Decidability and Expressiveness of Recursive Weighted Logic

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Abstract. Labelled weighted transition systems (LWSs) are transition systems labelled with actions and real numbers. The numbers represent the costs of the corresponding actions in terms of resources. Recursive Weighted Logic (RWL) is a multimodal logic that expresses qualitative and quantitative properties of LWSs. It is endowed with simultaneous recursive equations, which specify the weakest properties satisfied by the recursive variables. We demonstrate that RWL is sufficiently expressive to characterize weighted-bisimilarity of LWSs. In addition, we prove that the logic is decidable, i.e., the satisfiability problem for RWL can be algorithmically solved.

Keywords: labelled weighted transition system, maximal fixed point, Hennessy-Milner property, satisfiability

1 Introduction

The industrial practice has revealed lately the importance of model-driven and component-based development (MDD), in particular within the area of embedded systems. A key challenge is to handle the growing complexity of systems, while meeting requirements on correctness, predictability, performance and also resource-cost constraints. In this respect MDD is seen as a valuable approach, as it allows early design-space exploration and verification and may be used as the basis for systematic and unambiguous testing of a final product. However, for embedded systems, verification should not only address functional properties but also properties related to resource constraints. Within the area of model checking a number of state-machine based modelling formalisms have emerged, which allow for such quantitative aspects to be expressed, especially time-constraints. In particular the formalisms of timed automata [AD90], and the extensions to weighted timed automata [BFH$^+$01,ATP01] allow for such constraints to be modelled and efficiently analyzed.

Motivated by the needs from embedded systems, we consider Recursive Weighted Logic (RWL), which is an extension of *weighted modal logic* [LM13] with maximal fixed point (and without negation), for labelled weighted transition systems (LWS). It allows us to specify and reason about not only the qualitative behaviour of a system but also its quantitative consumption of resources, and to encode recursive properties. Our notion of weighted transition systems is more than a simple instance of *weighted automata* [DKV09], since we also study infinite and infinitely branching systems.

RWL is a multimodal logic defined for a semantics based on LWSs. It is endowed with modal operators that predicate about both the action and the values of transition labels. While in an LWS we can have real-valued labels, the modalities only encodes rational values. Often we need to characterize a transition using an infinite convergent sequences of rationals that approximate the real resource. The logic is also endowed with maximal fixed points defined by simultaneous recursive equations [Lar90,CKS92,CS93]. They specify the weakest properties satisfied by the recursive variables.

In the non-quantitative case, the modal μ-calculus [Koz82] allows for encoding both LTL and CTL. Moreover, the modal μ-calculus is obtained by extending a simple modal logic – the Hennessy Milner Logic (HML) [HM80] having a modality for each action of the underlying transition system – with the ability to define properties recursively. In particular, it was shown that HML is *adequate* in the sense that it completely characterizes bisimilarity between image-finite labelled transition systems (LTS), i.e. two LTSs are bisimilar if and only if they satisfy the same HML properties [Sti99,SR11].

As a first result, we demonstrate that RWL is adequate with respect to weighted bisimilarity between labelled weighted transition systems, i.e. RWL is sufficiently expressive to characterize weighted-bisimilarity of LWSs.

Secondly, we prove decidability of satisfiablity for RWL. Concretely, we present a model construction algorithm, which constructs an LWS for a given RWL formula (provided that the formula is not a contradiction, i.e., without any model).

To encode various resource-constrains in RWL, we use resource-variables, similar to the clock-variables used in timed logics [ACD93,HNSY92,AILS07]. These variables can be reset, meaning that we can consider, in various states, interpretations that will map certain variable to zero. This is useful in encoding various interesting scenarios. Nevertheless, in order to prove the decidability of our logic and to be able to have the finite model property, we restrict our attention to only one variable for each type of resources. This bounds the expressiveness of our logic while it guarantees its decidability.

The remainder of this paper is organized as follows: the next section is dedicated to the presentation of the notion of labelled weighted transition system; in Section 3, we introduce RWL with its syntax and semantics; Section 4 is dedicated to the Hennessy-Milner property of RWL; in Section 5 we prove the decidability of the satisfiability problem for RWL and we propose an algorithm to solve it. We also present a conclusive section where we summarize the results and describe future research directions.

2 Labelled Weighted Transition Systems

A *labelled weighted transition system* (LWS) is a transition system that has the transitions labelled both with real numbers and actions - as represented in Figure 1. The numbers are interpreted as the costs of the corresponding actions in terms of resources, e.g., energy consumption/production. Our intention is to remain as general as possible and for this reason we impose no restriction on the labels: they can be any real number, possibly negative. If the transition has a positive label, the system gains resources; negative labels encode consumption of resources.

Definition 1 (Labelled Weighted Transition System). *A labelled weighted transition system is a tuple*

$$\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$$

where M is a non-empty set of states*, Σ a non-empty set of* actions*, $\mathcal{K} = \{x_1, \ldots, x_k\}$ is the finite set of (k types of)* resource-variables *and $\theta : M \times (\Sigma \times \mathbb{R}^k) \to 2^M$ is the* labelled transition function*.*

For simplicity, hereafter we use a vector of real numbers instead of the function from the set of the resources $\mathcal{K}$ to real numbers, i.e., for $f : \mathcal{K} \to \mathbb{R}$ defined as $f(e_i) = r_i$ for all $i = 1, \ldots, k$, we write $\overline{u} = (r_1, \ldots, r_k) \in \mathbb{R}^k$ instead. On the other hand, for a vector of real numbers $\overline{u} \in \mathbb{R}^k$, $\overline{u}(e_i)$ denotes the i-th number of the vector $\overline{u}$, which represent the cost of the resource e_i during the transition.

Instead of $m' \in \theta(m, a, \overline{u})$ we write $m \xrightarrow{\overline{u}}_a m'$. For $r \in \mathbb{R}$ we write $r = \overline{u}(x_i)$ to denote that r is the i-th number of the vector $\overline{u}$.

An LWS is said to be *image-finite* if for each state and each action a with weight $\overline{u}$, there are finitely many outgoing a-transitions with weight $\overline{u}$.

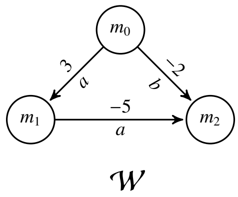

Fig. 1. labelled Weighted Transition System

Example 1. Figure 1 represent the LWS $\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$, where $M = \{m_0, m_1, m_2\}$, $\Sigma = \{a, b\}$, $\mathcal{K} = \{x\}$ and θ defined as follows: $m_0 \xrightarrow{3}_a m_1$, $m_0 \xrightarrow{-2}_b m_2$ and $m_1 \xrightarrow{-5}_a m_2$. $\mathcal{W}$ has three states m_0, m_1, m_2, one kind of resource x and two actions a, b. The state m_0 has two transitions - one a-transition which costs "3" units of x to m_1 and one b-transition which costs "−2" units of x to m_2. At m_0 the variable valuation l assigns "1" to x, which is the initial amount of the resource x at the state m_0. If the system does an a-transition from m_0 to m_1, the amount of the resource x increases with "3" units and becomes "4", which is the sum of the initial amount "1" and the value of the transition "3" - that the system gains by doing the a-transition. ∎

The concept of *weighted bisimulation* is a relation between the states of a given LWS that equates states with identical (action- and weighted-) behaviors.

