Metric Equational Theories

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ABSTRACT

Quantitative Equational Theories (QETs) have recently attracted significant interest as they extend powerful techniques from universal algebra to computation on quantitative models. For example, the fundamental Wasserstein and Kantorovich metrics on probability distributions can be given concise axiomatic presentations using QETs and thus these metrics can now be seen as algebraic structure on metric spaces. Crucially, QETs have a sound and complete proof system for proving bounds on the distances between terms. However, it transpires QETs are too weak to provide a comprehensive foundation in that there are examples which ought to be seen as algebraic structure on metric spaces but which cant be defined within QETs — the clearest example is that of Cauchy Completion. This is a fundamental problem as in the traditional setting there is a correspondence between finitary monads on the category Set and algebraic theories, and the monad corresponding to Cauchy completion in countably presentable and thus should be an example of algebraic structure over metric spaces. The alternative is to use Enriched Lawvere Theories specialised to a category of metric spaces as a foundational framework within which algebraic structure over metric spaces can be defined and developed. However, this also is insufficient because Enriched Lawvere theories don't give us proof systems and so one has no formal sound and complete system for gauging the distance between terms.

This paper solves the fundamental problem of what are appropriate sound and complete proof systems for algebraic structure over metric spaces by taking the best of each of the approaches outlined above. Thus we extend QETs to what we call Metric Equational Theories (METs) where operations no longer need to have arities which are merely finite sets (as in QETs) but rather arities are now drawn from countable metric spaces. This extension is exactly motivated by the theory of Enriched Lawvere Theories which suggests the arities of operations ought to be the λ-presentable objects of the underlying λ-accessible category. The technical challenge of the work then arises as these more general arities mean the validity of terms in METs can no longer be guaranteed independently of the validity of equations as is usually the case. Once we solve this problem, we can adapt the sound and complete proof system for QETs to these more general METs essentially taking advantage of the specific structure of metric spaces to ensure METs offer the best of both worlds — the generality and full expressive power of Enriched Lawvere Theories and sound and complete proof systems for the resulting equational theories.

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1 INTRODUCTION

Approximate computation arises naturally in a number of areas, e.g. in: i) stochastic systems, where one merely has probability distributions over values induced by inherent/simulated randomness; ii) resource limited environments, where exact computation is prohibitively expensive; iii) systems with imperfect/partial recall, where one only has limited information about what has happened or the intentions/trustworthiness of each agent; and iv) non-exact computation where primitive data (e.g. from sensors) is inexact and supplied with error bars. These scenarios arise in e.g. cyber-physical systems, machine learning, robotics, automotive engineering, aerospace, and energy systems.

Recently Mardare et al [11, 12] introduced Quantitative Algebra (QA) and Quantitative Equational Theories (QETs) to extend one of the central pillars of modern mathematics, namely universal algebra (UA), from the exact world to the approximate world. Central to their work was the introduction of approximate equations $s =_\epsilon t$ (for a positive real ϵ), formalising the intuition that ϵ measures the behavioural *similarity* between terms s and t. The generality of this new idea — replacing Boolean reasoning within equational logic with a more refined approximate form of reasoning — gives us a new paradigm which supports a rigorous logical framework for a proper approximation theory, where bounds can be handled, convergences proven and limits approximated. Quantitative Equational Theories [11] have been used to provide simple axiomatic presentations of well behaved metrics for several fundamental computational structures, e.g. the Kantorovich-Wasserstein metrics (resp. Hausdorff metrics) arise from convex structures (resp. semi-lattices). See 1.1 for more details.

Quantitative Algebra and Quantitative Equational Theories have been shown to have good meta-theoretic properties. For example, variety and quasi-variety results have been proven for Quantitative Algebra [12], revealing new insights of approximated reasoning. Similarly, compositionality principles have been studied for Quantitative Algebra, providing a formal tool to control the error propagation when computational systems interact [4, 5]. Conway and iteration theories have also been generalised to the quantitative case, to cover not only exact fixed-points $x = f(x)$, but also approximate fixed-points $x =_\epsilon f(x)$ [13]. However, despite all this work, there remains a fundamental and unanswered question:

> *What is the natural format/foundation for presenting algebraic structure in a metric setting?*

In the exact setting of traditional universal algebra, there are strong results which (when adapted to the metric setting) form desiderata for any answer to the above question. These include

- Every equational theory gives rise to a finitary monad on the category of Set which maps a set to the free algebra over that set.
- The models of an equational theory are the Eilenberg-Moore algebras of the associated finitary monad.
- Every finitary monad on Set arises via this construction.
- There is a sound and complete proof system for proving equations between terms.

