Indexed Induction and Coinduction, Fibrationally

Clément Fumex, Neil Ghani, and Patricia Johann
University of Strathclyde, Scotland

Abstract. This paper extends the fibrational approach to induction and coinduction pioneered by Hermida and Jacobs, and developed by the current authors, in two key directions. First, we present a sound coinduction rule for any data type arising as the final coalgebra of a functor, thus relaxing Hermida and Jacobs' restriction to polynomial data types. For this we introduce the notion of a *quotient category with equality* (QCE), which both abstracts the standard notion of a fibration of relations constructed from a given fibration, and plays a role in the theory of coinduction dual to that of a comprehension category with unit (CCU) in the theory of induction. Second, we show that indexed inductive and coinductive types also admit sound induction and coinduction rules. Indexed data types often arise as initial algebras and final coalgebras of functors on slice categories, so our key technical results give sufficent conditions under which we can construct, from a CCU (QCE) $U : \mathcal{E} \to \mathcal{B}$, a fibration with base $\mathcal{B}/I$ that models indexing by I and is also a CCU (QCE).

1 Introduction

Iteration operators provide a uniform way to express common and naturally occurring patterns of recursion over inductive data types. Categorically, iteration operators arise from initial algebra semantics: the constructors of an inductive data type are modelled as a functor F, the data type itself is modelled as the carrier μF of the initial F-algebra $in : F(\mu F) \to \mu F$, and the iteration operator $fold : (FA \to A) \to \mu F \to A$ for μF maps an F-algebra $h : FA \to A$ to the unique F-algebra morphism from in to h. Initial algebra semantics provides a comprehensive theory of iteration which is i) *principled*, in that it ensures that programs have rigorous mathematical foundations that can be used to give them meaning and prove their soundness; ii) *expressive*, in that it is applicable to *all* inductive types — i.e., all types that are carriers of initial algebras — rather than just to syntactically defined classes of data types such as polynomial ones; and iii) *sound*, in that it is valid in any model — set-theoretic, domain-theoretic, realizability, etc. — interpreting data types as carriers of initial algebras.

Final coalgebra semantics gives an equally comprehensive understanding of coinductive types: the destructors of a coinductive data type are modelled as a functor F, the data type itself is modelled as the carrier νF of the final F-coalgebra $out : \nu F \to F(\nu F)$, and the coiteration operator $unfold : (A \to FA) \to A \to \nu F$ for νF maps an F-coalgebra $k : A \to FA$ to the unique F-coalgebra morphism from k to out. Final coalgebra semantics thus provides a theory of coiteration which is as principled, expressive, and sound as that for induction.

Since induction and iteration are closely linked, we might expect initial algebra semantics to give a principled, expressive, and sound theory of induction as well. However, most theories of induction for a data type μF, where $F : \mathcal{B} \to \mathcal{B}$, are sound only under significant restrictions on the category $\mathcal{B}$, the functor F, or the property to be established. Recently a conceptual breakthrough in the theory of induction was made by Hermida and Jacobs [7]. They show how to lift an arbitrary functor F on a base category $\mathcal{B}$ of types to a functor $\hat{F}$ on a category of properties over those types. They take the premises of an induction rule for μF to be an $\hat{F}$-algebra, and their main theorem shows that such a rule is sound if the lifting $\hat{F}$ preserves truth predicates. Hermida and Jacobs work in a fibrational, and hence axiomatic, setting and treat *any* notion of property that can be suitably fibred over $\mathcal{B}$. Moreover, they place no stringent requirements on $\mathcal{B}$. Thus, they overcome two of the aforementioned limitations. But since they give sound induction rules only for polynomial data types, the limitation on the functors treated remains in their work. The current authors [3] subsequently removed this final restriction to give sound induction rules for all inductive types on the underlying fibration under conditions commensurate with those in [7].

In this paper, we extend the existing body of work in three key directions. First, Hermida and Jacobs developed a fibrational theory of coinduction to complement their theory of induction. But this theory, too, is sound only for polynomial data types, and so does not apply to final coalgebras of some key functors, such as the finite powerset functor. In this paper, we derive a sound fibrational coinduction rule for *every* coinductive data type, i.e., for every type that is the carrier of a final coalgebra. Second, data types arising as initial algebras of functors are fairly simple. More sophisticated data types — e.g., untyped lambda terms and red-black trees — are often modelled as inductive indexed types arising as initial algebras of functors on slice categories, presheaf categories, and similar structures. In this paper, we derive sound induction rules for such inductive indexed types. Finally, since we can derive sound induction rules for inductive types and inductive indexed types, and sound coinduction rules for coinductive types, we might expect to be able to derive sound coinduction rules for coinductive indexed types, too. In this paper, we confirm that this is the case.

This rest of this paper is structured as follows. In Section 2 we recall the fibrational approach to induction pioneered in [7] and extended in [3]. In Section 3 we extend the results of [7] to derive sound coinduction rules for *all* functors with final coalgebras. We give sound induction (coinduction) rules for inductive (resp., coinductive) indexed types in Section 4 (resp., Section 5). Section 6 summarises our conclusions, and discusses related work and possibilities for future work.

2 Induction in a Fibrational Setting

Fibrations support a uniform, axiomatic approach to induction and coinduction that is widely applicable and abstracts over the specific choices of category, functor, and predicate. This is advantageous because i) the semantics of data types in languages involving recursion and other effects usually involves categories other than Set; ii) in such circumstances, the standard set-based interpretations of predicates are no longer germane; iii) in any setting, there can be more than one reasonable notion of predicate; and iv) fibrations allow induction and coinduction rules for many classes of data types to be obtained by instantiation of a single, generic theory rather than developed an ad hoc, case-by-case basis.

2.1 Fibrations in a Nutshell

We begin with fibrations. More details can be found in, e.g., [9, 14].

Definition 2.1. *Let $U : \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $g : Q \to P$ in $\mathcal{E}$ is* cartesian *over a morphism $f : X \to Y$ in $\mathcal{B}$ if $Ug = f$ and, for every $g' : Q' \to P$ in $\mathcal{E}$ with $Ug' = fv$ for some $v : UQ' \to X$, there exists a unique $h : Q' \to Q$ in $\mathcal{E}$ such that $Uh = v$ and $gh = g'$.*

The cartesian morphism $f^{\S}_P$ over a morphism f with codomain UP is unique up to isomorphism. We write f^*P for the domain of $f^{\S}_P$, and omit the subscript P when it can be inferred from context.

Definition 2.2. *Let $U : \mathcal{E} \to \mathcal{B}$ be a functor. Then U is a* fibration *if for every object P of $\mathcal{E}$ and every morphism $f : X \to UP$ in $\mathcal{B}$ there is a cartesian morphism $f^{\S}_P : f^*P \to P$ in $\mathcal{E}$ over f.*

If $U : \mathcal{E} \to \mathcal{B}$ is a fibration, we call $\mathcal{B}$ the *base category* of U and $\mathcal{E}$ its *total category*. Objects of $\mathcal{E}$ are thought of as properties, objects of $\mathcal{B}$ are thought of as types, and U is thought to map each property P in $\mathcal{E}$ to the type UP about which it is a property. An object P in $\mathcal{E}$ is said to be *above* its image UP under U, and similarly for morphisms. For any object X of $\mathcal{B}$, we write $\mathcal{E}_X$ for the *fibre above* X, i.e., the subcategory of $\mathcal{E}$ comprising objects above X and morphisms above id_X. Morphisms within a fibre are said to be *vertical*. If $f : X \to Y$ is a morphism in $\mathcal{B}$, then the function mapping each object P of $\mathcal{E}$ to f^*P extends to a functor $f^* : \mathcal{E}_Y \to \mathcal{E}_X$ called the *reindexing functor induced by f*.