Definition 2 (Weighted Bisimulation). *Given an LWS $\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$, a* weighted bisimulation *is an equivalence relation $R \subseteq M \times M$ such that whenever $(m, m') \in R$,*

- *if* $m \xrightarrow{\overline{u}}_a m_1$*, then there exists* $m_1' \in M$ *s.t.* $m' \xrightarrow{\overline{u}}_a m_1'$ *and* $(m_1, m_1') \in R$*;*
- *if* $m' \xrightarrow{\overline{u}}_a m_1'$*, then there exists* $m_1 \in M$ *s.t.* $m \xrightarrow{\overline{u}}_a m_1$ *and* $(m_1, m_1') \in R$*.*

If there exists a weighted bisimulation relation R such that $(m, m') \in R$*, we say that m and* m' *are* bisimilar*, denoted by* $m \sim m'$*.*

As for the other types of bisimulation, the previous definition can be extended to define the weighted bisimulation between distinct LWSs by considering bisimulation relations on their disjoint union. *Weighted bisimilarity* is the largest weighted bisimulation relation; if $\mathcal{W}_i = (M_i, \Sigma_i, \mathcal{K}_i, \theta_i)$, $m_i \in M_i$ for $i = 1, 2$ and m_1 and m_2 are bisimilar, we write $(m_1, \mathcal{W}_1) \sim (m_2, \mathcal{W}_2)$. Example 2 shows the role of the weighted bisimilarity.

Example 2. In Figure 2, $\mathcal{W}_1 = (M_1, \Sigma_1, \mathcal{K}_1, \theta_1)$ is an LWS with five states, where $M_1 = \{m_0, m_1, m_2, m_3, m_4\}$, $\Sigma_1 = \{a, b, c, d\}$, $\mathcal{K}_1 = \{x\}$ and θ_1 is defined as: $m_0 \xrightarrow{3}_a m_1$, $m_0 \xrightarrow{-2}_b m_2$, $m_1 \xrightarrow{0}_d m_2$, $m_1 \xrightarrow{3}_c m_3$, $m_2 \xrightarrow{0}_c m_1$ and $m_2 \xrightarrow{3}_c m_4$.
It is easy to see that $m_3 \sim m_4$ because neither of them can perform any transition. Besides, $m_1 \sim m_2$ because both of them can do a c-transition with cost 3 to some states which are bisimular (m_3 and m_4), and a d-action transition with cost 0 to each other. m_0 is not bisimular to any states in $\mathcal{W}_1$.
$\mathcal{W}_2 = (M_2, \Sigma_2, \mathcal{K}_2, \theta_2)$ is an LWS with three states, where $M_2 = \{m_0', m_1', m_2'\}$, $\Sigma_2 = \Sigma_1$, $\mathcal{K}_2 = \mathcal{K}_1$ and θ_2 is defined as: $m_0' \xrightarrow{3}_a m_1'$, $m_0' \xrightarrow{-2}_b m_1'$, $m_1' \xrightarrow{0}_d m_1'$ and $m_1' \xrightarrow{3}_c m_2'$.
We can see that: $(m_0, \mathcal{W}_1) \sim (m_0', \mathcal{W}_2)$, $(m_1, \mathcal{W}_1) \sim (m_1', \mathcal{W}_2)$, $(m_2, \mathcal{W}_1) \sim (m_1', \mathcal{W}_2)$, $(m_3, \mathcal{W}_1) \sim (m_2', \mathcal{W}_2)$, $(m_4, \mathcal{W}_1) \sim (m_2', \mathcal{W}_2)$.
Notice that $(m_0'', \mathcal{W}_3) \nsim (m_0', \mathcal{W}_2)$, because $(m_1'', \mathcal{W}_3) \nsim (m_1', \mathcal{W}_2)$. Besides, $m_1'' \nsim m_2''$, because m_1'' can do a d-action with weight -1 while m_2'' cannot and m_2'' can do a d-action with weight 1 while m_1'' cannot. ∎

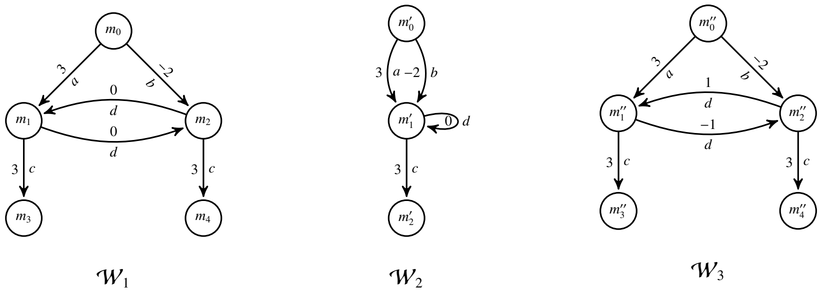

Fig. 2. Weighted Bisimulation

3 Recursive Weighted Logic

In this section we introduce a multimodal logic that encodes properties of LWSs called Recursive Weighted Logic (RWL). Our logic is endowed, in addition to the classic boolean operators (except negation), with a class of modalities of arity 0 called *state modalities* of type $x \bowtie r$ for $\bowtie \in \{\leq, \geq, <, >\}$, $r \in \mathbb{Q}$ and $x \in \mathcal{K}$, which predicates about the value of the resource x available at the current state; a class of modalities of arity 1, named *transition modalities*, of type $[x \bowtie r]_a$ or $\langle x \bowtie r\rangle_a$, for $\bowtie \in \{\leq, \geq, <, >\}$, $r \in \mathbb{Q}$, $a \in \Sigma$ and $x \in \mathcal{K}$, which approximates the transition labels; a class of modalities of arity 1, named *reset modalities*, of type $x \ \underline{\text{in}}$, which are inspired by timed logics [ACD93,HNSY92,LLW95] and refer to the fact that the resource x is interpreted to zero at the current state; and a class of recursive (formula) variables, $X \in \mathcal{X}$.

Hereafter, we fix a set Σ of actions and a set of $\mathcal{K}$ of resource variables and for simplicity, we omit them in the description of LWSs and RWL.

Firstly we define the basic formulas of RWL and their semantics. Based on them, we will eventually introduce the recursive definitions - the maximal equation blocks - which extend the semantics of the basic formulas.

Definition 3 (Syntax of Basic Formulas). *For arbitrary $r \in \mathbb{Q}$, $a \in \Sigma$, $x \in \mathcal{K}$ and $\bowtie \in \{\leq, \geq, <, >\}$, let*

$$\mathcal{L}: \quad \phi := \top \mid \bot \mid x \bowtie r \mid \phi \wedge \phi \mid \phi \vee \phi \mid [x \bowtie r]_a \phi \mid \langle x \bowtie r\rangle_a \phi \mid x \ \underline{in} \ \phi \mid X\ .$$

Before looking at the semantics for the basic formulas, we define the notion of *variable valuation* and *extended states*.

