to process it. For the application in this paper, this component is not required.

\(^2\)In [8], the relation \(\vdash\) also contains a third component \(\epsilon\), called a realizer of \(a\). It plays a role to measure the computation time of programs, and for cardinality issues when interpreting the second order universal quantification. We do not need these features, hence we also drop these realizers from our definitions.
\[\gamma \geq \alpha + \beta.\]

Again, we can check that this is a well-defined length space on \(\mathcal{M}\).

As mentioned earlier, length spaces together with the two operators \(\rightarrow\) and \(\otimes\) assemble into a symmetric monoidal closed category. The next Theorem was first proved in [S]. We reproduce the proof in details below; firstly because we slightly modified the definitions of resource monoids and length spaces, and secondly because we will need to know how the various maps are defined in order to compute the interpretation of a \(\lambda\)-term in this category.

**Theorem 1 (S).** Given any resource monoid \(\mathcal{M}\), the category of length spaces on \(\mathcal{M}\) is symmetric monoidal closed with respect to the tensor product and linear map defined above.

**Proof.** Recall that morphisms of length spaces simply consist of set-theoretic functions which can be majorized by some element of \(\mathcal{M}\). Thus, the symmetric monoidal closed structure is derived from the one in \(\text{Set}\), and all we need to do is find appropriate majorizers.

For example, to define composition of two morphisms \(f : \mathcal{A} \rightarrow \mathcal{B}\) and \(g : \mathcal{B} \rightarrow \mathcal{C}\), majorized respectively by \(\varphi\) and \(\psi\), we check that the function \(g \circ f := x \mapsto g(f(x)) : |\mathcal{A}| \rightarrow |\mathcal{C}|\) is majorized by \(\varphi + \psi\). Assume \(\alpha \vdash_\mathcal{A} a\), we want to show that \(\varphi + \psi + \alpha \vdash_\mathcal{C} g(f(a))\). Since \(\varphi\) majorizes \(f\), we know that \(\varphi + \alpha \vdash_\mathcal{B} f(a)\), and therefore since \(\psi\) majorizes \(g\), \(\psi + \varphi + \alpha \vdash_\mathcal{C} g(f(a))\). The identity function \(\text{id} : \mathcal{A} \rightarrow \mathcal{A}\) is majorized by \(0_\mathcal{M}\), the neutral element of \(\mathcal{M}\).

The unit object is defined as \(\mathcal{I} = (\{\ast\}, \vdash_\mathcal{I})\) where \(\alpha \vdash_\mathcal{I} \ast\) for all \(\alpha\). This is trivially a length space. The unitors, associator and braiding are all majorized by \(0_\mathcal{M}\). Let us check that this is the case for the associator \(\text{assoc}_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \rightarrow \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})\), whose underlying function is \(((a, b), c) \mapsto (a, (b, c))\). We assume that \(\mu \vdash_{(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}} ((a, b), c)\), i.e., that \(\mu \geq \nu + \gamma\) with \(\nu \vdash_{\mathcal{A} \otimes \mathcal{B}} (a, b)\) and \(\gamma \vdash_{\mathcal{C}} c\). Unfolding the definition of \(\vdash_{\mathcal{A} \otimes \mathcal{B}}\), we get that \(\nu \geq \alpha + \beta\), with \(\alpha \vdash_{\mathcal{A}} a\) and \(\beta \vdash_{\mathcal{B}} b\). Then we can show \(0_\mathcal{M} + \mu \vdash_{\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})} (a, (b, c))\), using the fact that \(\mu \geq \alpha + (\beta + \gamma)\).

Given two parallel morphisms \(f : \mathcal{A} \rightarrow \mathcal{A}'\) and \(g : \mathcal{B} \rightarrow \mathcal{B}'\), majorized by \(\varphi\) and \(\psi\) respectively, the functorial action of \(\otimes\) yields the morphism \(f \otimes g : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}' \otimes \mathcal{B}'\). The underlying function \((a, b) \mapsto (f(a), g(b))\) can be majorized by \(\varphi + \psi\). Indeed, given \(\gamma \vdash_{\mathcal{A} \otimes \mathcal{B}} (a, b)\), we need to prove that \(\varphi + \psi + \gamma \vdash_{\mathcal{A}' \otimes \mathcal{B}'} (f(a), g(b))\). By definition of \(\vdash_{\mathcal{A} \otimes \mathcal{B}}\), there are \(\alpha, \beta\) such that \(\gamma \geq \alpha + \beta\), with \(\alpha \vdash_{\mathcal{A}} a\) and \(\beta \vdash_{\mathcal{B}} b\). Then we know that \(\varphi + \alpha \vdash_{\mathcal{A}'} f(a)\) and \(\psi + \beta \vdash_{\mathcal{B}'} g(b)\) because \(f\) and \(g\) are morphisms. Hence \(\varphi + \psi + \alpha + \beta \vdash_{\mathcal{A}' \otimes \mathcal{B}'} (f(a), g(b))\), and we can conclude by the second property of length spaces.

The evaluation morphism \(\text{eval} : \mathcal{A} \otimes (\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{B}\), defined by \(\text{eval}(a, f) = f(a)\), can be majorized by \(0_\mathcal{M}\). Indeed, assume \(\varepsilon \vdash_{\mathcal{A} \otimes (\mathcal{A} \rightarrow \mathcal{B})} (a, f)\). By definition, there are \(\alpha, \varphi\) such that \(\alpha \vdash_{\mathcal{A}} a\), \(\varphi \vdash_{\mathcal{A} \rightarrow \mathcal{B}} f\) and \(\varepsilon \geq \alpha + \varphi\). Since \(\varphi\) majorizes \(f\) we have \(\alpha + \varphi \vdash_{\mathcal{B}} f(a)\), and since \(\varepsilon \geq \alpha + \varphi\), we conclude that \(\varepsilon \vdash_{\mathcal{B}} f(a)\) as required.

Finally to define currying, given any morphism \(f : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}\) majorized by \(\varphi\), the morphism \(\text{curry}(f) : \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})\) whose underlying function is \(a \mapsto b \mapsto f(a, b)\) can also be majorized by \(\varphi\). Indeed, assuming \(\alpha \vdash_{\mathcal{A}} a\), we want to show that \(\varphi + \alpha \vdash_{\mathcal{B} \rightarrow \mathcal{C}} b \mapsto f(a, b)\).
So suppose \( \beta \Vdash_B b \), we need to prove that \( \varphi + \alpha + \beta \Vdash_C f(a, b) \). But this is the case because \( \alpha + \beta \Vdash_{A \otimes B} \) and \( f \) is a morphism. \( \square \)

**Remark 2.** We actually have a little bit more than stated in Theorem 1: for all \( A \), there is a unique weakening map from \( A \) to \( I \) majorized by \( 0_M \). Thus, length spaces over a resource monoid actually provide a model of multiplicative affine logic. The missing ingredient to interpret \( \lambda \)-calculus is the duplication morphism \( \Delta : A \rightarrow A \otimes A \). Such a morphism might not exist in general, so we need to choose a suitable resource monoid.