Example 2.3. The category Fam(Set) has as objects pairs (X, P) with X a set and $P : X \to \mathrm{Set}$. We call X the *domain* of (X, P), and write P for (X, P) when convenient. A morphism from $P : X \to \mathrm{Set}$ to $P' : X' \to \mathrm{Set}$ is a pair $(f, f^\sim)$ of functions $f : X \to X'$ and $f^\sim : \forall x : X.\, P\,x \to P'(f\,x)$. The functor $U : \mathrm{Fam(Set)} \to \mathrm{Set}$ mapping (X, P) to X is called the *families fibration*.

Example 2.4. The *arrow category* of $\mathcal{B}$, denoted $\mathcal{B}^\to$, has morphisms of $\mathcal{B}$ as its objects. A morphism from $f : X \to Y$ to $f' : X' \to Y'$ in $\mathcal{B}^\to$ is a pair (α_1, α_2) of morphisms in $\mathcal{B}$ such that $f'\alpha_1 = \alpha_2 f$. The codomain functor $cod : \mathcal{B}^\to \to \mathcal{B}$ maps an object $f : X \to Y$ of $\mathcal{B}^\to$ to the object Y of $\mathcal{B}$. If $\mathcal{B}$ has pullbacks, then cod is a fibration, called the *codomain fibration over* $\mathcal{B}$. Indeed, given an object $f : X \to Y$ in the fibre above Y and a morphism $f' : X' \to Y$ in $\mathcal{B}$, the pullback of f along f' gives a cartesian morphism over f'.

We say $U : \mathcal{E} \to \mathcal{B}$ is an opfibration, if $U^{op} : \mathcal{E}^{op} \to \mathcal{B}^{op}$ is a fibration. Concretely:

Definition 2.5. *Let $U : \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $g : P \to Q$ in $\mathcal{E}$ is* opcartesian *over a morphism $f : X \to Y$ in $\mathcal{B}$ if $Ug = f$ and, for every $g' : P \to Q'$ in $\mathcal{E}$ with $Ug' = vf$ for some $v : Y \to UQ'$, there exists a unique $h : Q \to Q'$ in $\mathcal{E}$ such that $Uh = v$ and $hg = g'$.*

As for cartesian morphisms, the opcartesian morphism $f^P_{\S}$ over a morphism f with codomain UP is unique up to isomorphism. We write $\Sigma_f P$ for the domain of $f^P_{\S}$, and omit the superscript P when it can be inferred from context.

Definition 2.6. *If $U : \mathcal{E} \to \mathcal{B}$ is a functor, then U is an* opfibration *if for every object P of $\mathcal{E}$ and every morphism $f : UP \to Y$ in $\mathcal{B}$ there is an opcartesian morphism $f_{\S}^{P} : P \to \Sigma_f P$ in $\mathcal{E}$ over f. A functor U is a* bifibration *if it is simultaneously a fibration and an opfibration.*

If $f : X \to Y$ is a morphism in the base of an opfibration, then the function mapping each object P of $\mathcal{E}_X$ to $\Sigma_f P$ extends to a functor $\Sigma_f : \mathcal{E}_X \to \mathcal{E}_Y$ called the *opreindexing functor induced by* f. The following useful result is from [10].

Lemma 2.7. *Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration. Then U is a bifibration iff, for every morphism $f : X \to Y$ in $\mathcal{B}$, f^* is right adjoint to Σ_f.*

2.2 Fibrational Induction in Another Nutshell

At the heart of Hermida and Jacobs' approach to induction is the observation that if $U : \mathcal{E} \to \mathcal{B}$ is a fibration and $F : \mathcal{B} \to \mathcal{B}$ is a functor, then F can be lifted to a functor $\hat{F} : \mathcal{E} \to \mathcal{E}$ and the premises of the induction rule for μF can be taken to be an $\hat{F}$-algebra. Crucially, for this induction rule to be sound, the lifting must be truth-preserving. These terms are defined as follows.

Definition 2.8. *Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration and $F : \mathcal{B} \to \mathcal{B}$ be a functor. A* lifting *of F with respect to U is a functor $\hat{F} : \mathcal{E} \to \mathcal{E}$ such that $U\hat{F} = FU$. If each fibre $\mathcal{E}_X$ has a terminal object, and if reindexing preserves terminal objects, then we say that U* has fibred terminal objects*. In this case, the map assigning to every X in $\mathcal{B}$ the terminal object in $\mathcal{E}_X$ defines a functor K_U which is called the* truth functor *for U and is right adjoint to U. We omit the subscript on K_U when this can be inferred. A lifting $\hat{F}$ of F is called* truth-preserving *if $KF \cong \hat{F}K$.*

The codomain fibration *cod* from Example 2.4, for instance, has fibred terminal objects: the terminal object in $\mathcal{E}_X$ is id_X. A truth-preserving lifting $F^{\to}$ of F with respect to *cod* is given by the action of F on morphisms.

Definition 2.9. *A* comprehension category with unit *(CCU) is a fibration $U : \mathcal{E} \to \mathcal{B}$ with a truth functor K_U which itself has a right adjoint $\{-\}_U$. In this case, $\{-\}_U$ is called the* comprehension functor *for U.*

We omit the subscript on $\{-\}_U$ when this can be inferred. The fibration *cod* is the canonical CCU: the comprehension functor is the domain functor $dom : \mathcal{B}^{\to} \to \mathcal{B}$ mapping $f : X \to Y$ in $\mathcal{B}^{\to}$ to X. Truth-preserving liftings with respect to CCUs are used in [7] to state and prove soundness of induction rules.

Theorem 2.10. *Let $U : \mathcal{E} \to \mathcal{B}$ be a CCU, let $F : \mathcal{B} \to \mathcal{B}$ be a functor with initial algebra μF, and let $\hat{F}$ be a truth-preserving lifting of F. Then the following induction rule for F is sound:*

$$ind_F : \forall (P : \mathcal{E}).\ (\hat{F}P \to P) \to \mu F \to \{P\}$$

Proof. Because $\hat{F}$ is truth-preserving, the initial $\hat{F}$-algebra exists and has carrier $K(\mu F)$. Thus, for any $\hat{F}$-algebra $h : \hat{F}P \to P$, we have $fold\, h : K(\mu F) \to P$. Since $K \dashv \{-\}$, this map in turn gives the desired map from μF to $\{P\}$.

This very elegant theorem shows that fibrations provide just the right structure to derive sound induction rules for inductive data types whose underlying functors have truth-preserving liftings. Although Hermida and Jacobs gave such liftings only for polynomial functors, [3] showed that every functor has a truth-preserving lifting with respect to certain bifibrations, called *Lawvere categories.*

Definition 2.11. *A fibration $U : \mathcal{E} \to \mathcal{B}$ is a* Lawvere category *if it is a CCU which is also a bifibration.*

If ϵ is the counit of the adjunction $K \dashv \{-\}$ for a CCU U, then $\pi_P = U\epsilon_P$ defines a natural transformation $\pi : \{P\} \to UP$. (The domain of π_P really is $\{P\}$ since $UK = Id$.) Moreover, π extends to a functor $\pi : \mathcal{E} \to \mathcal{B}^{\to}$ in the obvious way.