Definition 4 (variable valuation). *A variable valuation is a function $l : \mathcal{K} \to \mathbb{R}$ that assigns a real numbers to all the resource variables in $\mathcal{K}$.*

A variable valuation assigns positive or negative values to resource-variables. The label is interpreted as the amount of resources available or required (depending of whether the number is positive or negative) in a given state of the system. We denote by L the class of variable valuations. If l is a resource valuation and $x \in \mathcal{K}$, $s \in \mathbb{R}$ we denote by $l[x \mapsto s]$ the resource valuation that associates the same values as l to all variables except x, to which it associates the value s, i.e., for any $y \in \mathcal{V}$,

$$l[x \mapsto s](y) = \begin{cases} s, & y = x \\ l(y), & \text{otherwise} \end{cases}$$

A pair (m, l) is called *extended state* of a given LWS $\mathcal{W} = (M, \mathcal{K}, \theta)$, where $m \in M$ and $l \in L$. Transitions between extended states are defined by:

$$(m, l) \to_a (m', l') \text{ iff } m \xrightarrow{\overline{u}}_a m' \text{ and } l' = l + \overline{u}.$$

Given an LWS $\mathcal{W} = (M, \mathcal{K}, \theta)$ and a class of variable valuation L, the *LWS-semantics* of RWL basic formulas is defined by the *satisfiability relation*, over an extended state (m, l) and an environment ρ which maps each recursive formula variables to subsets of $M \times L$, inductively as follows.

$\mathcal{W}, (m, l), \rho \models \top$ – always,

$\mathcal{W}, (m, l), \rho \models \bot$ – never,

$\mathcal{W}, (m, l), \rho \models x \bowtie r$ iff $l(x) \bowtie r$,

$\mathcal{W}, (m, l), \rho \models \phi \wedge \psi$ iff $\mathcal{W}, (m, l), \rho \models \phi$ and $\mathcal{W}, (m, l), \rho \models \psi$,

$\mathcal{W}, (m, l), \rho \models \phi \vee \psi$ iff $\mathcal{W}, (m, l), \rho \models \phi$ or $\mathcal{W}, (m, l), \rho \models \psi$,

$\mathcal{W}, (m, l), \rho \models [x \bowtie r]_a \phi$ iff for arbitrary $(m', l') \in M \times L$ such that $(m, l) \rightarrow_a (m', l')$ and $l' - l \bowtie r$, we have $\mathcal{W}, (m', l'), \rho \models \phi$,

$\mathcal{W}, (m, l), \rho \models \langle x \bowtie r \rangle_a \phi$ iff exists $(m', l') \in M \times L$ such that $(m, l) \rightarrow_a (m', l')$, $l' - l \bowtie r$ and $\mathcal{W}, (m', l'), \rho \models \phi$,

$\mathcal{W}, (m, l), \rho \models x \underline{\text{in}}\, \phi$ iff $\mathcal{W}, (m, l[x \mapsto 0]), \rho \models \phi$,

$\mathcal{W}, (m, l), \rho \models X$ iff $m \in \rho(X)$.

Definition 5 (Maximal Equation Blocks). *Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of recursive variables. A maximal equation block B is a list of (mutually recursive) equations:*

$$X_1 = \phi_1$$
$$\vdots$$
$$X_n = \phi_n$$

in which X_i are pairwise-distinct over $\mathcal{X}$ and ϕ_i are basic formulas over $\mathcal{X}$, for all $i = 1, \ldots, n$.

Each maximal equation block B defines an *environment* for the recursive formula variables $X_1, \ldots, X_n$, which is the weakest property that the variables satisfy. We say that an arbitrary formula ϕ is *closed with respect to a maximal equation block B* if all the recursive formula variables appearing in ϕ are defined in B by some of its equations. If all the formulas ϕ_i that appear in the right hand side of some equation in B are closed with respect to B, we say that B is *closed*.

Given an environment ρ and $\overline{\Upsilon} = \langle \Upsilon_1, \ldots, \Upsilon_n \rangle \in (2^{M \times L})^n$, let

$$\rho_{\overline{\Upsilon}} = \rho[X_1 \mapsto \Upsilon_1, \ldots, X_n \mapsto \Upsilon_n]$$

be the environment obtained from ρ by updating the binding of X_i to Υ_i.

Given a maximal equation block B and an environment ρ, consider the function

$$f_B^\rho : (2^{M \times L})^n \longrightarrow (2^{M \times L})^n$$

defined as follows:

$$f_B^\rho(\overline{\Upsilon}) = \langle [\![\phi_1]\!]\rho_{\overline{\Upsilon}}, \ldots, [\![\phi_n]\!]\rho_{\overline{\Upsilon}} \rangle,$$

where $[\![\phi]\!]\rho = \{(m, l) \in M \times L \mid \mathcal{W}, (m, l), \rho \models \phi\}$.

Observe that $(2^{M \times L})^n$ forms a complete lattice with the ordering, join and meet operations defined as the point-wise extensions of the set-theoretic inclusion, union and intersection, respectively. Moreover, for any maximal equation block B and environment ρ, f_B^ρ is monotonic with respect to the order of the lattice and therefore, according to the Tarski fixed point theorem [Tar55], it has a greatest fixed point that we denote by $\nu\overline{X}.f_B^\rho$. This fixed point can be characterized as follows:

$$\nu\overline{X}.f_B^\rho = \bigcup\{\overline{\Upsilon} \mid \overline{\Upsilon} \subseteq f_B^\rho(\overline{\Upsilon})\}.$$

When the transition system is finite-state f_B^ρ is continuous, and the fixed points also have an iterative characterization given as follows. Let

$$f_0 = \langle M \times L, \dots, M \times L \rangle,$$
$$f_{i+1} = f_B^\rho(f_i).$$

Then, $\nu\overline{X}.f_B^\rho = \bigcap_{i=0}^{\infty} f_i$.

Consequently, a maximal equation block defines an environment that satisfies all its equations, i.e., $[\![B]\!]\rho = \nu\overline{X}.f_B^\rho$.

When B is closed, i.e. there is no free recursive formula variable in B, it is not difficult to see that for any ρ and ρ', $[\![B]\!]\rho = [\![B]\!]\rho'$. So, we just take $\rho = 0$ and write $[\![B]\!]$ instead of $[\![B]\!]0$. In the rest of the paper we will only discuss this kind of equation blocks. (For those that are not closed, we only need to have the initial environment which maps the free variables to subsets of the state set.)

Now we are ready to define the general semantics of RWL: for an arbitrary LWS $\mathcal{W} = (M, \theta)$ with $m \in M$, an arbitrary variable valuation $l \in L$ and arbitrary RWL-formula ϕ closed w.r.t. a maximal equation block B,

$$\mathcal{W}, (m, l) \models_B \phi \text{ iff } \mathcal{W}, (m, l), [\![B]\!] \models \phi.$$

The symbol $\models_B$ is interpreted as satisfiability for the block B. Whenever it is not the case that $\mathcal{W}, (m, l) \models_B \phi$, we write $\mathcal{W}, (m, l) \not\models_B \phi$. We say that a formula ϕ is *B-satisfiable* if there exists at least one LWS that satisfies it for the block B in one of its states under at least one variable valuation; ϕ is a *B-validity* if it is satisfied in all states of any LWS under any variable valuation - in this case we write $\models_B \phi$.

To exemplify the expressiveness of RWL, we propose the following example of a bank transaction system with recursively-defined properties.

Example 3. Consider a models of a bank transaction scenarios that involves two basic actions w (withdraw) and d (deposit). The specifications of the system are as follows:
1. The system is in a safe range, i.e., the amount of resource x is always above 0;
2. The system is never deadlocked, i.e. it should always do an w-action or a d-action;
3. There is at least one w-action in three steps;
4. Every w-action consumes at least 3 units of resource x and every d-action produces at most 2 units of resource x.