### 3 A resource monoid measuring the size of terms

We now define our resource monoid of interest, \( M \), that we will use later to prove the three Claims of Section 2.1. The elements of \( M \) will serve as bounds the “potential” of a \( \lambda \)-term for producing large integers. Alternatively, one can think of it as measuring the size of the normal form of a \( \lambda \)-term. Indeed, the natural number represented by a closed terms of type \( o \) is the size of its normal form.

#### 3.1 The resource monoid

The elements of \( M \) are sequences of natural numbers with finite support, that is, ending with infinitely many 0’s. For ease of notation, we write them as finite lists of natural numbers: an element \( \alpha = (a_n)_{n \in \mathbb{N}} \) of \( |M| \) such that \( a_k \neq 0 \) and \( \forall n > k, a_n = 0 \) is denoted by \( \alpha = [a_0, \ldots, a_k] \). We write \(|\alpha|\) the length of that list (in this case, \( k + 1 \)). Note that we always assume that the last element of a list is non-zero, to avoid having several lists representing the same sequence. The constant zero sequence (corresponding to the empty list) is written \( 0_M \).

**Definition 4.** Let \( M = (|M|, +, \leq_M) \) be the following structure:

- \( |M| \) is the set of sequences of natural numbers with finite support.
- \( +_M \) is defined componentwise: we take the max on the first component, and the sum on the others. For ease of notation, we denote \( \text{max}(a, b) \) by \( a \lor b \).

\[ (a_n) +_M (b_n) = (c_n) \quad \text{where} \quad \begin{cases} c_0 = a_0 \lor b_0 \\ c_n = a_n + b_n \quad \text{for} \quad n > 0 \end{cases} \]

- \( (a_n) \leq_M (b_n) \) iff there exists \( (d_n) \in M \) such that:

\[ \begin{cases} a_0 \leq b_0 + d_0 \\ a_n + d_{n-1} \leq b_n \cdot 2^{d_n} \quad \text{for} \quad n > 0 \end{cases} \]

The intuition behind this ordering is the following. First, notice that if we take \( (d_n) \) to be the constant zero sequence \( 0_M \), then the condition above reduces to the componentwise
ordering, \( a_n \leq b_n \) for all \( n \). However, if \( a_0 \) happens to be greater than \( b_0 \), we can accommodate for it by taking a “debt” \( d_0 \). This debt must be paid back in the next step, where we compare \( a_1 \) and \( b_1 \): if it is not the case that \( a_1 + d_0 \leq b_1 \), then we must accumulate some more debt \( d_1 \), and so on. However, notice that the debt must eventually be paid back: since the sequence \((b_n)\) ultimately becomes null, so must \((d_n)\). Also note that the debt decreases exponentially at each step (but never reaches 0 unless \( a_i < b_i \) at some point).

**Example 1.**

(a) To check that \([7, 12, 1] \leq_\mathcal{M} [0, 2, 5]\), first we take a debt \( d_0 = 7 \) so that \( 7 \leq 0 + d_0 \). Then the condition on the second component is \( 12 + 7 \leq 2 \cdot 2^{d_1}; \) take \( d_1 = 4 \). Finally, the last condition is \( 1 + 4 \leq 5 \cdot 2^{d_2} \); we can take \( d_2 = 0 \) and the debt is paid off.

(b) Similarly, one can check that \([65537] \leq_\mathcal{M} [1, 1, 1, 1, 1, 1]\).

(c) However, \([65538] \not\leq_\mathcal{M} [1, 1, 1, 1, 1, 1]\).

**Lemma 1.** If \( \alpha \leq_\mathcal{M} \beta \), then \(|\alpha| \leq |\beta|\).

**Proof.** Let \( \beta = [b_0, \ldots, b_k] \) (the case \( \beta = 0_\mathcal{M} \) is trivial). Assuming \( \alpha \leq_\mathcal{M} \beta \), for every index \( n > k \), the condition \( a_n + d_{n-1} \leq 0 \) is verified because \( b_n = 0 \). Since both \( a_n \) and \( d_{n-1} \) are natural numbers, we must have \( a_n = d_{n-1} = 0 \).

In the rest of the paper, we often use Lemma 1 implicitly: whenever there is an assumption of the form \( \alpha \leq_\mathcal{M} [b_0, \ldots, b_k] \), only the first \((k + 1)\) inequalities matter. In particular, the last inequality is just \( a_k + d_{k-1} \leq b_k \), since we must have \( d_k = 0 \).

**Proposition 1.** \( \mathcal{M} \) is a resource monoid.

**Proof.** First, we check that \((|\mathcal{M}|, +_\mathcal{M})\) is a commutative monoid. The neutral element is 0_\mathcal{M}. Associativity and commutativity are clear since \( +_\mathcal{M} \) is defined componentwise, and \( + \) and \( \lor \) on natural number are both associative and commutative.

Next, we check that \( \leq_\mathcal{M} \) is a pre-order. Reflexivity is clear by taking \((d_n) = 0_\mathcal{M} \). For transitivity, assume \((a_n) \leq_\mathcal{M} (b_n)\) with debt \((d_n)\), and \((b_n) \leq_\mathcal{M} (c_n)\) with debt \((d'_n)\). We show that \((a_n) \leq_\mathcal{M} (c_n)\) with debt \((d_n + d'_n)\).

- \( a_0 \leq b_0 + d_0 \) and \( b_0 \leq c_0 + d'_0 \), thus \( a_0 \leq c_0 + d_0 + d'_0 \).
- For all \( n > 0 \), we have \( a_n + d_{n-1} \leq b_n \cdot 2^{d_n} \) and \( b_n + d'_{n-1} \leq c_n \cdot 2^{d_n} \). Then,

\[
a_n + d_{n-1} + d'_{n-1} \leq b_n \cdot 2^{d_n} + d'_{n-1}
\]

\[
\leq (b_n + d'_{n-1}) \cdot 2^{d_n}
\]

\[
\leq c_n \cdot 2^{d_n} \cdot 2^{d_n}
\]

\[
\leq c_n \cdot 2^{d_n + d'_n}
\]
Finally, \( \leq_M \) is compatible with \(+_M\). Assume that \((a_n) \leq_M (b_n)\) with debt \((d_n)\). We show that \((a_n) +_M (c_n) \leq_M (b_n) +_M (c_n)\) with debt \((d_n)\).