Lemma 2.12. *Let $U : \mathcal{E} \to \mathcal{B}$ be a Lawvere category. Then π has a left adjoint $I : \mathcal{B}^{\to} \to \mathcal{E}$ defined by $I\,(f : X \to Y) \,=\, \Sigma_f\,(KX)$.*

For any functor F, the composition $\hat{F} = IF^{\to}\pi : \mathcal{E} \to \mathcal{E}$ defines a truth-preserving lifting with respect to the Lawvere category U [4]. Here, $F^{\to}$ is the lifting given after Definition 2.8. If F also has an initial algebra, then Theorem 2.10 guarantees that it has a sound induction rule as well.

If $\mathcal{B}$ has pullbacks, the following diagram shows that we have actually given a modular construction of a lifting with respect to a Lawvere category by factorisation through the lifting for the codomain fibration:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\pi"] \arrow[rr, phantom, "\top"] \arrow[dr, "U"'] & & \mathcal{B}^{\to} \arrow[ll, bend left, "I"] \arrow[dl, "cod"] \\
& \mathcal{B} &
\end{tikzcd}$$

3 Coinduction

In [7], a sound fibrational coinduction rule is given for final coalgebras of polynomial functors. The development is based on a fibration U, but since coinduction is concerned with relations, a new fibration $Rel(U)$ of relations is first constructed.

Definition 3.1. *Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration, assume $\mathcal{B}$ has products, and let $\Delta : \mathcal{B} \to \mathcal{B}$ be the diagonal functor sending an object X to $X \times X$. Then the fibration $Rel(U) : Rel(\mathcal{E}) \to \mathcal{B}$ is obtained by the pullback of U along Δ.*

That the pullback of a fibration along any functor is a fibration is well-known [11]. The process of pulling back a fibration along a functor F, called *change of base along* F, is also well-known to preserve fibred terminal objects [6]. The fibration $Rel(U)$ therefore has a truth functor. Below we denote the pullback of *any* functor $F : \mathcal{A} \to \mathcal{B}$ along a functor $G : \mathcal{B}' \to \mathcal{B}$ by $G^*F : G^*\mathcal{A} \to \mathcal{B}'$. The objects of $G^*\mathcal{A}$ are pairs (X, Y) such that $GX = FY$, and G^*F maps the pair (X, Y) to the object X. We write Y for (X, Y) in G^*F when convenient. Definition 3.1 entails that the fibre of $Rel(\mathcal{E})$ above X is the fibre $\mathcal{E}_{X\times X}$. A morphism from (X, Y) to (X', Y') in $Rel(\mathcal{E})$ consists of a pair of morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that $U\beta = \alpha \times \alpha$. Finally, if U has truth functor K, then the truth functor for $Rel(U)$ is given by $K_{Rel(U)}X = K(X \times X)$.

In the inductive setting, truth-preserving liftings were needed. In the coinductive setting, we need equality-preserving liftings, where equality is given by:

Definition 3.2. *Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with a truth functor and assume $\mathcal{B}$ has products. The* equality functor *for U is the functor $Eq_U : \mathcal{B} \to Rel(\mathcal{E})$ mapping an object X to $\Sigma_\delta\, K_{Rel(U)}\, X$ and a morphism $f : X \to I$ to the unique morphism above $f \times f$ induced by the naturality of δ at f and the opcartesian map $\delta_{\S}^{KX}$. Here, $\delta : Id_{\mathcal{B}} \to \Delta$ is the diagonal natural transformation with components $\delta_X : X \to X \times X$, and $\Sigma_\delta : \mathcal{E} \to Rel(\mathcal{E})$ maps an object P in $\mathcal{E}_X$ to an object $\Sigma_{\delta_X} P$ in $\mathcal{E}_{X \times X}$. If Eq_U has a left adjoint Q_U, then Q_U is called the* quotient functor *for U. We suppress the subscripts on Eq_U and Q_U when convenient.*

Definition 3.3. *Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration which has a truth functor, assume $\mathcal{B}$ has products, and let $F : \mathcal{B} \to \mathcal{B}$ be a functor. A lifting $\check{F}$ of F with respect to $Rel(U)$ is called* equality-preserving *if $Eq\, F \cong \check{F}\, Eq$.*

That every polynomial functor has an equality-preserving lifting is shown in [7]. Theorem 3.4 is Hermida and Jacobs' main theorem about coinduction. Note the duality: in the inductive, setting the truth functor K must have a right adjoint, while in the coinductive one, the equality functor Eq must have a left adjoint.

Theorem 3.4. *Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration which has a truth functor, assume $\mathcal{B}$ has products, let $F : \mathcal{B} \to \mathcal{B}$ be a functor with final coalgebra νF, and let $\check{F}$ be an equality-preserving lifting of F. If Eq has a left adjoint Q, then the following coinduction rule for F is sound:*

$$coind_F : \forall (R : Rel(\mathcal{E})).\ (R \to \check{F} R) \to QR \to \nu F$$

Proof. Because $\check{F}$ is equality-preserving, the final $\check{F}$-coalgebra exists and has carrier $Eq(\nu F)$. Thus, for any $\check{F}$-coalgebra $k : R \to \check{F}R$, we have $unfold\, k : R \to Eq(\nu F)$. Since $Q \dashv Eq$, this map in turn gives the desired map from QR to νF.

3.1 Generic Coinduction For All Final Coalgebras

The first contribution of this paper is to give a sound coinduction rule for any functor with a final coalgebra. To do so, we show how to generate liftings that can be instantiated to give both the truth-preserving liftings required for induction and, by duality, the equality-preserving liftings required for coinduction.

Lemma 3.5. *Consider a* quotient category with equality *(QCE) over $\mathcal{B}$, i.e., a fibration $U : \mathcal{E} \to \mathcal{B}$ with a full and faithful functor $E : \mathcal{B} \to \mathcal{E}$ such that $UE = Id_{\mathcal{B}}$ and E has left adjoint Q with unit η. Define functors ρ, J, and $\check{F}$ by*

$$\begin{array}{lll} \rho : \mathcal{E} \to \mathcal{B}^{\to} & J : \mathcal{B}^{\to} \to \mathcal{E} & \check{F} : \mathcal{E} \to \mathcal{E} \\ \rho P = U \eta_P & J\,(f : X \to Y) = f^* E Y & \check{F} = J\, F^{\to}\, \rho \end{array}$$

Then $U\check{F} = FU$ and $\check{F}E \cong EF$.

Proof. To prove $U\check{F} = FU$, note that the morphisms $\rho\, P$ each have domain UP, that $dom\, F^{\to}\, \rho = FU$, and that $UJ = dom$. Together these give $U\check{F} = UJF^{\to}\rho = FU$. To prove $\check{F}E \cong EF$, we first assume that i) for every X in $\mathcal{B}$, ρEX is an isomorphism in $\mathcal{B}$, and ii) for every isomorphism f in $\mathcal{B}$,

$J\,f \cong E(dom\,f)$. Then since $UE = Id_{\mathcal{B}}$, i) and ii) imply that $\check{F}E = JF^{\rightarrow}\rho E \cong E\,dom\,F^{\rightarrow}\rho E = EFUE = EF$. To discharge i), note that, since E is full and faithful, $\eta E : E \rightarrow EQE$ is $E\kappa$ for a natural transformation $\kappa : Id_{\mathcal{E}} \rightarrow QE$, where each κ_X is an isomorphism with inverse ϵ_X and ϵ is the counit of $Q \dashv E$. Then $\rho EX = U\eta_{EX} = UE\kappa_X = \kappa_X$, so that ρEX is indeed an isomorphism. To discharge ii), let f be an isomorphism in $\mathcal{B}$. Since cartesian morphisms over isomorphisms are isomorphisms, we have $Jf = f^*(E\,(cod f)) \cong E\,(cod\,f) \cong E\,(dom\,f)$. Here, the first isomorphism is witnessed by $f^{\S}$ and the second by Ef^{-1}.