In our logic the above mentioned requirements can be encoded as follows, where $[a]\phi = \bigwedge_{x \in \mathcal{K}}([x \geq 0]_a\phi \wedge [x \leq 0]_a\phi)$ and $\langle a\rangle\phi = \bigvee_{x \in \mathcal{K}}(\langle x \geq 0\rangle_a\phi \vee \langle x \leq 0\rangle_a\phi)$, for $a = w, d$:

$$X = (x \geq 0) \wedge [w]X \wedge [d]X,$$
$$Y = (\langle w\rangle\top \vee \langle d\rangle\top) \wedge [w]Y \wedge [d]Y,$$
$$Z = [d][d][d]\bot \wedge [w]Z \wedge [d]Z,$$
$$W = [x > -3]_w\bot \wedge [w]W \wedge [d]W,$$
$$D = [x \leq 0]_d\bot \wedge [x > 2]_d\bot \wedge [w]D \wedge [d]D.$$

4 Hennessy-Milner Property

The standard theory of fixed points tells us that if f is a monotone function on a lattice, we can construct the greatest fixed point of f by repeatedly applying f on the largest element to form a decreasing chain whose limit is the greatest fixed point [Tar55]. The stages of this iteration $\nu^\alpha\overline{X}.f$ can be be defined as follows:

$$\nu^0\overline{X}.f = \top$$
$$\nu^{\beta+1}\overline{X}.f = f\{\nu^\beta\overline{X}.f/\overline{X}\}$$
$$\nu^\lambda\overline{X}.f = \bigwedge_{\beta<\lambda}\nu^\beta\overline{X}.f$$

where $\bigwedge$ is the countable conjunction.

We use this characterization to prove that RWL satisfies the Hennessy-Milner property for LWSs. To do this we firstly define a non-recursive version of RWL in which we allow countable conjunctions.

Definition 6 (Weighted Modal Logic with countable conjunction). *For arbitrary $r \in \mathbb{Q}$, $a \in \Sigma$, $x \in \mathcal{K}$, $\bowtie \in \{\leq, \geq, <, >\}$ and I a finite or countable set of indexes, let $\mathcal{L}^c$ be the set of the formulas inductively defined as follows:*

$$\phi^c := \top \mid \bot \mid x \bowtie r \mid \bigwedge_{i\in I}\phi_i^c \mid \phi^c \vee \phi^c \mid [x \bowtie r]_a\phi^c \mid \langle x \bowtie r\rangle_a\phi^c \mid x\ \underline{in}\ \phi^c\ .$$

Excepting the infinite conjunction, the semantics of the above logic is defined similarly to that of RWL with no environment. In addition,

$\mathcal{W}, (m,l) \models \bigwedge_{i\in I}\phi_i^c$ iff for any $i \in I$, $\mathcal{W}, (m,l) \models \phi_i^c$.

We first demonstrate that $\mathcal{L}^c$ satisfies the Hennessy-Milner property.

Lemma 1. *Let $\mathcal{W} = (M, \mathcal{K}, \theta)$ be an image-finite labelled weighted transition system. Then, for any $m, m' \in M$:*

$m \sim m'$ *iff* $\forall\phi^c \in \mathcal{L}^c$ *and* $l \in L$, *we have* $\mathcal{W}, (m,l) \models \phi^c \Leftrightarrow \mathcal{W}, (m',l) \models \phi^c$.

Proof. "$\Longrightarrow$": Induction on ϕ^c. The cases $\top$, $\bot$ and $\phi^c \vee \psi^c$ are easy.

- **Case** $x \bowtie r$**:** $\mathcal{W}, (m,l) \models x \bowtie r$ implies $l(x) \bowtie r$, which implies $\mathcal{W}, (m',l) \models x \bowtie r$. Hence, $\mathcal{W}, (m,l) \models x \bowtie r$ implies $\mathcal{W}, (m',l) \models x \bowtie r$.
Similarly $\mathcal{W}, (m',l) \models x \bowtie r$ implies $\mathcal{W}, (m,l) \models x \bowtie r$.

- **Case** $\bigwedge_{i\in I}\phi_i^c$**:** $\mathcal{W}, (m,l) \models \bigwedge_{i\in I}\phi_i^c$ implies for any $i \in I$, $\mathcal{W}, (m,l) \models \phi_i^c$. By inductive hypothesis, for any $i \in I$, $\mathcal{W}, (m',l) \models \phi_i^c$, which implies $\mathcal{W}, (m',l) \models \bigwedge_{i\in I}\phi_i^c$. Hence, $\mathcal{W}, (m,l) \models \bigwedge_{i\in I}\phi_i^c$ implies $\mathcal{W}, (m',l) \models \bigwedge_{i\in I}\phi_i^c$.
Similarly $\mathcal{W}, (m',l) \models \bigwedge_{i\in I}\phi_i^c$ implies $\mathcal{W}, (m,l) \models \bigwedge_{i\in I}\phi_i^c$.

- **Case** $[x \bowtie r]_a\phi^c$**:** $\mathcal{W}, (m,l) \models [x \bowtie r]_a\phi^c$ implies for any $(m_1, l_1) \in M \times L$ s.t. $(m,l) \rightarrow_a (m_1, l_1)$ and $l_1 - l \bowtie r$, $\mathcal{W}, (m_1, l_1) \models \phi^c$. $(m,l) \rightarrow_a (m_1, l_1)$ implies $m \xrightarrow{\overline{u}}_a m_1$ and $l_1 = l + \overline{u}$. Since $m \sim m'$, for any $m_1' \in M$ s.t. $m' \xrightarrow{\overline{u}}_a m_1'$, there exists $m_1 \in M$ s.t. $m \xrightarrow{\overline{u}}_a m_1$ and $m_1 \sim m_1'$. By inductive hypothesis, $\mathcal{W}, (m_1', l_1) \models \phi^c$. So for any $(m_1', l_1) \in M$ s.t. $(m', l) \rightarrow_a (m_1', l_1)$ and $l_1 - l \bowtie r$, $\mathcal{W}, (m_1', l_1) \models \phi^c$. Then $\mathcal{W}, (m', l) \models [x \bowtie r]_a\phi^c$. Hence, $\mathcal{W}, (m,l) \models [x \bowtie r]_a\phi^c$ implies $\mathcal{W}, (m',l) \models [x \bowtie r]_a\phi^c$.
Similarly $\mathcal{W}, (m',l) \models [x \bowtie r]_a\phi^c$ implies $\mathcal{W}, (m,l) \models [x \bowtie r]_a\phi^c$.

