Lecture 5: Proof and Infinite Sets

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CS103 Machines, Languages and Computation
October 9th 2015

Homework for Tutorial next week

- Read Chapter 2 of Gödel, Escher, Bach
- Do Assignment 2, parts (d) and (e)
  - Part (d) is a proof by induction
  - Part (e) asks why proof by induction is like toppling an infinite line of dominoes
- Try to make progress with Assignment 1, part (d)
  - What is the decision procedure for MIU?
- Hand in your workbooks by 3pm on Tue (13th Oct)

Non-Computable Functions

- The first 5 weeks of the course will build up towards the proof that non-computable functions exist:

  If P is the set of all computer programs and F is the set of all functions $f: n \rightarrow m$ such that n,m are members of the set of natural numbers N, then $|F| > |P|$ and therefore there are some functions in F for which no computer program can exist.

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What is a Set?

- In mathematics, a set can be thought of as any collection of distinct things considered as a whole.
- Though a simple idea, it is nevertheless one of the most important and fundamental concepts in modern mathematics.
- It is the language in which modern mathematics is described.
- Set theory can be viewed as the foundation upon which nearly all of mathematics can be built.

What is a Set?

- Example set: $S = \{8, 6, 2\}$
- Or in words: S = the set of ages of John's children
- Order is not important: $\{2, 8, 6\} = \{8, 6, 2\}$
- If a is a *member* of S, we write $a \in S$
- If two sets, R and S, have exactly the same members, they are identical, and we write $R = S$
- If S contains all the elements of R, then R is a *subset* of S, and we write $R \subseteq S$

Describing Sets

- We can describe sets by listing the items explicitly
- Or we can use words: “Let S be the set of all even numbers less than 10”
- Or we can use “…” notation: $S = \{1, 2, 3, \ldots , 100\}$
- Sets can contain elements other than integers: the set of all colours on the French flag = {red, white, blue}
- Sets can contain other sets: $S = \{\{1, 2, 3\}, \{\text{a, b, c}\}\}$
- Using “such that” notation: $S = \{n^2 \mid n \text{ is an integer} < 5\}$

Infinite Sets

- Sets can have an infinite number of elements.
- The most important is the set of *natural numbers*, *N.*
- $N = \{0, 1, 2, 3, \ldots\}$
- Each element has a natural successor: if I'm looking at element a then I know the next element is $a+1$.
- This set is certainly infinite, but it is described as being *countably* infinite.
- The size or *cardinality* of a countably infinite set is the "smallest" sort of infinity there is.

Dealing with Infinity in Proofs

If we want to prove that some property holds for all elements of *N*, then we have a number of strategies:

- Use letters instead of numbers; if we can show that the property holds without choosing a specific integer, then it must be true for all integers.
- Use a “counting” argument and show that all cases are covered: e.g. prove the proposition for even numbers, then prove it for odd numbers, then prove it for zero.
- Use Proof by Induction (the domino proof)

Proof by Induction

- Often used to prove statements of the form:

  "for all $n \in N$, some property holds of n"

- First, prove true for the first case, which is easy enough: just find the smallest possible value of n for which the formula makes sense, and see if the formula holds.

- Next, assume the formula is true for some arbitrary value, k. The inductive step consists of proving that the formula must be true for the value $k+1$.

- The "domino effect" makes the formula true for all n.

Example of Proof by Induction

- The sum of the first n non-zero integers = $n.(n+1)/2$
- In other words, $1 + 2 + 3 + \ldots + n = n.(n+1)/2$
- Proof by induction:
- First case is $n = 1$. $1 = 1.(1+1)/2$, which is true.
- Assume kth case is true:

$$1 + 2 + 3 + \ldots + k = k.(k+1)/2 \qquad (1)$$

- Now prove the $(k+1)$th case is true:

$$1 + 2 + 3 + \ldots + k + (k+1) = (k+1).(k+2)/2 \qquad (2)$$

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Example of Proof by Induction

- Substitute the right-hand side of (1) into (2):
- $1 + 2 + 3 + \ldots + k + (k+1) = (k+1).(k+2)/2 \qquad (2)$
- $k.(k+1)/2 + (k+1) = (k+1).(k+2)/2$
- $(k^2 + k)/2 + (k+1) = (k+1).(k+2)/2$
- $k^2/2 + k/2 + k+1 = (k+1).(k+2)/2$
- $k^2/2+3k/2+1 = (k+1).(k+2)/2$
- $k^2/2+3k/2+1 = (k^2+3k+2)/2$
- $k^2/2+3k/2+1 = k^2/2+3k/2+1. \ \blacksquare$

Assignment 2, Part (d)

- Use induction to produce a proof that the sum of the first n odd numbers is n^2:

$$
\begin{aligned}
n &= 1: \quad 1 = 1 \\
n &= 2: \quad 1 + 3 = 4 \\
n &= 3: \quad 1 + 3 + 5 = 9 \\
n &= 4: \quad 1 + 3 + 5 + 7 = 16
\end{aligned}
$$

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Assignment 2, Part (e)

- Why is proof by induction like toppling an infinite line of dominoes?