The lifting $\check{F}$ has as its dual the lifting $\hat{F}$ given in the following lemma.

Lemma 3.6. *Let $U : \mathcal{E} \rightarrow \mathcal{B}$ be an opfibration, let $K : \mathcal{B} \rightarrow \mathcal{E}$ a full and faithful functor such that $UK = Id_{\mathcal{B}}$, and let $C : \mathcal{E} \rightarrow \mathcal{B}$ be a right adjoint to K with counit ϵ. Define functors π, I, and $\hat{F}$ by*

$$\begin{array}{lll} \pi : \mathcal{E} \rightarrow \mathcal{B}^{\rightarrow} & I : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{E} & \hat{F} : \mathcal{E} \rightarrow \mathcal{E} \\ \pi P = U\epsilon_P & I\,(f : X \rightarrow Y) = \Sigma_f KY & \hat{F} = I\,F^{\rightarrow}\,\pi \end{array}$$

Then $U\hat{F} = FU$ and $\hat{F}K \cong KF$.

Proof. By dualisation of Lemma 3.5. The setting on the left below with U an opfibration is equivalent to the setting on the right with U a fibration.

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "U"] \arrow[dl, bend right, "C"'] \\ \mathcal{B} \arrow[ur, "K"'] \arrow[ur, phantom, shift left=3, "\vdash"] \arrow[r, "Id_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{E}^{op} \arrow[d, "U"] \arrow[dl, bend right, "C"'] \\ \mathcal{B}^{op} \arrow[ur, "K"'] \arrow[ur, phantom, shift left=3, "\dashv"] \arrow[r, "Id_{\mathcal{B}^{op}}"'] & \mathcal{B}^{op} \end{tikzcd}$$

We can instantiate Lemmas 3.5 and 3.6 to derive both the truth-preserving lifting for all functors from [3] and an equality-preserving lifting for all functors. The former gives the sound induction rule for all inductive types presented in [3], and the latter gives a sound coinduction rule for all coinductive types. To obtain the lifting for induction, let $U : \mathcal{E} \rightarrow \mathcal{B}$ be a Lawvere category, K be the truth functor for U, and C be the comprehension functor for U. Since a Lawvere category is an opfibration, Lemma 3.6 ensures that any functor $F : \mathcal{B} \rightarrow \mathcal{B}$ lifts to a truth-preserving lifting $\hat{F} : \mathcal{E} \rightarrow \mathcal{E}$. This is exactly the lifting of [3]. To obtain the lifting for coinduction, let $U : \mathcal{E} \rightarrow \mathcal{B}$ be a bifibration with a truth functor and products in $\mathcal{B}$, and let E be the equality functor $Eq = \Sigma_\delta\, K_{Rel(U)}$ for U. Since both $K_{Rel(U)}$ and Σ_δ are full and faithful, so is Eq. Moreover, since EqX is in the fibre of $Rel(U)$ above X we have $Rel(U)\,Eq = Id_{\mathcal{B}}$. We can therefore take E to be Eq in Lemma 3.5 provided Eq has a left adjoint Q. In this case, every functor $F : \mathcal{B} \rightarrow \mathcal{B}$ has an equality-preserving lifting $\check{F} : Rel(\mathcal{E}) \rightarrow Rel(\mathcal{E})$. Thus, if F has a final coalgebra, then νF has a sound coinduction rule.

The domain functor $dom : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$ is actually a fibration called the *domain fibration over* $\mathcal{B}$. No conditions on $\mathcal{B}$ are required. Just as *cod* is the canonical CCU, *dom* is the canonical QCE. A QCE $Rel(U)$ over $\mathcal{B}$ which is obtained by change of base along Δ, and for which the functor E is the equality functor for U, is called a *relational QCE*.

Example 3.7. We can take U to be $dom : \mathcal{B}^{\to} \to \mathcal{B}$, E to map each X in $\mathcal{B}$ to id_X, and Q to be cod in Lemma 3.5. Then $\check{F}$ is exactly $F^{\to}$, so that $F^{\to}$ and $\check{F}$ are interdefinable. Thus, just as the lifting $\hat{F}$ with respect to an arbitrary fibration U satisfying the hypotheses of Lemma 3.6 can be modularly constructed from the specific lifting $F^{\to}$ with respect to cod [3], so the lifting $\check{F}$ with respect to an arbitrary fibration U satisfying the hypotheses of Lemma 3.5 can be modularly constructed from the specific lifting $F^{\to}$ with respect to dom.

So dom plays a role role in the coinductive setting similar to that played by cod in the inductive one. We think of a morphism $f : X \to Y$ in the total category of cod as a predicate on Y whose proofs constitute X. Intuitively, f maps each p in X to the element y in Y about which it is a proof. Similarly, we think of a morphism $f : X \to Y$ in the total category of dom as a relation on X, the quotient of X by which has equivalence classes comprising Y. Intuitively, f maps each x in X to its equivalence class in that quotient.

Example 3.8. If U is the families fibration, then the fibre above X in $Rel(U)$ consists of functions $R : X \times X \to Set$. We think of these as constructive relations, where $R(x, x')$ gives the set of proofs that x is related to x'. In Lemma 3.5 we can take U to be the families fibration, E to map each set X to the relation eq_X defined by $eq_X(x, x') = 1$ if $x = x'$ and $eq_X(x, x') = 0$ otherwise, and Q to map each relation $R : X \times X \to Set$ to the quotient X/R of X by the least equivalence relation containing R. We get this instantiation of the definition of $\check{F}$, for $F : Set \to Set$, from Lemma 3.5: $\rho : Rel(U) \to Set^{\to}$ maps a relation $R : X \times X \to$ Set to the quotient map $\rho_R : X \to X/R$, $F^{\to}$ maps f to Ff, and $J : Set^{\to} \to Rel(U)$ maps $f : X \to Y$ to the relation $\bar{f}$ mapping (x, x') to 1 if $fx = fx'$ and to 0 otherwise. Thus $\check{F} : FA \times FA \to Set$ is given by $\check{F}R = \overline{F\rho_R}$.

We now derive the coinduction rule, for the functor $\mathscr{P}_{\!fin}$, which maps a set to its finite powerset, with respect to the fibration of relations constructed from the families fibration in Example 3.8. Since $\mathscr{P}_{\!fin}$ is not polynomial, it lies outside the scope of [7], but it is important, since a number of canonical coalgebras are built from it. For example, a finitely branching labelled transition system with state space S and labels from an alphabet A is a coalgebra $S \to \mathscr{P}_{\!fin}(A \times S)$.