- **Case** $\langle x \bowtie r\rangle_a\phi^c$**:** $\mathcal{W}, (m,l) \models \langle x \bowtie r\rangle_a\phi^c$ implies there exists $(m_1, l_1) \in M \times L$ s.t. $(m,l) \rightarrow_a (m_1, l_1)$, $l_1 - l \bowtie r$ and $\mathcal{W}, (m_1, l_1) \models \phi^c$. $(m,l) \rightarrow_a (m_1, l_1)$ implies $m \xrightarrow{\overline{u}}_a m_1$ and $l_1 = l + \overline{u}$. Since $m \sim m'$, there exists m_1' s.t. $m' \xrightarrow{\overline{u}}_a m_1'$ and $m_1 \sim m_1'$. By inductive

hypothesis, $\mathcal{W}, (m'_1, l_1) \models \phi^c$. So we have that there exists $(m'_1, l_1) \in M \times L$ s.t. $(m', l) \rightarrow_a (m'_1, l_1)$, $l_1 - l \bowtie r$ and $\mathcal{W}, (m'_1, l_1) \models \phi^c$, which implies $\mathcal{W}, (m', l) \models \langle x \bowtie r\rangle_a \phi^c$. Hence, $\mathcal{W}, (m, l) \models \langle x \bowtie r\rangle_a \phi^c$ implies $\mathcal{W}, (m', l) \models \langle x \bowtie r\rangle_a \phi^c$.
Similarly $\mathcal{W}, (m', l) \models \langle x \bowtie r\rangle_a \phi^c$ implies $\mathcal{W}, (m, l) \models \langle x \bowtie r\rangle_a \phi^c$.

- **Case** $x\ \underline{\text{in}}\ \phi^c$**:** $\mathcal{W}, (m, l) \models x\ \underline{\text{in}}\ \phi^c$ implies $\mathcal{W}, (m, l[x \mapsto 0]) \models \phi^c$. By inductive hypothesis, $\mathcal{W}, (m', l[x \mapsto 0]) \models \phi^c$. Hence, $\mathcal{W}, (m, l) \models x\ \underline{\text{in}}\ \phi^c$ implies $\mathcal{W}, (m', l) \models x\ \underline{\text{in}}\ \phi^c$.
Similarly $\mathcal{W}, (m', l) \models x\ \underline{\text{in}}\ \phi^c$ implies $\mathcal{W}, (m, l) \models x\ \underline{\text{in}}\ \phi^c$.

"$\Longleftarrow$": Let $R = \{(m, m') \mid \forall \phi^c \in \mathcal{L}^c, \mathcal{W}, (m, l) \models \phi \Leftrightarrow \mathcal{W}, (m', l) \models \phi\}$. We prove that R is a weighted bisimulation relation.
* If $m \xrightarrow{\overline{u}}_a m_1$:
If there exists no $m'_1 \in M$ s.t $m' \xrightarrow{\overline{u}}_a m'_1$, $\mathcal{W}, (m', l) \models [x \bowtie r]_a \bot$ for any $x \in \mathcal{K}$ and $r \in \mathbb{Q}$ s.t. $\overline{u}(x) \bowtie r$. Then $\mathcal{W}, (m, l) \models [x \bowtie r]_a \bot$ since $(m, m') \in R$, which contradicts the premise.

Suppose $F = \{m'_i \mid m' \xrightarrow{\overline{u}}_a m'_i\}$ and $(m_1, m'_i) \notin R$ for any i, i.e. for any i, there exists l_i and ϕ_i s.t. $\mathcal{W}, (m_1, l_i) \models \phi_i$ and $\mathcal{W}, (m'_i, l_i) \not\models \phi_i$. For every $x \in \mathcal{K}(\phi^i)$, introduce x_i. Let $\phi'_i = \phi_i\{x_i/x\}$ for every ϕ_i. Let $l'(x_i) = l_i(x)$ for any i and x_i. We have: $\mathcal{W}, (m_1, l') \models \bigwedge_i \phi_i$ and $\mathcal{W}, (m'_i, l') \not\models \phi'_i$ for all i. Then $\mathcal{W}, (m, l) \models [a] \bigwedge_i \phi'_i$ and $\mathcal{W}, (m', l) \not\models [a] \bigwedge \phi'_i$ - contradiction. Hence, there exists $m'_1 \in M$ s.t. $m' \xrightarrow{\overline{u}}_a m'_1$ and $m_1 \sim m'_1$.
* If $m' \xrightarrow{\overline{u}}_a m'_1$: similar as above. ∎

We previously noticed that every maximal fixed point in RWL can be translated into a formula in weighted modal logic with countable conjunction. Hence, the previous lemma ensures that RWL enjoys the Hennessy-Milner property as well.

Theorem 1 (Hennessy-Milner Theorem). *Let $\mathcal{W} = (M, \mathcal{K}, \theta)$ be an image-finite labelled weighted transition system. Then, for any $m, m' \in M$:*

$m \sim m'$ *iff*

for any equation block B, any ϕ closed w.r.t. B and any $l \in L$,

$\mathcal{W}, (m, l) \models_B \phi \Leftrightarrow \mathcal{W}, (m', l) \models_B \phi.$

Notice that in Example 2 we have already seen that$(m''_1, \mathcal{W}_3) \nsim (m'_1, \mathcal{W}_2)$. There exists, however, a RWL formula that distinguishes them. This is $[x \geq 0]_d \bot$.

5 Satisfiability of Recursive Weighted Logic

In this section we prove that it is decidable whether a given formula ϕ which is closed w.r.t. a maximal equation block B of RWL is satisfiable. We also present a decision procedure for the satisfiability problem of RWL. The results rely on a syntactic characterization of satisfiability that involves a notion of *mutually-consistent sets*.

Before going through the formal definitions, we consider the property in Example 3. Since we require that after any transition X, Y, Z, W, D still hold, X, Y, Z, W, D will hold in all the states. Let's start from the state m_0, where X, Y, Z, W, D hold and the label of x is 0. m_0 needs to do a d-action with weight at most 2 to a state m_1, since it cannot do any w-action with weight at most -3 to a state where $x \geq 0$ still holds. For m_1, the label of x can be $1, 2$ or some value in the interval $(1, 2)$. If m_0 does a d-transition with weight less than 1, after the next step there will be no next movement (no d-transition because of the constraint stated in Z and no w-transition because of the constraint stated in W). And we can also find out the transitions of m_1 and so on so forth. In this way, we can construct a finite model for the required properties. This is only a very informal discussion. We will see how to construct the model in the following.

Consider an arbitrary formula $\phi \in \mathcal{L}$ which is closed w.r.t. a maximal equation block B. In this context we define the following notions:
- For any $x \in \mathcal{K}$, let $R^B_\phi(x) \subseteq \mathbb{Q}$ be the set of all $r \in \mathbb{Q}$ such that r is in the label of some state or transition modality of type $x \bowtie r$, $\langle x \bowtie r\rangle_a$ or $[x \bowtie r]_a$ that appears in the syntax of ϕ or B. Let $Q^B_\phi(x)$ be the largest interval centred in zero that contains $R^B_\phi(x)$. If $R^B_\phi(x) = \emptyset$, then $Q^B_\phi(x) = \emptyset$.
- Let Σ^B_ϕ be the set of all actions $a \in \Sigma$ such that a appears in some transition modality of type $\langle x \bowtie r\rangle_a$ or $[x \bowtie r]_a$ in ϕ or B.
- We denoted by $G^B_\phi(x)$ the *granularity of* ϕ, defined as the least common denominator of the elements of $R^B_\phi(x)$.
- Let $I^B_\phi(x)$ be the set of all rationals of type $\frac{p}{G^B_\phi(x)}$ in $Q^B_\phi(x)$, for $p \in \mathbb{Z}$. Let

$$\Lambda^B_\phi(x) = \{\{q\} \mid q \in I^B_\phi(x)\} \cup \{(q, q + \tfrac{1}{G^B_\phi(x)}) \mid q, q + \tfrac{1}{G^B_\phi(x)} \in I^B_\phi(x)\} \cup$$
$$\{(-\infty, min I^B_\phi(x_i))\} \cup \{(max I^B_\phi(x_i), +\infty)\}.$$