- \(a_0 \lor c_0 \leq (b_0 + d_0) \lor c_0 \leq (b_0 + c_0) + d_0\).
- For all \(n > 0\), \((a_n + c_n) + d_{n-1} \leq b_n \cdot 2^{dn} + c_n \leq (b_n + c_n) \cdot 2^{dn}\).

\[\square\]

**Remark 3.** The relation \(\leq_M\) is actually an order. Indeed, if \(\alpha \leq_M \beta\) and \(\beta \leq_M \alpha\), then \(|\alpha| = |\beta|\) and we can then prove by induction on their size that \(\alpha = \beta\). However, this property is not needed to be able to use the framework of resource monoids.

### 3.2 The collapse functions

An element \(\alpha = [a_0, \ldots, a_k]\) of \(M\) can be associated with a natural number \(a_0 + a_1 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots\). We call this operation “collapsing” the sequence \(\alpha\). The order relation \(\leq_M\) on these sequences almost amounts to comparing these associated numbers, but not exactly. In this section, we make precise the relationship between the two. To this end, we define a function \(\text{collapse}_n : |M| \rightarrow |M|\) for every \(n \in \mathbb{N}\). The idea is that, given an element \(\alpha = [a_0, \ldots, a_k]\) of \(M\), we decrease its length by one by plugging the last component into the penultimate one (either by addition or by multiplying by a power of 2, depending on the case). We then repeat this process until the size of the list becomes \(n + 1\) or smaller. Formally:

\[
\begin{align*}
\text{collapse}_0([a_0, a_1]) &= [a_0 + a_1] \\
\text{collapse}_n(\alpha) &= \alpha & \text{if } |\alpha| \leq n + 1 \\
\text{collapse}_n([a_0, \ldots, a_k]) &= \text{collapse}_n([a_0, \ldots, a_{k-2}, a_{k-1} \cdot 2^{ak}]) & \text{otherwise}
\end{align*}
\]

**Example 2.** For instance, \(\text{collapse}_3([0, 1, 2, 3, 4, 5, 6]) = [0, 1, 2, 3 \cdot 2^{4} \cdot 2^{5}]\) is a list of size 4, while \(\text{collapse}_0([6, 3, 11, 4]) = [6 + 3 \cdot 2^{11} \cdot 2^4]\) is a list of size 1.

**Proposition 2.** The collapse functions have the following properties:

(i) For all \(n\), \(|\text{collapse}_n(\alpha)| \leq n + 1\).

(ii) For all \(n\), \(\text{collapse}_n(\alpha) \leq_M \alpha\).

(iii) If \(\alpha \leq_M \beta\) and \(|\alpha| \leq n + 1\) then \(\alpha \leq_M \text{collapse}_n(\beta)\).

(iv) For all \(n\), if \(\alpha \leq_M \beta\) then \(\text{collapse}_n(\alpha) \leq_M \text{collapse}_n(\beta)\).

(v) If \(n \leq m\), then \(\text{collapse}_n(\alpha) \leq_M \text{collapse}_m(\alpha)\).

(vi) If \(|\alpha| \leq n\) then \(\text{collapse}_n(\alpha + \beta) = \alpha + \text{collapse}_n(\beta)\).

**Proof.**

(i) The first property is straightforward by induction on the size of the list.

(ii) Proceed by induction on the size of \(\alpha\). We distinguish the three cases of the definition.

- If \(|\alpha| \leq n + 1\), then \(\text{collapse}_n(\alpha) = \alpha \leq_M \alpha\).
- If $\alpha = [a_0, a_1]$ and $n = 0$, then $[a_0 + a_1] \leq_M [a_0, a_1]$ by taking the debt $d_0 = a_1$.
- Otherwise $\alpha = [a_0, \ldots, a_k]$ with $k > n$, and the induction hypothesis gives us:

\[
collapse_n([a_0, \ldots, a_{k-2}, a_{k-1} : 2^a]) \leq_M [a_0, \ldots, a_{k-2}, a_{k-1} : 2^a]
\]

Moreover,

\[
[a_0, \ldots, a_{k-2}, a_{k-1} : 2^a] \leq_M [a_0, \ldots, a_k]
\]

by taking the debt $d_{k-1} = a_k$ and $d_i = 0$ otherwise. By transitivity,

\[
collapse_n([a_0, \ldots, a_k]) \leq_M [a_0, \ldots, a_k].
\]

(iii) By induction on the size of $\beta$. Let $(d_i)$ be the debt that proves $\alpha \leq_M \beta$.

- Case $|\beta| \leq n + 1$: trivial since $\text{collapse}_n(\beta) = \beta$.
- Case $\beta = [b_0, b_1]$ and $n = 0$: we have $a_0 \leq b_0 + d_0$ and $a_1 + d_0 \leq b_1$. Since $|\alpha| \leq 1$, we have $a_1 = 0$ and therefore $d_0 \leq b_1$. So $a_0 \leq b_0 + b_1$, from which we can deduce that $\alpha \leq_M [b_0 + b_1] = \text{collapse}_0(\beta)$ without debt.
- Case $\beta = [b_0, \ldots, b_k]$ for $k > n$. By induction hypothesis, we just need to prove:

\[
\alpha \leq_M [b_0, \ldots, b_{k-2}, b_{k-1} : 2^{b_k}].
\]

We use the debt $(d'_i)$ defined by $d'_{k} = 0$ and $d'_i = d_i$ otherwise. The conditions at ranks 0 to $k-2$ stay the same. We need to check the condition at rank $k-1$, which is $a_{k-1} + d_{k-1} \leq b_{k-1} : 2^{d_k}$. From the hypothesis $\alpha \leq_M \beta$, we get $a_{k-1} + d_{k-1} \leq b_{k-1} : 2^{d_k}$ and $a_k + d_k \leq b_k$. But we assumed $|\alpha| \leq n + 1$ and $k > n$, so $a_k = 0$. Hence $d_k \leq b_k$ and we are done.

(iv) By property (ii), $\text{collapse}_n(\alpha) \leq_M \alpha \leq_M \beta$. But $|\text{collapse}_n(\alpha)| \leq n + 1$, so by property (iii) we conclude that $\text{collapse}_n(\alpha) \leq_M \text{collapse}_n(\beta)$.