Example 3.9. By Example 3.8, the lifting $\check{\mathscr{P}}_{\!fin}$ maps a relation $R : A \times A \to Set$ to the relation $\check{\mathscr{P}}_{\!fin}R : \mathscr{P}_{\!fin}A \times \mathscr{P}_{\!fin}A \to Set$ defined by $\check{\mathscr{P}}_{\!fin}R = \overline{\mathscr{P}_{\!fin}\rho_R}$. Thus, if X and Y are finite subsets of A, then $(X, Y) \in \check{\mathscr{P}}_{\!fin}R$ iff $\mathscr{P}_{\!fin}\rho_R X = \mathscr{P}_{\!fin}\rho_R Y$. Since the action of $\mathscr{P}_{\!fin}$ on a morphism f maps any subset of the domain of f to its image under f, $\mathscr{P}_{\!fin}\rho_R X = \mathscr{P}_{\!fin}\rho_R Y$ iff $(\forall x : X).(\exists y : Y).\, xRy \;\wedge\; (\forall y : Y).(\exists x : X).\, xRy$. From $\check{\mathscr{P}}_{\!fin}$ we have that the resulting coinduction rule has as its premises a $\check{\mathscr{P}}_{\!fin}$-coalgebra, i.e., a relation $R : A \times A \to Set$ and a map from R to $\check{\mathscr{P}}_{\!fin}R$ in $Rel(U)$. An object of $Rel(U)$ is a pair $(X, (Y, P))$ where X is a set, (Y, P) is an object of $Fam(Set)$, and $Y = X \times X$. A morphism in $Rel(U)$ from $(X, (Y, P))$ to $(X', (Y', P'))$ consists of a morphism $\phi : X \to X'$ in Set and a morphism $(\psi, \psi^{\sim}) : (Y, P) \to (Y', P')$ in $Fam(Set)$ such that $\psi = \phi \times \phi$. Thus, a $\check{\mathscr{P}}_{\!fin}$-coalgebra consists of a function $\alpha : A \to \mathscr{P}_{\!fin}A$ together with a function $\alpha^\sim : (\forall a, a' : A).\, aRa' \to (\alpha a)\ \check{\mathscr{P}}_{\mathit{fin}} R\ (\alpha a')$. If we regard $\alpha : A \to \mathscr{P}_{\mathit{fin}} A$ as a transition function, i.e., if we define $a \to b$ iff $b \in \alpha a$, then $\alpha^\sim$ captures the condition that R is a bisimulation over α. The coinduction rule thus asserts that any two bisimilar states have the same interpretation in the final coalgebra.

4 Indexed Induction

Data types arising as initial algebras and final coalgebras on traditional semantic categories such as Set and $\omega cpo_\perp$ are of limited expressivity. More sophisticated data types arise as initial algebras of functors on their indexed versions. To build intuition about the resulting *inductive indexed types*, first consider the inductive type $\mathsf{List}\,X$ of lists of X. It is clear that the definition of $\mathsf{List}\,X$ does not require an understanding of $\mathsf{List}\,Y$ for for any $Y \neq X$. Since, each type $\mathsf{List}\,X$ is, in isolation, inductive, List can be considered a *family of inductive types.* By contrast, for each n in Nat, let $\mathsf{Fin}\,n$ be the data type of n-element sets, and consider the inductive definition of the Nat-indexed type $\mathsf{Lam} : \mathsf{Nat} \to Set$ of untyped λ-terms up to α-equivalence with free variables in $\mathsf{Fin}\,n$ given by

$$\frac{i : \mathsf{Fin}\,n}{Var\,i : \mathsf{Lam}\,n} \qquad \frac{f : \mathsf{Lam}\,n \quad a : \mathsf{Lam}\,n}{App\,f\,a : \mathsf{Lam}\,n} \qquad \frac{b : \mathsf{Lam}\,(n+1)}{Abs\,b : \mathsf{Lam}\,n}$$

Unlike $\mathsf{List}\,X$, the type $\mathsf{Lam}\,n$ cannot be defined in isolation using only the elements of $\mathsf{Lam}\,n$ that have already been constructed. Indeed, elements of $\mathsf{Lam}\,(n+1)$ are needed to construct elements of $\mathsf{Lam}\,n$ so that, in effect, all of the types $\mathsf{Lam}\,n$ must be inductively constructed simultaneously. The indexed type Lam is thus an *inductive family of types*, rather than a family of inductive types.

There is considerable interest in inductive and coinductive indexed types. If types are interpreted in a category $\mathcal{B}$, and if I is a set of indices considered as a discrete category, then an inductive I-indexed type can be modelled by the initial algebra of a functor on the functor category $I \to \mathcal{B}$. Alternatively, indices can be modelled by objects of $\mathcal{B}$, and inductive I-indexed types can be modelled by initial algebras of functors on slice categories $\mathcal{B}/I$. Coinductive indexed types can similarly be modelled by final colagebras of functors on slice categories.

Initial algebra semantics for inductive indexed types has been developed extensively [2, 12]. Pleasingly, no fundamentally new insights were required: the standard initial algebra semantics needed only to be instantiated to categories such as $\mathcal{B}/I$. By contrast, the theory of induction for inductive indexed types has received comparatively little attention. The second contribution of this paper is to use our fibrational framework to derive sound induction rules for such types by similarly instantiating initial algebra semantics to appropriate categories. The key technical question to be solved turns out to be: given a Lawvere category of properties fibred over types, can we construct a new Lawvere category fibred over indexed types from which induction rules for the indexed types can be derived? To answer it, we make the simplifying assumption that the inductive indexed types of interest arise as initial algebras of functors over slice categories, i.e., of functors $F : \mathcal{B}/I \to \mathcal{B}/I$, where I is an object of $\mathcal{B}$. Let U/I denote the Lawvere category to be constructed. We conjecture that the total category of U/I

should be a slice category of $\mathcal{E}$, and so make the canonical choice to slice over KI, where K is the truth functor for U. We then define $U/I : \mathcal{E}/KI \to \mathcal{B}/I$ by $U/I\,(f : P \to KI) = Uf : UP \to I$. Here, *cod* (Uf) really is I because $UK = Id$.

We first show that U/I is indeed a bifibration. We give a concrete proof before indicating how the same result can be derived from a more abstract treatment.

Lemma 4.1. *If $U : \mathcal{E} \to \mathcal{B}$ is a fibration (bifibration) and I is an object of $\mathcal{B}$, then U/I is a fibration (resp., bifibration).*

Proof. Let $\alpha : Y \to I$ and $\beta : X \to I$ be objects of $\mathcal{B}/I$, and let $\phi : Y \to X$ be a morphism in $\mathcal{B}/I$ from α to β, i.e., be such that $\alpha = \beta\phi$. First, let $f : P \to KI$ be an object of $\mathcal{E}/KI$ such that $(U/I)f = Uf = \beta$, and let $\phi^{\S}_P : \phi^* P \to P$ be the cartesian morphism in $\mathcal{E}$ over ϕ with respect to U. Then $\phi^{\S}_P$ is a morphism in $\mathcal{E}/KI$ with domain $f\phi^{\S}_P$ and codomain f, and it is cartesian over ϕ with respect to U/I. Thus, U/I is a fibration if U is. Now, let $g : Q \to KI$ be an object of $\mathcal{E}/KI$ such that $(U/I)g = Ug = \alpha$, and let $\phi^Q_{\S} : Q \to \Sigma_\phi Q$ be the opcartesian morphism in $\mathcal{E}$ over ϕ with respect to U. Since $\alpha = \beta\phi$, the opcartesianness of $\phi^Q_{\S}$ ensures that there is a unique map $k : \Sigma_\phi\, Q \to KI$ in $\mathcal{E}$ above β such that $g = k\phi^Q_{\S}$. Then $\phi^Q_{\S}$ is a morphism in $\mathcal{E}/KI$ with domain g and codomain k, and it is opcartesian over ϕ with respect to U/I. Thus, U/I is an opfibration if U is. Combining these results gives that if U is a bifibration then so is U/I.