- Let $\mathcal{R}^B_\phi = \{\overline{\delta} = (\delta_1, \ldots, \delta_k) \mid \delta_i \in \Lambda^B_\phi(x_i),\ \text{where } \mathcal{K} = \{x_1, .., x_k\}\}$. For $r \in \mathbb{R}$, we write $r \in \overline{\delta}(x_j)$ to denote that $r \in \delta_j$, for arbitrary $j \in \{1, \ldots, k\}$.
- The *modal depth* of ϕ, denoted by $md(\phi, B)$, is defined inductively by

$$md(\phi, B) = \begin{cases} 0, & \text{if } \phi = \top, \phi = \bot \text{ or } \phi = x \bowtie r \\ max\{md(\psi, B), md(\psi', B)\}, & \text{if } \phi = \psi \wedge \psi' \text{ or } \phi = \psi \vee \psi' \\ md(\psi, B) + 1, & \text{if } \phi = [x \bowtie r]_a \psi \text{ or } \phi = \langle x \bowtie r\rangle_a \psi \\ md(\psi, B), & \text{if } \phi = x\ \underline{\text{in}}\ \psi \\ md(B)(X), & \text{if } \phi = X \text{ and } X = \psi \in B \\ 0, & \text{if } \phi = X \text{ and } X \notin B \end{cases}$$
$$md(B) = \quad (md(\psi_1, B - \{X_1 = \psi_1\}), \ldots, md(\psi_n, B - \{X_n = \psi_n\}))$$

Observe that $R^B_\phi(x)$, Σ^B_ϕ, $I^B_\phi(x)$ and $\mathcal{R}^B_\phi$ are all finite (or empty). These sets will be used to construct the Fischer-Ladner closure of a given formula.

At this point we can start our model construction. We fix a formula $\phi_0 \in \mathcal{L}$ that is closed w.r.t. a given maximal equation block B and, supposing that the formula admits a model, we construct a model for it. Let

$$\mathcal{L}[\phi_0, B] = \{\phi \in \mathcal{L} \mid I^B_\phi(x) \subseteq I^B_{\phi_0}(x) \text{ for any } x \in \mathcal{K}, md(\phi, B) \leq md(\phi_0, B), \Sigma^B_\phi \subseteq \Sigma^B_{\phi_0}\}.$$

To construct the model we will use as states sets of tuples of type $(\phi, \overline{\delta}) \in \mathcal{L}[\phi_0, B] \times \mathcal{R}^B_{\phi_0}$, which are required to be maximal in a precise way. The intuition is that a state $\Gamma \subseteq \mathcal{L}[\phi_0, B] \times \mathcal{R}^B_{\phi_0}$ shall satisfy the formula ϕ in our model with the variable valuation l, whenever $(\phi, (\delta_1, .., \delta_k)) \in \Gamma$ and $l(x_j) \in \delta_j$, $j = 1, ..k$. Our construction is inspired from the region construction proposed in [LLW95] for timed automata, which adapts of the classical filtration-based model construction used in modal logics [HKT01,Wal00].

Let $\Omega[\phi_0, B] \subseteq \mathcal{L}[\phi_0, B] \times \mathcal{R}^B_{\phi_0}$. Since $\mathcal{L}[\phi_0, B]$ and $\mathcal{R}^B_{\phi_0}$ are finite, $\Omega[\phi_0, B]$ is finite.

Definition 7. *For any $\Gamma \subseteq \Omega[\phi_0, B]$, Γ is said to be* maximal *iff:*

1. For any $\overline{\delta} \in \mathcal{R}^B_{\phi_0}$, $(\top, \overline{\delta}) \in \Gamma$, $(\bot, \overline{\delta}) \notin \Gamma$;

2. $(x \bowtie r, \overline{\delta}) \in \Gamma$ iff for any $w \in \mathbb{R}$ s.t. $w \in \overline{\delta}(x)$, $w \bowtie r$;

3. $(\phi \wedge \psi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta}) \in \Gamma$ and $(\psi, \overline{\delta}) \in \Gamma$;
$(\phi \vee \psi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta}) \in \Gamma$ or $(\psi, \overline{\delta}) \in \Gamma$;

4. $(\langle x < r\rangle_a \phi, \overline{\delta}) \in \Gamma$ implies $(\langle x \leq r\rangle_a \phi, \overline{\delta}) \in \Gamma$;
$(\langle x > r\rangle_a \phi, \overline{\delta}) \in \Gamma$ implies $(\langle x \geq r\rangle_a \phi, \overline{\delta}) \in \Gamma$;

5. $(\langle x \leq r\rangle_a \phi, \overline{\delta}) \in \Gamma$ implies $(\langle x < r + s\rangle_a \phi, \overline{\delta}) \in \Gamma$;
$(\langle x \geq r\rangle_a \phi, \overline{\delta}) \in \Gamma$ implies $(\langle x > r - s\rangle_a \phi, \overline{\delta}) \in \Gamma$, for $s > 0$;

6. $(x \, \underline{in} \, \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta}[x \mapsto 0]) \in \Gamma$;

7. $(X, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta}) \in \Gamma$, for $X = \phi \in B$.

The maximal subsets of $\Omega[\phi_0, B]$ will be used as states in our model and for this reason we have to guarantee that their mutual relations allow us to do the construction. This is what the next lemma states.

Lemma 2. *For arbitrary $\Gamma, \Gamma' \in \Omega[\phi_0, B]$ and $r, s \in \mathbb{Q}$ with $s > 0$,*

1. If $[([x \leq r]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma']$ and there exist $w, w' \in \mathbb{R}$ s.t. $w \in \overline{\delta}(x)$, $w' \in \overline{\delta'}(x)$ and $w' - w \leq r$, then $[([x \leq r + s]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma']$;

2. If $[([x \geq r]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma']$ and there exist $w, w' \in \mathbb{R}$ s.t. $w \in \overline{\delta}(x)$, $w' \in \overline{\delta'}(x)$ and $w' - w \geq r$, then $[([x \geq r - s]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma']$;

3. If $r \leq \inf\{t \in \mathbb{Q} \mid ([x \leq t]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma$ and there exist $w, w' \in \mathbb{R}$ s.t. $w \in \overline{\delta}(x), w' \in \overline{\delta'}(x)$ and $w' - w \leq t\}$, then

$$([x \leq r]_a \phi, \overline{\delta}) \in \Gamma \text{ implies } (\phi, \overline{\delta''}) \in \Gamma',$$

for any $\overline{\delta''}$ s.t. there exist $w, w'' \in \mathbb{R}$ with $w \in \overline{\delta}(x)$, $w'' \in \overline{\delta''}(x)$ and $w'' - w \leq r$;

4. If $r \geq \sup\{t \in \mathbb{Q} \mid ([x \geq t]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma$ and there exist $w, w' \in \mathbb{R}$ s.t. $w \in \overline{\delta}(x), w' \in \overline{\delta'}(x)$ and $w' - w \geq t\}$, then

$$([x \geq r]_a \phi, \overline{\delta}) \in \Gamma \text{ implies } (\phi, \overline{\delta''}) \in \Gamma',$$

for any $\overline{\delta''}$ s.t. there exist $w, w'' \in \mathbb{R}$ with $w \in \overline{\delta}(x)$, $w'' \in \overline{\delta''}(x)$ and $w'' - w \leq r$.