(v) By (ii), we have $\text{collapse}_n(\alpha) \leq_M \alpha$. Moreover, since $|\text{collapse}_n(\alpha)| \leq n + 1 \leq m + 1$, we can apply (iii) to get $\text{collapse}_n(\alpha) \leq_M \text{collapse}_m(\alpha)$.

(vi) This is obvious since the $\text{collapse}_n$ function does not alter the first $n$ components of a list. Formally, a proof by induction on the size of $\beta$ is straightforward. \qed

4 A quantitative model for $\lambda$-calculus

We can now interpret the $\lambda$-calculus with constants of Section 2.1 in the category of length spaces over $\mathcal{M}$. To do this, we need to define the interpretation of the constants; and most importantly, define a duplication morphism to interpret the contraction rule.

10
4.1 Interpretations of \( o, 0, S \)

In our model, we would like the underlying sets of the length spaces, and the underlying functions of the morphisms, to be the usual set-theoretic semantics. Thus, we need a length space whose underlying set is \( N \) in order to interpret the type \( o \).

**Definition 5.** Define \( N = (N, \models_N) \), where \( \alpha \models_N n \) iff \( [n] \leq_M \alpha \).

This is easily seen to be a length space on \( M \). Indeed, for all \( n \in N \) we have \( [n] \models_N n \). Moreover, if \( \alpha \models_N n \) and \( \alpha \leq_M \beta \), by transitivity \( [n] \leq_M \alpha \) so \( \beta \models_N n \).

**Lemma 2.** We have another characterization of \( \models_N \): \( \alpha \models_N n \) iff \( [n] \leq_M \text{collapse}_0(\alpha) \). If we write \( \text{collapse}_0(\alpha) = [a_0] \), then \( \alpha \models_N n \) iff \( n \leq a_0 \).

**Proof.** If \( [n] \leq_M \alpha \), then by Proposition 2 (iii), \( [n] \leq_M \text{collapse}_0(\alpha) \). Conversely, if \( [n] \leq_M \text{collapse}_0(\alpha) \), then \( [n] \leq_M \alpha \) by Proposition 2 (ii) and transitivity.

Next, we want to interpret the function symbols \( 0 \) and \( S \). Recall that the unit of the category of length spaces is \( I = (\{ \ast \}, \models_I) \) where \( \alpha \models_I \ast \) for all \( \alpha \).

- We interpret \( 0 : o \) as the morphism in \( I \to N \) that sends \( \ast \) to \( 0 \in N \). It is easy to check that this morphism is majorized by \( 0_M \).

- \( S : o \to o \) is a little bit less trivial: we want to interpret it as the successor function \( n \mapsto n + 1 : N \to N \). To prove that it is indeed a morphism in \( N \to N \), we need to find a majorizer. Let us check that \( [0, 1] \models_N n \mapsto n + 1 \).

Suppose that \( \alpha \models_N n \), we want to prove that \( \alpha + [0, 1] \models_N n + 1 \). By assumption, we have \( [n] \leq_M \alpha \), so \( [n] + [0, 1] \leq_M \alpha + [0, 1] \). And by Proposition 2 (ii) we get \( [n + 1] \leq_M \alpha + [0, 1] \).

4.2 The duplication morphism

In Section 2.2, we have seen that we can interpret multiplicative linear logic with full weakening in the category of length spaces on \( M \). What remains to be done is to define a duplication morphism of type \( A \to A \otimes A \) whose underlying function is \( a \mapsto (a, a) \). Unfortunately, this is not possible in general: we would have to find an element \( \varphi \in M \) such that \( \alpha + \alpha \leq_M \varphi + \alpha \) for all \( \alpha \), and no such element exists (take \( |\alpha| > |\varphi| \)).

To fix this issue, we restrict ourselves to the full subcategory \( S \) of the category of length spaces on \( M \), whose objects are the length spaces built inductively from \( N, I, \to \) and \( \otimes \). The objects of \( S \) are the length spaces generated by the following grammar:

\[ A, B ::= I \mid N \mid A \to B \mid A \otimes B \]
Theorem 3. Let \( A \in \mathcal{S} \), and let \( n = \text{Ord}(A) \). If \( \alpha \vdash_A a \), then \( \text{collapse}_n(\alpha) \vdash_A a \).

Proof. By induction on the structure of \( A \).
- The case \( A = I \) is trivial.
- The case \( A = N \) follows from Lemma 2.
- Case \( A = B \vdash C \): assume \( \varphi \vdash_B f \). By definition of the order of \( B \vdash C \), we must have \( \text{Ord}(B) \leq n - 1 \) and \( \text{Ord}(C) \leq n \).
  Assume \( \beta \vdash_B b \). By induction hypothesis, \( \text{collapse}_{\text{Ord}(B)}(\beta) \vdash_B b \) and by Proposition 2 (vi) and upward-closure of \( \vdash \), \( \text{collapse}_{n-1}(\beta) \vdash_B b \).
  Since \( \varphi \) majorizes \( f \), we have \( \varphi \vdash \text{collapse}_{n-1}(\beta) \vdash_C f(b) \). By the second induction hypothesis (and upward-closure), \( \text{collapse}_n(\varphi \vdash \text{collapse}_{n-1}(\beta)) \vdash_C f(b) \). Then:
  \[
  \begin{align*}
  \text{collapse}_n(\varphi \vdash \text{collapse}_{n-1}(\beta)) & \\
  \leq_M \text{collapse}_n(\varphi) + \text{collapse}_{n-1}(\beta) & \text{by Proposition 2 (vi)} \\
  \leq_M \text{collapse}_n(\varphi) + \beta & \text{by Proposition 2 (ii)}
  \end{align*}
  \]
  Hence \( \text{collapse}_n(\varphi) + \beta \vdash_C f(b) \), so \( \text{collapse}_n(\varphi) \vdash_B \vdash_C f \).
- Case \( A = B \otimes C \): assume \( \alpha \vdash_{B \otimes C} (b, c) \). There are \( \beta \) and \( \gamma \) such that \( \beta \vdash_B b, \gamma \vdash_C c \) and \( \beta + \gamma \leq_M \alpha \). Since \( \text{Ord}(B) \leq n \) and \( \text{Ord}(C) \leq n \), we can use the induction hypothesis (and upward-closure of \( \vdash \)) to obtain \( \text{collapse}_n(\beta) \vdash_B b \) and \( \text{collapse}_n(\gamma) \vdash_C c \).
  Moreover, \( \text{collapse}_n(\beta) + \text{collapse}_n(\gamma) \leq_M \beta + \gamma \leq_M \alpha \). By Proposition 2 (iii), \( \text{collapse}_n(\beta) + \text{collapse}_n(\gamma) \leq_M \text{collapse}_n(\alpha) \). Therefore, \( \text{collapse}_n(\alpha) \vdash_{B \otimes C} (b, c) \).