There is an alternative characterisation of $\mathcal{E}/KI$ which both clarifies the conceptual basis of our treatment of indexed induction and simplifies our calculations. The next lemma is the key observation underlying this characterisation.

Lemma 4.2. *Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration with truth functor K, let I be an object of $\mathcal{B}$, and let $\alpha : X \to I$. Then $(\mathcal{E}/KI)_\alpha \cong \mathcal{E}_X$.*

Proof. One half of the isomorphism maps the object $f : P \to KI$ of $(\mathcal{E}/KI)_\alpha$ to P. For the other half, note that since truth functors map objects to terminal objects, and since reindexing preserves terminal objects, we have $KX \cong \alpha^* KI$. Thus, for any object Q above X, we get a morphism from Q to KI by composing $\alpha^{\S}_{KI}$ and the unique morphism $!$ from Q to KX. Since $!$ is vertical and $\alpha^{\S}_{KI}$ is above α, this composition is above α. Thus each object Q in $\mathcal{E}_X$ maps to an object of $(\mathcal{E}/KI)_\alpha$. It is routine to verify that these maps constitute an isomorphism.

By Lemma 4.2 we may identify objects (morphisms) of $(\mathcal{E}/KI)_\alpha$ and objects (resp., morphisms) of $\mathcal{E}_X$. This gives our abstract characterisation of U/I:

Lemma 4.3. *Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration and I be an object of $\mathcal{B}$. Then U/I can be obtained by change of base by pulling U back along $dom : \mathcal{B}/I \to \mathcal{B}$.*

Proof. As noted in Section 3, the pullback of a fibration along a functor is a fibration. The objects (morphisms) of the fibre above $\alpha : X \to I$ of the pullback of U along *dom* are the objects (resp., morphisms) of $\mathcal{E}_X$. By Lemma 4.2, the pullback of U along *dom* is therefore U/I.

As observed in Section 3, pulling back a fibration along a functor preserves fibred terminal objects, so U/I has fibred terminal objects if U does by Lemma 4.3. Concretely, the truth functor $K_{U/I} : B/I \to \mathcal{E}/KI$ maps an object $f : X \to I$ to $Kf : KX \to KI$. To see that U/I is a Lawvere category if U is, we thus need to show that $K_{U/I}$ has a right adjoint. For this, we use an abstract theorem due to Hermida [5] to transport adjunctions to pullbacks along fibrations.

Lemma 4.4. *Let $F \dashv G : \mathcal{A} \to \mathcal{B}$ be an adjunction with counit ϵ, and let $U : \mathcal{E} \to \mathcal{B}$ be a fibration. Then the functor $U^*F : U^*\mathcal{A} \to \mathcal{E}$ has a right adjoint $G_U : \mathcal{E} \to U^*\mathcal{A}$ whose action maps each object E to the object $(\epsilon^*_{UE}E, GUE)$.*

Lemma 4.5. *Change of base along a fibration preserves CCUs, i.e., if $U : \mathcal{E} \to \mathcal{B}$ is a CCU and $U' : \mathcal{E}' \to \mathcal{B}$ is a fibration, then the pullback U'^*U is a CCU.*

Proof. We already have that U'^*U is a fibration with fibred terminal objects. To see that $K_{U'^*U}$ has a right adjoint, consider the pullback of U^*U' and K_U. This pullback is given by $\mathcal{E}'$, $K_{U'^*U} : \mathcal{E}' \to U'^*\mathcal{E}$, and $U' : \mathcal{E}' \to \mathcal{B}$. Note that U^*U' is a fibration since it is obtained by pulling U' back along U. Lemma 4.4 then ensures that, since K_U has a right adjoint, so does $K_{U'^*U}$. Thus U'^*U is a CCU.

When $U : \mathcal{E} \to \mathcal{B}$, I is an object of $\mathcal{B}$, and U' is $dom : \mathcal{B}/I \to \mathcal{B}$, the comprehension functor for U'^*U — i.e., for U/I — maps an object P in $\mathcal{E}_X$ above $\alpha : X \to I$ to $\alpha\pi_P : \{P\} \to I$. Combining Lemmas 4.1 and 4.5 and the fact that U^{op} is a fibration if U is an opfibration, we have

Lemma 4.6. *Let $U : \mathcal{E} \to \mathcal{B}$ be a Lawvere category and $U' : \mathcal{E}' \to \mathcal{B}$ be a fibration. Then U'^*U is a Lawvere category.*

Thus, if F is a functor on $\mathcal{E}'$ with initial algebra μF, then Theorem 2.10 guarantees the existence of a sound induction rule for μF. We use this observation to derive an induction rule for the indexed containers of Morris and Altenkirch [12].

Example 4.7. If I is a set, then the *category of I-indexed sets* is the fibre $\mathrm{Fam}(\mathrm{Set})_I$. An *$I$-indexed set* is thus a function $X : I \to \mathrm{Set}$, and a morphism h from X to X', written $h : X \to_I X'$, is a function of type $(\Pi i\!:\!I).\, Xi \to X'i$. Morris and Altenkirch denote this category $I \to \mathrm{Set}$ and define an *I-indexed container* to be a pair (S, P) with $S : I \to \mathrm{Set}$ and $P : (\Pi i\!:\!I).\, Si \to I \to \mathrm{Set}$. An I-indexed container defines a functor $[S, P] : (I \to \mathrm{Set}) \to I \to \mathrm{Set}$ by $[S, P]Xi = (\Sigma s : Si).\, P\,i\,s \to_I X$. Thus, if $t\!:\![S, P]\,X\,i$, then t is of the form (s, f), with projections ρ_0 and ρ_1 defined by $\rho_0\,t = s$ and $\rho_1\,t = f$. The action of $[S, P]$ on a morphism $g : X \to_I Y$ maps a pair (s, f) to (s, gf). The initial algebra of $[S, P]$ is denoted $in : [S, P]W_{S,P} \to_I W_{S,P}$. Since $I \to \mathrm{Set}$ is equivalent to Set/I, we can use the results of this section to extend those of [12] by deriving an induction rule for $W_{S,P}$. A predicate over an I-indexed set X is a function $Q : (\Pi i : I).\, Xi \to \mathrm{Set}$. To simplify notation, this is written $Q : X \to_I \mathrm{Set}$. The lifting $\widehat{[S, P]}$ of $[S, P]$ maps each $Q : X \to_I \mathrm{Set}$ to the predicate $\widehat{[S, P]}Q : [S, P]X \to_I \mathrm{Set}$ defined by $\widehat{[S, P]}\,Q\,i\,(s, f) = (\Pi j\!:\!I).\,(\Pi p\!:\!P\,i\,s\,j).\,Q\,j\,(f\,j\,p)$. Altogether, this gives the following induction rule for establishing a predicate $Q : W_{S,P} \to_I \mathrm{Set}$:

$$(\Pi i\!:\!I).\,(\Pi(s,f)\!:\![S,P]\,W_{S,P}\,i).\,((\Pi j\!:\!I).\,(\Pi p\!:\!P\,i\,s\,j).\,Q\,j(f\,j\,p) \rightarrow Q\,i(in\,i\,(s,f))))$$
$$\rightarrow (\Pi i\!:\!I).\,(\Pi t\!:\!W_{S,P}\;i).\,Q\,i\,t$$