Proof. 1. From Definition 7, we have that $([x \leq r + s]_a \phi, \overline{\delta}) \in \Gamma$ implies $([x \leq r]_a \phi, \overline{\delta}) \in \Gamma$. So $([x \leq r + s]_a \phi, \overline{\delta}) \in \Gamma$ implies $(\phi, \overline{\delta'}) \in \Gamma'$. Similarly for 2.

3. It is a direct consequence of case 1 when we consider the infimum. Similarly for 4. ∎

Notice that sup and inf above might be irrationals and cannot be used to index modalities. Nevertheless, they are limits of some monotone sequences of rationals.

The following definition establishes the framework on which we will define our model.

Definition 8. *Let $C \subseteq 2^{\Omega[\phi_0,B]}$. C is said to be mutually-consistent if whenever $\Gamma \in C$:*

$$[\forall \Gamma', \Gamma \xrightarrow{\overline{u}}_a \Gamma' \text{ and } \overline{u}(x) \bowtie r \Rightarrow (\phi, \overline{\delta}) \in \Gamma'] \text{ implies } ([x \bowtie r]_a \phi, \overline{\delta} - \overline{u}) \in \Gamma.$$

We say that Γ is *consistent* if it belongs to some mutually-consistent set.

Lemma 3. *Let $\phi \in \mathcal{L}$ be a formula closed w.r.t. a maximal equation block B. Then ϕ is satisfiable iff there exist $\Gamma \in \Omega[\phi_0, B]$ and $\overline{\delta} \in \mathcal{R}^B_\phi$ s.t. Γ is consistent and $(\phi, \overline{\delta}) \in \Gamma$.*

Proof. ($\Longrightarrow$): Suppose ϕ is satisfied in the LWS $\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$. We construct

$$C = \{\Gamma \in \Omega[\phi_0, B] \mid \exists m \in M \text{ and } l \in \overline{\delta}, \mathcal{W}, (m, l) \models_B \psi \text{ for all } (\psi, \overline{\delta}) \in \Gamma\}.$$

It is not difficult to verify that C is a mutually-consistent set.

($\Longleftarrow$): Let C be a mutually-consistent set. We construct an LWS $\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$, where: $M = C$, and the transition relation $\Gamma \xrightarrow{\overline{u}}_a \Gamma'$ is defined whenever

$$\overline{u}(x) = \sup\{r \in \mathbb{Q} \mid ([x \geq r]_a \phi, \overline{\delta}) \in \Gamma \text{ implies } (\phi, \overline{\delta'}) \in \Gamma', \text{ with } \overline{\delta'}(x) - \overline{\delta}(x) \geq r\}$$
$$= \inf\{r \in \mathbb{Q} \mid ([x \leq r]_a \phi, \overline{\delta}) \in \Gamma \text{ implies } (\phi, \overline{\delta'}) \in \Gamma', \text{ with } \overline{\delta'}(x) - \overline{\delta}(x) \leq r\} \in \mathbb{R}.$$

Let $\rho(X) = \{\Gamma \mid (X, \overline{\delta}) \in \Gamma\}$ for $X \in \mathcal{X}$. With this construction we can prove the following implication by a simple induction on the structure of ϕ, where $\Gamma \in M$ and $l \in \overline{\delta}$:

$$(\phi, \overline{\delta}) \in \Gamma \text{ implies } \mathcal{W}, \Gamma, l, \rho \models \phi.$$

We prove that ρ is a fixed point of B under the assumption that $X = \phi_X \in B$: $\Gamma \in \rho(X)$ implies $(X, \overline{\delta}) \in \Gamma$ by the construction of ρ, which implies $(\phi_X, \overline{\delta}) \in \Gamma$. Then, by the implication we just proved above, $\mathcal{W}, \Gamma, l, \rho \models \phi_X$.

Thus ρ is a fixed point of B. Since $[\![B]\!]$ is the maximal fixed point, $\rho \subseteq [\![B]\!]$. So for any $(\psi, \overline{\delta}) \in \Gamma \in C$, we have $\mathcal{W}, \Gamma, l, \rho \models \psi$ with $l \in \overline{\delta}$. Then $\mathcal{W}, \Gamma, l, [\![B]\!] \models \psi$ because $\rho \subseteq [\![B]\!]$.

Hence, $(\psi, \overline{\delta}) \in \Gamma \in C$ implies $\mathcal{W}, \Gamma, l \models_B \psi$ with $l \in \overline{\delta}$. ■

The above lemma allows us to conclude the finite model construction.

Theorem 2. *[Finite Model Property] For any satisfiable RWL formula ϕ closed w.r.t. a maximal equation block B, there exists a finite LWS $\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta)$ and a variable valuation l such that $\mathcal{W}, m, l \models_B \phi$ for some $m \in M$.*

Lemma 3 and Theorem 2 provide a decision procedure for the satisfiability problem of RWL. Given a RWL formula ϕ_0 closed w.r.t. a maximal equation block B, the algorithm constructs the model with $\Sigma = \Sigma^B_{\phi_0}$:

$$\mathcal{W} = (M, \Sigma, \mathcal{K}, \theta).$$

If ϕ_0 is satisfiable, then it is contained in some consistent set. Hence, ϕ_0 will be satisfied at some state m of $\mathcal{W}$. If ϕ_0 is not satisfiable, then the attempt to construct a model will fail; in this case the algorithm will halt and report the failure.

We start with a superset of the set of states of $\mathcal{W}$, then repeatedly delete states when we discover some inconsistency. This will give a sequence of approximations

$$\mathcal{W}_0 \supseteq \mathcal{W}_1 \supseteq \mathcal{W}_2 \supseteq \ldots$$

converging to $\mathcal{W}$.

The domains M_i, $i = 0, 1, 2, \ldots$, of these structures are defined below and they are s.t.

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots.$$

The transition relation for $\mathcal{W}_i$ are defined as follows: $\Gamma \xrightarrow{\overline{u}}_a \Gamma'$ whenever

$$\begin{aligned}\overline{u}(x) &= \sup\{r \in \mathbb{Q} \mid ([x \geq r]_a\phi, \overline{\delta}) \in \Gamma \text{ implies } (\phi, \overline{\delta'}) \in \Gamma',\ \text{with } \overline{\delta'}(x) - \overline{\delta}(x) \geq r\} \\ &= \inf\{r \in \mathbb{Q} \mid ([x \leq r]_a\phi, \overline{\delta}) \in \Gamma \text{ implies } (\phi, \overline{\delta'}) \in \Gamma',\ \text{with } \overline{\delta'}(x) - \overline{\delta}(x) \leq r\} \in \mathbb{R}.\end{aligned}$$

Here is the algorithm for constructing the domains M_i of $\mathcal{W}_i$.