Lemma 4. Let \( \varphi = [0, 0, 1, \ldots, 1] \) where \( |\varphi| = n + 1 \). Then, for all \( \alpha \) such that \( |\alpha| \leq n \), \( \alpha + \alpha \leq_M \varphi + \alpha \).

Proof. Write \( \alpha = (a_i) \) and take the debt \( d_i = 1 \) when \( 1 \leq i \leq n - 1 \), and \( d_i = 0 \) otherwise. On the first component, \( a_0 \vee a_0 \leq (a_0 \vee 0) + 0 \) is verified. On the second component, \( a_1 + a_1 \leq (a_1 + 0) \cdot 2^1 \) is verified. Then for \( 2 \leq i \leq n - 1 \), \( a_i + a_i + 1 \leq (a_i + 1) \cdot 2^1 \) is verified. The last condition is just \( 1 \leq 1 \) since \( a_n = 0 \).
We can now define the duplication in $S$:

**Proposition 3.** Let $\mathcal{A} \in S$ with $\text{Ord}(\mathcal{A}) = n$, the duplication function $a \mapsto (a, a)$ is majorized by $\varphi_n = [0, 0, 1, \ldots, 1]$ where $|\varphi_n| = n + 2$.

**Proof.** Assume $\alpha \parallel \mathcal{A} a$, we must prove that $\varphi_n + \alpha \parallel \mathcal{A} \otimes \mathcal{A} (a, a)$. By Lemma 3 we know that $\text{collapse}_n(\alpha) \parallel \mathcal{A} a$, and since $|\text{collapse}_n(\alpha)| \leq n + 1$, we can apply Lemma 4 to get $\text{collapse}_n(\alpha) + \text{collapse}_n(\alpha) \leq \mathcal{M} \varphi_n + \text{collapse}_n(\alpha)$. Then by Proposition 2 (ii) we obtain $\text{collapse}_n(\alpha) + \text{collapse}_n(\alpha) \leq \mathcal{M} \varphi_n + \alpha$, which concludes the proof. □

### 4.3 Interpreting $\lambda$-calculus

Putting together the results of Sections 2.2, 4.1 and 4.2 we get the next Theorem, allowing us to interpret our simply typed $\lambda$-calculus in the category of length spaces on $\mathcal{M}$.

**Theorem 2.** The category $S$ is cartesian closed.

We denote by $[-]_S$ the interpretation of types and terms in $S$, and by $[-]$ the standard set-theoretic interpretation. Thus:

$[0]_S := \mathbb{N}$

$[\sigma \to \tau]_S := [\sigma]_S \to [\tau]_S$

$[\sigma \times \tau]_S := [\sigma]_S \otimes [\tau]_S$

Recall that $[0]_S : \mathcal{I} \to \mathbb{N}$ is majorized by $0_M$; $[S]_S : \mathbb{N} \to \mathbb{N}$ is majorized by $[0, 1]$; and the duplication morphism $\Delta_\mathcal{A} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ with $\text{Ord}(\mathcal{A}) = n$ is majorized by $[0, 0, 1, \ldots, 1]$ of size $n + 2$. Moreover, recall from the proof of Theorem 1 that if $f$ and $g$ are majorized by $\varphi$ and $\psi$, then both $f \circ g$ and $f \times g$ are majorized by $\varphi + \psi$. All the other structure maps of $S$ are majorized by $0_M$.

Thus, given a term $t : \tau$, we can effectively compute a majorizer of the element $[t]_S$ of the length space $[\tau]_S$. It is straightforward to check that the underlying set of $[\tau]_S$ is $[\tau]$, and the underlying function of $[t]_S$ is $[t]$.

**Example 3.** Let us compute the majorizers of a few $\lambda$-terms.

(a) Let $t = S (S (\ldots (S 0)))$, with $k$ occurrences of $S$. Then $[t]_S : \mathcal{I} \to \mathbb{N}$ is majorized by $[0, k]$. By Lemma 2 this yields the upper bound $[t] \leq k$, which is a tight bound since the denotation of $t$ is $[t] = k$.

(b) The Church encoding $\bar{2} = \lambda f. \lambda x. f (f x) : (o \to o) \to o \to o$ has a contraction on the variable $f$ of order 1. Thus, its denotation $[\bar{2}]_S$ is majorized by $[0, 0, 1]$.

(c) Let $u = \bar{2} (\bar{2} (\ldots (\bar{2} S))) 0$, with $k$ occurrences of $\bar{2}$. Its denotation $[u]_S : \mathcal{I} \to \mathbb{N}$ is majorized by $[0, 1, k]$. By Lemma 2 we can collapse it to get the bound $[u] \leq 2^k$, which is once again a tight bound.

(d) Let $v = (((\bar{2} \bar{2}) \bar{2}) \ldots \bar{2}) S 0$, with $k$ occurrences of $\bar{2}$. Here, each occurrence of $\bar{2}$ has a different type, $\bar{2} : (\tau \to \tau) \to \tau \to \tau$, where the order of $\tau$ increases by 1 at
Moreover, the constant $J$ we have
\[ \leq 1 \cdot 2^{k-2} \cdot 2^{2^1}. \]
By collapsing it using Lemma 2, we get the upper bound $[v] \leq 1 \cdot 2^{k-2} \cdot 2^{2^1}$. This time the bound is not tight: the real value of $[v]$ is a tower of exponentials of 2 with the same height, but without the multiplicative constants $k$, $k-1$, etc. For example with $k = 3$, $[v] = 2^{2^2} = 16$, while the majorizer $[0, 1, 3, 2, 1]$ collapses to $[2^{48}]$.

We can already prove Claims 2 and 3 of Section 2.1.

**Claim 2.** For every closed term $f : o \rightarrow o$, there exists a constant $C \in \mathbb{N}$ such that its denotation $[f] : \mathbb{N} \rightarrow \mathbb{N}$ is bounded by $n \mapsto n + C$.