5 Indexed Coinduction

We now present our third contribution: we derive coinduction rules for coinductive indexed types. Examples of such types are infinitary versions of inductive indexed types, such as infinitary untyped lambda terms and the interaction structures of Hancock and Hyvernat [8]. If $U : \mathcal{E} \rightarrow \mathcal{B}$ supports coinduction for the final coalgebra of any functor on $\mathcal{B}$ having one, and if $U' : \mathcal{E}' \rightarrow \mathcal{B}$ gives a change of base to an indexed notion of data described by $\mathcal{E}'$, then is there a fibration over $\mathcal{E}'$ supporting indexed coinduction for the final coalgebra of any functor on $\mathcal{E}'$ having one. However, the details in the coinductive setting are much more involved than in the inductive one, here we present only the following simpler result, showing that for any relational QCE over a base category $\mathcal{B}$ and object I of $\mathcal{B}$, change of base along $dom : \mathcal{B}/I \rightarrow \mathcal{B}$ yields a relational QCE over $\mathcal{B}/I$.

If $\mathcal{B}$ has products and $U : \mathcal{E} \rightarrow \mathcal{B}$ is a bifibration with truth functor K, then the equality functor Eq for U is given by $Eq = \Sigma_\delta K$. Let $Rel(U) : Rel(E) \rightarrow \mathcal{B}$ be a QCE, i.e., let Eq have a left adjoint Q. To define a relational QCE over $\mathcal{B}/I$ we must first see that $\mathcal{B}/I$ has products. But the product of f and g in $\mathcal{B}/I$ is determined by their pullback: if W, $j : W \rightarrow Z$, and $i : W \rightarrow X$ give the pullback of f and g, then their product in $\mathcal{B}/I$ is the morphism fi or, equivalently, gj. Below, we write f^2 for the product of f with itself in $\mathcal{B}/I$ and $X_f X$ for the domain of f^2. Then, if $\mathcal{B}$ has pullbacks, we can construct the relation fibration $Rel(U/I) : Rel(\mathcal{E}/KI) \rightarrow \mathcal{B}/I$ from the pullback of U/I along the product functor $\Delta/I : \mathcal{B}/I \rightarrow \mathcal{B}/I$ mapping f to f^2. Concretely, an object of $Rel(\mathcal{E}/KI)$ above $f : X \rightarrow I$ is an object of $\mathcal{E}/KI$ above f^2 with respect to U/I. This is, in turn, equivalent to an object of $\mathcal{E}$ above $X_f X$ with respect to U.

5.1 The Equality Functor for U/I

If U is a bifibration with a truth functor then, for any object I of $\mathcal{B}$, U/I is as well, and so U/I has an equality functor $Eq_{U/I}$. To define this functor concretely, note that the component of the diagonal natural transformation $\delta/I : Id \rightarrow \Delta/I$ at $f : X \rightarrow I$ is given by the diagram on the left. Thus, $Eq_{U/I}$ maps an object $f : X \rightarrow I$ of $\mathcal{B}/I$ to the unique morphism above f^2 in the diagram on the right induced by the opcartesian map m above $(\delta/I)_f$:

$$\begin{tikzcd}
X \arrow[dr, dotted, "(\delta/I)_f"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X_f X \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
 & KI \\
KX \arrow[ur, "Kf"] \arrow[r, "m"'] & \Sigma_{(\delta/I)_f} KX \arrow[u, dotted, "Eq_{U/I} f"']
\end{tikzcd}$$

5.2 The Quotient Functor for U/I

Whereas defining the equality functor for U/I was straightforward, defining its quotient functor is actually tricky. We have not (yet!) found any abstract fibrational results to deliver it, so we give a concrete construction. For each object I of $\mathcal{B}$, we define another fibration, denoted $Rel(U)/I : Rel(\mathcal{E})/Eq\,I \to \mathcal{B}/I$, where $Eq : \mathcal{B} \to Rel(\mathcal{E})$ is the equality functor for U. The objects of $Rel(\mathcal{E})/Eq\,I$ above $f : X \to I$ are morphisms $\alpha : P \to Eq\,I$ for some object P of $Rel(\mathcal{E})$ such that $U\alpha = \Delta f$. Our first result identifies conditions under which $Rel(U)/I$ is a QCE.

Lemma 5.1. *Let $\mathcal{B}$ have pullbacks, let I be an object of $\mathcal{B}$, and let $Rel(U) : Rel(\mathcal{E}) \to \mathcal{B}$ be a relational QCE. Then $Rel(U)/I$ is a QCE.*

Proof. Let $Eq : \mathcal{B} \to Rel(\mathcal{E})$ and $Q : Rel(\mathcal{E}) \to \mathcal{B}$ be the equality and quotient functors for U, respectively. We construct a full and faithful functor $E' : \mathcal{B}/I \to Rel(\mathcal{E})/Eq\,I$ such that $(Rel(U)/I)\,E' = Id_{\mathcal{B}/I}$ and a left adjoint Q' for E' as follows. Take E' to be Eq. Then E' is full and faithful since Eq is. Moreover, for any $f : X \to I$, Definition 3.2 ensures that $Eq\,f$ is above $f \times f$ with respect to U, so $(Rel(U)/I)\,E'f = f$, and thus $(Rel(U)/I)\,E' = Id_{\mathcal{B}/I}$. Finally, we define Q' to map each object $\alpha : P \to Eq\,I$ of $Rel(\mathcal{E})/Eq\,I$ to its transpose $\alpha' : QP \to I$ under the adjunction $Q \dashv Eq$. That $Q' \dashv E'$ follows directly from $Q \dashv Eq$.

We can now define the quotient functor for $Rel(U/I)$ using the functor Q' from the proof of Lemma 5.1. The key step is to define an adjunction $\tau \dashv \sigma$ so that the diagram below commutes. Then if E' and Q' are the functors witnessing the fact that $Rel(U)/I$ is a QCE, the compositions $\sigma E'$ and $Q'\tau$ give equality and quotient functors for $Rel(U/I)$.

$$\begin{tikzcd}
Rel(\mathcal{E}/KI) \arrow[rr, bend left=10, "\tau"] \arrow[rr, phantom, "\perp"] \arrow[dr, "Rel(U/I)"'] & & Rel(\mathcal{E})/E\,I \arrow[ll, bend left=10, "\sigma"] \arrow[dl, "Rel(U)/I"] \\
& \mathcal{B}/I &
\end{tikzcd}$$

To define τ and σ, let $f : X \to I$, let i and j be the projections for the pullback square defining $X_f X$. The universal property of the product $X \times X$ ensures the existence of a morphism $v : X_f X \to X \times X$ such that $\pi_1 v = i$ and $\pi_2 v = j$. By the universal property of the pullback of f along itself, v is a monomorphism; we will use this in the proof of Lemma 5.3. From the diagram on the left below, we have that $\delta_X = v\,(\delta/I)_f$, which also gives the diagram on the right:

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "id"'] \arrow[d, "(\delta/I)_f"'] \arrow[dd, bend left, "\delta_X"] \arrow[ddr, bend left, "id"] & \\
& X_f X \arrow[d, "v"'] & \\
X & X \times X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X_f X \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "f^2"'] & & \\
& X \arrow[ul, "(\delta/I)_f"'] \arrow[r, "\delta_X"'] \arrow[d, "f"'] & X \times X \arrow[d, "f \times f"] \\
& I \arrow[r, "\delta_I"'] & I \times I
\end{tikzcd}$$