Algorithm

Step 1: Construct $M_0 = \Omega[\phi_0, B]$.
Step 2: Repeat the following for $i = 0, 1, 2, \ldots$ until no more states are deleted. Find a formula $[x \bowtie r]_a\phi \in \mathcal{L}[\phi_0, B]$ and a state $\Gamma \in M_i$ violating the property

$$\begin{aligned}&[\forall \Gamma', \Gamma \xrightarrow{\overline{u}}_a \Gamma' \text{ and } \overline{u}(x) \bowtie r \Rightarrow (\phi, \overline{\delta}) \in \Gamma'] \\ &\qquad \text{implies } [([x \bowtie r]_a\phi, \overline{\delta'}) \in \Gamma \text{ and } \overline{\delta'} = \overline{\delta} - \overline{u}].\end{aligned}$$

Pick such an $[x \bowtie r]_a\phi$ and Γ. Delete Γ from M_i to get M_{i+1}. ∎

Step 2 can be justified intuitively as follows. To say that Γ violates the above mentioned condition, means that Γ requires an a-transition at cost $\overline{u}$ to some state that does not satisfy ϕ; however, the left-hand side of the condition above guarantees that all the outcomes of an a-transition at cost $\overline{u}$ satisfy ϕ. This demonstrates that Γ cannot be in M, since every state Γ in M satisfies ψ, whenever $(\psi, \overline{\delta}) \in \Gamma$.

The algorithm must terminate, since there are only finitely many states initially, and at least one state must be deleted during each iteration of step 2 in order to continue. Then ϕ is satisfiable if and only if, upon termination there exists $\Gamma \in M$ such that $(\phi, \overline{\delta}) \in \Gamma$. Obviously, M is a mutually-consistent set upon termination. The correctness of this algorithm follows from Lemma 3. The $\Leftarrow$ direction of the proof guarantees that all formulas in any $\Gamma \in M$ are satisfiable. The $\Rightarrow$ direction of the proof guarantees that all satisfiable Γ will not be deleted from M.

The finite model property also supported by the above algorithm demonstrates the decidability of the B-satisfiability problem for RWL.

Theorem 3 (Decidability of B-satisfiability). *For an arbitrary maximal equation block B, the B-satisfiability problem for RWL is decidable.*

6 Conclusion

In this paper we develop a recursive version of the weighted modal logic [LM13] that we call Recursive Weighted Logic (RWL). It uses a semantics based on labelled weighted transition systems (LWSs). This type of transition systems describes systems where the computations have some costs that must be paid in terms of the resources available in its states: positive transitions means that the system gains some resources during the transition, while negative ones represent resource consumption.

RWL encodes qualitative and quantitative properties of LWSs. With respect to the weighted logics studied before, RWL has recursive variables that allow us to encode circular properties. These features reflect concrete requirements from applications where liveness and safeness properties including cost information are essential.

We first prove that RWL enjoys the Hennessy-Milner property and it is consequently appropriate for describing LWSs up to bisimilarity. This result is particularly interesting because it shows that the Hennessy-Milner property can be satisfied in the absence of negation.

Our second major result is the decidability of the satisfiability problem for RWL which derives directly from our model construction. This is a novel construction that we design for RWL, which also provides a satisfiability-checking algorithm. We will discuss the complexity in a future paper.

For future work we consider to extend RWL. The current version only allows one variable for each type of resource in the syntax of the logic. This represents an important expressiveness restriction, but a necessary one if we want the satisfiability problem to be decidable. Nevertheless, we believe that one can adapt our model construction to the extended case, where we will not get a finite model any more, but one with certain types of regularity that will be properly described by some concept of weighted automata.

References

[ACD93] Rajeev Alur, Costas Courcoubetis, and David L. Dill. Model-checking in dense real-time. *Inf. Comput.*, 104(1):2–34, 1993.

[AD90] Rajeev Alur and David L. Dill. Automata for modeling real-time systems. In Mike Paterson, editor, *ICALP*, volume 443 of *Lecture Notes in Computer Science*, pages 322–335. Springer, 1990.

[AILS07] Luca Aceto, Anna Ingólfsdóttir, Kim Guldstrand Larsen, and Jiri Srba. *Reactive Systems: modelling, specification and verification*. Cambridge University Press, 2007.

[ATP01] Rajeev Alur, Salvatore La Torre, and George J. Pappas. Optimal paths in weighted timed automata. In Benedetto and Sangiovanni-Vincentelli [BSV01], pages 49–62.

[BFH+01] Gerd Behrmann, Ansgar Fehnker, Thomas Hune, Kim Guldstrand Larsen, Paul Pettersson, Judi Romijn, and Frits W. Vaandrager. Minimum-cost reachability for priced timed automata. In Benedetto and Sangiovanni-Vincentelli [BSV01], pages 147–161.

[BSV01] Maria Domenica Di Benedetto and Alberto L. Sangiovanni-Vincentelli, editors. *Hybrid Systems: Computation and Control, 4th International Workshop, HSCC 2001, Rome, Italy, March 28-30, 2001, Proceedings*, volume 2034 of *Lecture Notes in Computer Science*. Springer, 2001.

[CKS92] Rance Cleaveland, Marion Klein, and Bernhard Steffen. Faster model checking for the modal mu-calculus. In Gregor von Bochmann and David K. Probst, editors, *CAV*, volume 663 of *Lecture Notes in Computer Science*, pages 410–422. Springer, 1992.

[CS93] Rance Cleaveland and Bernhard Steffen. A linear-time model-checking algorithm for the alternation-free modal mu-calculus. *Formal Methods in System Design*, 2(2):121–147, 1993.

[DKV09] Manfred Droste, Werner Kuich, and Heiko Vogler, editors. *Handbook of Weighted Automata*. Springer Verlag, 2009.

[HKT01] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic Logic*. The MIT Press, 2001.

[HM80] Matthew Hennessy and Robin Milner. On observing nondeterminism and concurrency. In J. W. de Bakker and Jan van Leeuwen, editors, *ICALP*, volume 85 of *Lecture Notes in Computer Science*, pages 299–309. Springer, 1980.

[HNSY92] Thomas A. Henzinger, Xavier Nicollin, Joseph Sifakis, and Sergio Yovine. Symbolic model checking for real-time systems. In *LICS*, pages 394–406, 1992.

[Koz82] Dexter Kozen. Results on the propositional μ-calculus. In Mogens Nielsen and Erik Meineche Schmidt, editors, *ICALP*, volume 140 of *Lecture Notes in Computer Science*, pages 348–359. Springer, 1982.

[Lar90] Kim Guldstrand Larsen. Proof systems for satisfiability in hennessy-milner logic with recursion. *Theor. Comput. Sci.*, 72(2&3):265–288, 1990.

[LLW95] François Laroussinie, Kim Guldstrand Larsen, and Carsten Weise. From timed automata to logic - and back. In Jirí Wiedermann and Petr Hájek, editors, *MFCS*, volume 969 of *Lecture Notes in Computer Science*, pages 529–539. Springer, 1995.

[LM13] Kim G. Larsen and Radu Mardare. Complete proof system for weighted modal logic. *Theoretical Computer Science*, to appear, 2013.

[SR11] D. Sangiorgi and J. Rutten, editors. *Advanced topics in Bisimulation and Coinduction*. Cambridge University Press, 2011.

[Sti99] Colin Stirling. Bisimulation, modal logic and model checking games. *Logic Journal of the IGPL*, 7(1):103–124, 1999.

[Tar55] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.

[Wal00] Igor Walukiewicz. Completeness of kozen's axiomatisation of the propositional μ-calculus. *Inf. Comput.*, 157(1-2):142–182, 2000.