**Proof.** Let $f : o \rightarrow o$, and $[f] : \mathbb{N} \rightarrow \mathbb{N}$ its set-theoretic denotation. Thus, there is a morphism $[f]_S : \mathbb{N} \rightarrow \mathbb{N}$ in $\mathcal{S}$ whose underlying function is $[f]$. Let $\varphi$ be a majorizer of $[f]$. By Lemma 3 collapse$_1(\varphi) = [f_0, f_1]$ is also a majorizer. Then, for any $n \in \mathbb{N}$, since $[n] \models \mathbb{N} n$, we have $[n] + [f_0, f_1] \models \mathbb{N} f(n)$. By Lemma 2 $f(n) \leq (n \lor f_0) + f_1 \leq n + (f_0 + f_1) = n + C_f$. \hfill $\Box$

**Claim 3.** For every closed term $f : o^k \rightarrow o$, there exists a constant $C \in \mathbb{N}$ such that its denotation $[f] : \mathbb{N}^k \rightarrow \mathbb{N}$ is bounded by $(n_1, \ldots, n_k) \mapsto \max(n_1, \ldots, n_k) + C$.

**Proof.** Let $f : o^k \rightarrow o$, and $[f]_S : \mathbb{N}^k \rightarrow \mathbb{N}$ in $\mathcal{S}$ whose underlying function is $[f]$. By the same reasoning, $[f]_S$ has a majorizer of size 2, say $\varphi = [f_0, f_1]$. Given $k$ arguments $n_1, \ldots, n_k$ majorized by $[n_1], \ldots, [n_k]$ respectively, the tuple $(n_1, \ldots, n_k)$ is majorized by $[n_1] + \mathcal{M} \ldots + \mathcal{M} [n_k] = [\max(n_1, \ldots, n_k)]$. Then the same reasoning as before yields $f(n_1, \ldots, n_k) \leq \max(n_1, \ldots, n_k) + C_f$. \hfill $\Box$

## 5 Bounding the constant

To prove the remaining Claim 1 we want to find a bound on the constant $C_f$ defined in the proof of Claim 2 above, depending on the size of $f$ and the maximal order of a $\lambda$-abstraction. Moreover, the constant $C_f$ is defined by $[C_f] = \text{collapse}_0(\varphi)$, where $\varphi$ is a majorizer of $f$. Thus, what we want to prove is an upper bound on the majorizer of a $\lambda$-term.

**Definition 7.** First we define notations to write towers of exponentials more compactly:
\[
\begin{align*}
\cdot 2^n &= 2 \cdot 2 \cdot \ldots \cdot 2 \\
&\text{with } n \text{ occurrences of } 2, \text{ i.e., } \begin{cases} 2^0 = a \\
2^{a+1} = 2^{2^a} \end{cases} \\
\end{align*}
\]
\[
\begin{align*}
\mathcal{C}([a_1, \ldots, a_n]) &= a_1 \cdot 2^{a_2 \cdot 2^{a_3 \cdot \cdots \cdot 2^{a_n}}} \\
&\text{i.e., } \begin{cases} \mathcal{C}([]) = 0 \\
\mathcal{C}([a_1, \ldots, a_n]) = a_1 \cdot 2^{\mathcal{C}([a_2, \ldots, a_n])} \end{cases} \\
\end{align*}
\]

With this notation, we have collapse$_0([a_0, \ldots, a_n]) = [a_0 + \mathcal{C}([a_1, \ldots, a_n])]$.

**Lemma 5.** Let $\alpha = [a_0, \ldots, a_n] \in \mathcal{M}$, and let $a = \max(a_0, \ldots, a_n)$. Then collapse$_0(\alpha) \leq 2^n$.


Proof. Since \(a_i \leq a\) for all \(i\), \([a_0, \ldots, a_n]\) \(\leq [a, \ldots, a]\) with \((n+1)\)-many \(a\)'s, and by Proposition 2 (iv), it is enough to prove that \(\text{collapse}_0([a, \ldots, a]) \leq 2^n\). To keep track of the number of \(a\)'s, we write \(\mathcal{E}_n^a = \mathcal{E}([a, \ldots, a])\) with \(n\)-many \(a\)'s. Let us show that \(a + \mathcal{E}_n^a \leq 2^n\). We proceed by induction on \(n\).

- \(n = 0\) is trivial: \(\mathcal{E}_0^a = 0\) and \(2^0 = a\).
- \(2^{n+1}_n + 2^{n}_n \geq 2^{a + \mathcal{E}_n^a} = 2^a \cdot 2^{\mathcal{E}_n^a}\) by induction hypothesis. Moreover, since \(a\) is a natural number, we have \(2^a \geq a + a\). So \(2^a \cdot 2^{\mathcal{E}_n^a} \geq (a + a) \cdot 2^{\mathcal{E}_n^a} \geq a + a \cdot 2^{\mathcal{E}_n^a} = a + \mathcal{E}_n^a + 1\). \(\square\)

With Lemma 5 in mind, we are going to bound the size of \(\text{collapse}_0(\varphi)\) by relying on two parameters: the size of \(\varphi\) and its maximal coefficient. When \(\varphi\) is the majorizer of some \(\lambda\)-term \(f\), these two parameters correspond respectively to the \(\text{rank}\) and the size of \(f\). Proving this fact will be the aim of Lemma 6.

**Definition 8.** We define the \(\text{rank}\) of a well-typed term \(t\). The idea is that \(\text{rank}(t)\) is the maximal order of a contraction that occurs in the typing derivation of \(t\). When \(t\) is closed, the rank is defined as follows:

\[
\begin{align*}
\text{rank}(x) &= 0 \\
\text{rank}(0) &= 0 \\
\text{rank}(S) &= 0 \\
\text{rank}(t \ u) &= \max(\text{rank}(t), \text{rank}(u)) \\
\text{rank}(\lambda x^\tau. t) &= \begin{cases} 
\text{rank}(t) & \text{if } x \text{ appears at most once in } t \\
\max(\text{Ord}(\tau), \text{rank}(t)) & \text{otherwise}
\end{cases}
\end{align*}
\]

When \(t\) has free variables \(x_1, \ldots, x_n\) and is typed in context \(\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n\),

\[
\text{rank}(\Gamma \vdash t : \tau) = \max(|\{\text{Ord}(\tau_i) \mid x_i \text{ appears at least twice in } t\}| \cup \{\text{rank}(t)\})
\]

For \(\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n\), we write \(|\Gamma| = n\) and \(\text{var}(\Gamma) = \{x_1, \ldots, x_n\}\). For a \(\lambda\)-term \(t\), we denote by \(\text{FV}(t)\) the set of free variables of \(t\), and its size \(|t|\) is defined inductively by

\[
\begin{align*}
|0| &= 0 \\
|S| &= 1 \\
|x| &= 1 \\
|t \ u| &= |t| + |u| \\
|\lambda x. t| &= |t|.
\end{align*}
\]

For \(\alpha = [a_0, \ldots, a_n] \in \mathcal{M}\), we write \(\max(\alpha) = \max(a_0, \ldots, a_n)\). Our goal is now to prove the following Lemma:

**Lemma 6.** If \(\Gamma \vdash t : \tau\) and \(\text{var}(\Gamma) \subseteq \text{FV}(t)\), then there is a majorizer \(\alpha\) of \([t]_\mathcal{S}\) such that

\[
|\alpha| \leq \text{rank}(\Gamma \vdash t : \tau) + 2 \text{ and } \max(\alpha) + |\Gamma| \leq |t|.
\]

Before we can prove Lemma 6, we need to make sure that the derivation of \(\Gamma \vdash t : \tau\) does not contain unnecessary contractions, i.e., contractions that introduce a new variable which is later weakened. This is the aim of two Facts below.