We use the right diagram and the opcartesianness of $v_{\S}^{A}$ to define the functor τ. First recall that, if $R : A \to KI$ is an object of $Rel(\mathcal{E}/KI)$ above $f : X \to I$

with respect to $Rel(U/I)$, then $Rel(U/I)R = (U/I)R = UR = f^2$. The following diagram then defines the morphism h above $f \times f$ in $\mathcal{E}$ to which τ maps R:

$$\begin{tikzcd}
A \arrow[r, "v_\S^A"] \arrow[d, "R"'] & \Sigma_v A \arrow[d, dotted, "h"] \\
KI \arrow[r, "(\delta_I)_\S^{KI}"'] & Eq\, I
\end{tikzcd}$$

So, assuming U satisfies the Beck-Chevalley condition [9], this all gives:

Definition 5.2. *The functors τ and σ are given by:*

$$\begin{array}{ll}
\tau : Rel(\mathcal{E}/KI) \to Rel(\mathcal{E})/Eq\, I & \quad \sigma : Rel(\mathcal{E})/Eq\, I \to Rel(\mathcal{E}/KI) \\
\tau(R : A \to KI) = h & \quad \sigma(S : B \to Eq\, I) = v^* B
\end{array}$$

Lemma 5.3. *τ is a full and faithful left adjoint to σ.*

Proof. We exhibit the unit η of the adjunction $\tau \dashv \sigma$ and show that it is an isomorphism. Because v is a monomorphism, the unit η' of the adjunction $\Sigma_v \dashv v^*$ is an isomorphism. Moreover, since objects of $Rel(\mathcal{E}/KI)$ above $f : X \to I$ in the fibration $Rel(U/I)$ can be seen as objects of $\mathcal{E}$ above $X_f X$ in U, η must assign to every object R above $X_f X$ in U a morphism from R to $v^*\Sigma_v A$. We define $\eta = \eta'$. The universality of η follows from the fact that η' is an isomorphism.

Recall that our candidate for the quotient functor for $Rel(U/I)$ is $Q'\tau$. To see that $Q'\tau \dashv Eq_{U/I}$, note that $Q'\tau \dashv \sigma E'$, so we need only verify that $Eq_{U/I}$ is $\sigma E'$. It is routine to check that $\tau Eq_{U/I} = E'$, from which $Eq_{U/I} = \sigma E'$ follows.

We now use the results of this section to give a coinduction rule for final coalgebras of indexed containers that is dual to the induction rule of Example 4.7.

Example 5.4. Let (S, P) be an I-indexed container with final coalgebra $out : M_{S,P} \to_I [S, P]M_{S,P}$. A relation over an I-indexed set $X : I \to \mathrm{Set}$ is an I-indexed family of relations Ri on Xi. The relational lifting of $[S, P]$ maps a relation R over an I-indexed set X to the relation R' over the I-indexed set $[S, P]X$ that relates $(s, f) \in [S, P]Xi$ and $(s', f') \in [S, P]Xi$ iff $s = s'$ and, for all $j : I$ and $p : P\, i\, s\, j$, $f\, j\, p$ is related in Rj to $f'\, j\, p$. This gives the following notion of bisimulation for $[S, P]$ coalgebras $k : X \to_I [S, P]X$: if $x, x' \in Xi$, then $x \sim_i x'$ iff $\rho_0(kx) = \rho_0(kx')$ and, for all $j : I$ and $p : P\, i\, (\pi_0(kx))\, j$, $\rho_1(fx)p \sim_j \rho_1(fx')p$.

6 Conclusions, Related Work, and Future Work

In this paper, we have extended the fibrational approach to induction and coinduction pioneered by Hermida and Jacobs, and further developed by the current authors, in three key directions: We gave sound coinduction rules for all functors having final coalgebras provided the fibration interpreting them is a QCE, and we gave similarly sound generic induction and coinduction rules for all functors over slice categories having initial algebras and final coalgebras.

The work of Hermida and Jacobs is most closely related to ours, but there is, of course, a large body of work on induction and coinduction. in a broader setting. In dependent type theory, for example, data types are usually presented with elimination rules that are exactly induction rules. Along these lines, [13] has heavily influenced the development of induction in Coq. Another important strand of related work concerns inductive families and their induction rules [2]. On the coinductive side, papers such as [1, 15, 16] have had immense impact in bringing bisimulation into the mainstream of theoretical computer science.

There are several evident directions for future work. The most immediate is showing that change of base along a fibration preserves QCEs, just as it does CCUs; this would yield a compact derivation of the results in Section 5 analogous to that in Section 4. We also expect to exploit the predictive power of our theory to provide induction and coinduction rules for advanced data types — such as inductive recursive types — whose rules are not discernible by sheer intuition. In such circumstances our generic fibrational approach should provide rules whose use is justified by their soundness proofs. Finally, we would like to see our induction and coinduction rules for advanced data types incorporated into implementations such as Agda and Coq.

References

1. P. Aczel and P. Mendler. A Final Coalgebra Theorem. *Proceedings, Category Theory and Computer Science*, pp. 357–365, 1989.
2. P. Dybjer. Inductive Families. *Formal Aspects of Computing* 6(4), pp. 440–465, 1994.
3. N. Ghani, P. Johann, and C. Fumex. Fibrational Induction Rules for Initial Algebras. *Proceedings, Computer Science Logic*, pp. 336–350, 2010.
4. N. Ghani, P. Johann, and C. Fumex. Generic Fibrational Induction. Submitted, 2011.
5. C. Hermida. Some properties of Fib as a Fibred 2-Category. *Journal of Pure and Applied Algebra* 134(1), pp. 83–109, 1993.
6. C. Hermida. Fibrations, Logical Predicates and Related Topics. Dissertation, University of Edinburgh, 1993.
7. C. Hermida and B. Jacobs. Structural Induction and Coinduction in a Fibrational Setting. *Information and Computation* 145, pp. 107–152, 1998.
8. P. G. Hancock and P. Hyvernat. Programming Interfaces and Basic Topology. *Annals of Pure and Applied Logic* 137(1-3), pp. 189–239, 2006.
9. B. Jacobs. *Categorical Logic and Type Theory.* Studies in Logic and the Foundations of Mathematics, Volume 141, 1999.
10. B. Jacobs. Comprehension Categories and the Semantics of Type Dependency. *Theoretical Computer Science* 107, pp. 169–207, 1993.
11. B. Jacobs. Quotients in Simple Type Theory. Mathematics Institute, 1994.
12. P. Morris and T. Altenkirch. Indexed Containers. *Proceedings, Logic in Computer Science*, pp. 277–285, 2009.
13. Frank Pfenning and C. Paulin-Mohring. Inductively Defined Types in the Calculus of Constructions. *Proceedings, Mathematical Foundations of Programming Semantics*, pp. 209–228, 1989.
14. D. Pavlovič. *Predicates and Fibrations.* Dissertation, University of Utrecht, 1990.
15. J. Rutten. Universal Coalgebra: A Theory of Systems. *Theoretial Computer Science* 249(1), pp. 3–80, 2000.
16. D. Turi and J. Rutten. On the Foundations of Final Coalgebra Semantics. *Mathematical Structures in Computer Science* 8(5), pp. 481–540, 1998.