**Fact 1.** For every derivation of \(\Gamma, \ x : \sigma \vdash t : \tau\) with \(x \notin \text{FV}(t)\), there is a derivation of \(\Gamma \vdash t : \tau\) whose height is smaller or equal.
Proof. This is a straightforward induction on the derivation of $\Gamma, x : \sigma \vdash t : \tau$. 

Fact 2. If $\Gamma \vdash t : \tau$ with $\text{var}(\Gamma) \subseteq \text{FV}(t)$, then there is a derivation of $\Gamma \vdash t : \tau$ where the weakening rule only occurs right after the lambda rule that introduced the weakened variable.

Proof. By induction on the height of the derivation.

- **VAR, ZERO, SUCC**: these derivations already satisfy the property.

- **APP**: $\Gamma, \Delta \vdash t \ u : \tau$ comes from $\Gamma \vdash t : \sigma \to \tau$ and $\Delta \vdash u : \sigma$.

  Since $\text{var}(\Gamma, \Delta) \subseteq \text{FV}(t \ u)$ by assumption and $\Gamma \cap \Delta = \emptyset$, we have $\text{var}(\Gamma) \subseteq \text{FV}(t)$ and $\text{var}(\Delta) \subseteq \text{FV}(u)$, so we can use the induction hypothesis on both premises. Then by applying the APP rule we get a derivation that satisfies the property.

- **LAM**: $\Gamma \vdash \lambda x. t : \sigma \to \tau$ comes from $\Gamma, x : \sigma \vdash t : \tau$.

  Either $x \in \text{FV}(t)$ or $x \notin \text{FV}(t)$. In the first case, we can use our induction hypothesis and we are done. In the second case, Fact 1 gives us a derivation of $\Gamma \vdash t : \tau$ whose height is smaller, so we can use the induction hypothesis to get a derivation of $\Gamma \vdash t : \tau$ satisfying the desired property. Using weakening on $x$ and the LAM rule, we get a derivation of $\Gamma \vdash \lambda x. t : \sigma \to \tau$ that satisfies the property.

- **CONTR**: $\Gamma, z : \sigma \vdash t[x, y \leftarrow z] : \tau$ comes from $\Gamma, x : \sigma, y : \sigma \vdash t : \tau$.

  At least one of the variables $x$ and $y$ is in $\text{FV}(t)$, otherwise $z$ would not be either. If they both are in $\text{FV}(t)$, we use the induction hypothesis and we are done. Otherwise, if $x \notin \text{FV}(t)$, then Fact 1 gives us a smaller derivation of $\Gamma, y : \sigma \vdash t : \tau$. We can then use the induction hypothesis on this derivation, and by renaming $y$ into $z$, we are done. Same reasoning if $y \notin \text{FV}(t)$ instead.

- **WEAK**: $\Gamma, x : \sigma \vdash t : \tau$ comes from $\Gamma \vdash t : \tau$.

  This case is not possible since by assumption $x \notin \text{FV}(t)$. 

We can now prove Lemma 6 with the extra assumption that the derivation does not contain weakening rules, except right after the corresponding lambda rule.

Proof of Lemma 6. By induction on the typing derivation of $\Gamma \vdash t : \tau$.

- **VAR, ZERO, SUCC**: the morphisms $\text{id} : [\tau]_\mathcal{S} \to [\tau]_\mathcal{S}$, $[0]_\mathcal{S} : I \to N$ and $[S]_\mathcal{S} : N \to N$ are majorized by $0_M, 0_M$ and $[0, 1]$ respectively, which satisfy the two conditions.

- **APP**: $\Gamma, \Delta \vdash t \ u : \tau$ comes from $\Gamma \vdash t : \sigma \to \tau$ and $\Delta \vdash u : \sigma$.

  Since $\Gamma \cap \Delta = \emptyset$, we have $\text{var}(\Gamma) \subseteq \text{FV}(t)$ and $\text{var}(\Delta) \subseteq \text{FV}(u)$, so we can use the induction hypothesis on both premises to get two majorizers $\alpha$ and $\beta$ of $[t]_\mathcal{S}$ and $[u]_\mathcal{S}$ which satisfy the property. Then, the morphism $[t \ u]_\mathcal{S}$ is obtained by composing
Contr

By currying, we obtain a morphism $\Gamma \vdash t : \sigma \rightarrow \tau$ majorized by $\alpha$ and $\max$. Then,

$$|\alpha + \beta| \leq \max(|\alpha|, |\beta|)$$

$$\leq \max(\text{rank}(\Gamma \vdash t : \sigma \rightarrow \tau) + 2, \text{rank}(\Delta \vdash u : \sigma) + 2)$$

$$= \max(\text{rank}(\Gamma \vdash t : \sigma \rightarrow \tau), \text{rank}(\Delta \vdash u : \sigma)) + 2$$

$$= \text{rank}(\Gamma, \Delta \vdash t \ u : \tau) + 2$$

and

$$\max(\alpha + \beta) + |\Gamma, \Delta| \leq \max(\alpha) + |\Gamma| + \max(\beta) + |\Delta|$$

$$\leq |t| + |u|$$

$$\leq |t \ u|$$

- **LAM:** $\Gamma \vdash \lambda x. t : \sigma \rightarrow \tau$ comes from $\Gamma, x : \sigma \vdash t : \tau$. If $x \in \text{FV}(t)$, the induction hypothesis gives a majorizer $\alpha$ for $[t]_S : [\Gamma]_S \otimes [\sigma]_S \rightarrow [\tau]_S$. By currying, we obtain a morphism $[\lambda x. \tau]_S : [\Gamma]_S \rightarrow [\sigma]_S \rightarrow [\tau]_S$ which is also majorized by $\alpha$. Then $|\alpha| \leq \text{rank}(\Gamma, x : \sigma \vdash t : \tau) + 2 = \text{rank}(\Gamma \vdash \lambda x. t : \sigma \rightarrow \tau) + 2$, and $\max(\alpha) + |\Gamma| \leq \max(\alpha) + |\Gamma, x : \sigma| - 1 \leq |t| - 1 \leq |\lambda x. t|$, as required.

If $x \notin \text{FV}(t)$, the rule is immediately followed by a weakening rule whose premise is $\Gamma \vdash t : \tau$. The induction hypothesis on this premise gives us a majorizer $\alpha$ for $[t]_S : [\Gamma]_S \rightarrow [\tau]_S$. By weakening and currying, we get a morphism $[\lambda x. \tau]_S : [\Gamma]_S \rightarrow [\sigma]_S \rightarrow [\tau]_S$ which is also majorized by $\alpha$. The two conditions on the size and the max still hold: $|\alpha| \leq \text{rank}(\Gamma \vdash t : \tau) + 2 = \text{rank}(\Gamma \vdash \lambda x. t : \sigma \rightarrow \tau) + 2$ and $\max(\alpha) + |\Gamma| \leq |t| \leq |\lambda x. t|$. 

- **CONTR:** $\Gamma, z : \sigma \vdash t[x, y \leftarrow z] : \tau$ comes from $\Gamma, x : \sigma, y : \sigma \vdash t : \tau$. Both $x$ and $y$ are necessarily in $\text{FV}(t)$; otherwise, it would need to be weakened later, and we assumed this does not happen. So we can use the induction hypothesis to get a majorizer $\alpha$ of $[t]_S : [\Gamma]_S \otimes [\sigma]_S \otimes [\sigma]_S \rightarrow [\tau]_S$. To obtain $[t[x, y \leftarrow z]]_S$, we compose $[t]_S$ with the duplication morphism $[\sigma]_S \rightarrow [\sigma]_S \otimes [\sigma]_S$. Thus, it is majorized by $\alpha + \varphi_n$, where $n = \text{Ord}(\sigma)$ and $\varphi_n = [0, 0, 1, \ldots, 1]$ of size $n + 2$. Then,

$$|\alpha + \varphi_n| \leq \max(|\alpha|, |\varphi_n|)$$

$$\leq \max(\text{rank}(\Gamma, x : \sigma, y : \sigma \vdash t : \tau) + 2, n + 2)$$

$$= \max(\text{rank}(\Gamma, x : \sigma, y : \sigma \vdash t : \tau), \text{Ord}(\sigma)) + 2$$

$$= \text{rank}(\Gamma, x : \sigma \vdash t[x, y \leftarrow z] : \tau) + 2$$

and

$$\max(\alpha + \varphi_n) + |\Gamma, z : \sigma| \leq \max(\alpha) + 1 + |\Gamma, x : \sigma, y : \sigma| - 1 \leq |t|$$
We can finally apply Lemma 6 to prove Claim 1.

**Claim 1.** For every closed term \( t : o \), we have \( \lceil t \rceil \leq 2^{\lceil |t| \rceil + 1} \).

**Proof.** Applying Lemma 6 to a closed term \( t : o \) gives a majorizer \( \alpha \) of \( \lceil t \rceil \) with \( \max(\alpha) \leq |t| \) and \( |\alpha| \leq \text{rank}(t) + 2 \). By Lemma 2, this means that \( \lceil t \rceil \leq a_0 \), where \( \text{collapse}_0(\alpha) = [a_0] \).

And by Lemma 5, we deduce the desired upper bound, \( \lceil t \rceil \leq 2^{\lceil |t| \rceil} \).

Note that with the same reasoning, this bound also applies to the constants that appear in Claims 2 and 3. As can be seen in the Examples below, this bound is far from being tight: there is roughly one spare level in the tower of exponents of 2. This is because powers of 2 are hardwired into our model, while \( \lambda \)-terms using the church encoding \( \bar{3} \) might compute a tower of 3's instead.

**Example 4.**

(a) \( t = S (S (\ldots (S 0))) \), with \( k \) occurrences of \( S \). The size of \( t \) is \( |t| = k \) and its rank is \( \text{rank}(t) = 0 \), so our bound gives \( \lceil t \rceil \leq 2^{k+1} \), while its real denotation is \( \lceil t \rceil = k \).

(b) \( u = \bar{2} (\bar{2} (\ldots (\bar{2} S))) 0 \), with \( k \) occurrences of \( \bar{2} \). The size of \( u \) is \( |u| = 3k + 1 \) and its rank is \( \text{rank}(u) = 1 \), so our bound gives \( \lceil u \rceil \leq 2^{2^{3k+1}} \), while the denotation is \( \lceil u \rceil = 2^k \).

(c) \( v = \bar{2} \bar{2} \ldots \bar{2} S 0 \), with \( k \) occurrences of \( \bar{2} \). The size of \( v \) is \( |v| = 3k + 1 \) and its rank is \( \text{rank}(v) = k \), so our bound gives \( \lceil v \rceil \leq 2^{3k+1} \), while the denotation is \( \lceil v \rceil = 2^k \).

(d) \( w = \bar{3} \bar{3} \ldots \bar{3} S 0 \), with \( k \) occurrences of \( \bar{3} \). The size of \( w \) is \( |w| = 4k + 1 \) and its rank is \( \text{rank}(w) = k \), so our bound gives \( \lceil w \rceil \leq 2^{4k+1} \), while the denotation is \( \lceil w \rceil = 3^k \).

**6. Conclusion**

We have shown a new use-case of Dal Lago and Hofmann’s semantic framework based on resource monoids. Our resource monoid is new in two aspects: unlike other resource monoids found in the literature, which are concerned with time complexity bounds, our model measures the potential of a \( \lambda \)-terms for producing large natural numbers. The second difference with previous instances of resource monoids is that it is able to model the contraction rule of \( \lambda \)-calculus, by bounding the combinatorial explosion caused by duplication of variables.

It would be interesting to try to extend this model to tackle programming languages with more computational power such as Gödel’s System T. Another motivating research direction would be to find other use-cases of resource monoids, measuring yet another kind of quantitative property of programs, such as space complexity, probability, and so on.
Acknowledgements. The authors would like to thank Robert Atkey, Lê Thành Dũng (Tito) Nguyên and Igor Walukiewicz for helpful discussions and comments.

References