Unfortunately QETs fail these desiderata, for example the monad mapping a metric space to its Cauchy Completion does not arise from a QET as described above. We discuss this example in section 7. Note that the above desiderata can (and perhaps should) be framed in terms of Lawvere Theories as Lawvere theories: i) smoothly generalise to models in other categories; and ii) give syntax-independent presentations of algebraic theories, in the sense that distinct syntactical presentations of equivalent algebraic theories give rise to isomorphic Lawvere theories.

Turning from the case of sets with algebraic structure to metric spaces with algebraic structure, one approach would be to turn to the abstract setting of Enriched Lawvere Theories [14] which we cover in greater detail in section 2. For now, we note that the category Met of extended metric spaces (ie where the distance between two points can be infinity) and non-expansive maps satisfied the preconditions of that paper. A key point though is that Met is not locally finitely presentable but countably presentable. That means arities of operators and equations ought to be not merely finite sets as in the case of traditional algebraic theories, nor countable (discrete) metric spaces and one ought to consider countably-presentable monads in the desiderata above. The good news is that, once these changes have been made, the work of [7, 14] seem to give us exactly what we want. That is i) a notion of algebraic theory consisting of operations and equations; ii) free algebras for such algebraic theories correspond to countably presentable **enriched** monads on Met such that the models of the former are the Eilenberg Moore algebras of the latter; and iii) a notion of Lawvere theory giving a syntax-free presentation of algebraic theories.

However, this abstract approach leaves a lot to be desired, because (as with finitary monads on Set) the presentations one gets from this framework while theoretically elegant are very cumbersome to the point of being of little practical use. In particular, given a countably presentable monad, the associated algebraic theory has as operators of a given arity all elemwents of the monad applied to that arity. In the case of the cauchy completion, where we woukld like one operator, we get a countyably infinite number of operators! And thats before we even consider equations. A further limitation of the work on Lawvere theories is they dont cover sound and complete proof systems to enable proof of when one term equals another.

This paper therefore starts from the premise of wanting the best of both worlds, ie the theoretic clarity of Lawvere theories and the concise presentations and proof systems of QETs. Thus we introduce *Metric Equational Theories* which extend QETs by allowing operators to have as arities countable metric spaces as opposed to just countable sets. This allows us to cover for example the countable presentable Cauchy Completion monad. This generalisation however, creates a difficulty. In both traditional equation theories and QETs one can first define the definable terms over a given arity and then the derivable equations better such terms. However this is no longer possible with METs as metric arities mean the definabililty of a term might require subterms to be within a certain distance of each other, but that might only be provable by using some of the equations of the theory. However, once we devise mechanisms for handling this increased complexity in METs, this paper shows we do indeed get what we want, that is all of the above dissiderata when suitably generalised to Met together with sound and complete proof systems for deriving the equality between terms.

1.1 Background on Quantitative Equational Theories

In [11, 12] the authors develop the theory of *quantitative algebras*, an extension of the theory of universal algebras that describe algebraic structures over metric spaces. To do this, they propose an equational logic in which the basic bricks are *quantitative equalities* of type $s =_\varepsilon t$, where s and t are Ω algebraic terms defined given an algebraic similarity type Ω, and $\varepsilon \geq 0$ is a positive real. The models are Ω-algebras supported by metric spaces with all the operators non-expansive, and the interpretation of the aforementioned quantitative equality is that the distance between the interpretation of s and t is upper-bounded by ε. Replacing the classical equality with a quantitative equality of type $=_\varepsilon$ requires fundamental changes in the metalogical reasoning. For instance, $=_\varepsilon$ is not transitive and the transitivity rule is replaced by the triangle inequality rule. Moreover, all the algebraic operators are required to be non-expansive. All these differences make the quantitative equational logic a novel logic, fundamentally different from the classic equational logic used to define monads on Set in the theory of universal algebras. Instead, the quantitative equational theories define monads on Met.

However, the presentation in [11, 12] does not involve operators with metric arities, that would naturally point towards Met-enriched Lawvere theories. Instead they use the classic arities and propose a theory that resemble the classic one.

Hereafter we present two examples studied in [11].

Example 1.1 (Quantitative Semilattices with 0*).* Consider the monad P_f : Met → Met which carries a metric space to the set of finite subsets equipped with the Hausdorff metric (with the unit being the inclusion of singleton subsets, the multiplication being union, and the action on morphisms being given by direct image).

In [11], this monad is axiomatized using a quantitative equational theory over the signature of semilattices with 0. Hence we have one constant 0 and one binary operator + satisfying the following axioms, where $\vdash$ is the classic logical implication indicating, as usual, that the set of quantitative equalities on its left (hypotheses) entail the conclusion on its right:

- $\vdash x + 0 =_0 x$
- $\vdash x + x' =_0 x' + x$
- $\vdash x + x =_0 x$
- $\vdash x + (x' + x'') =_0 (x + x') + x''$
- $x =_\epsilon y,\ x' =_{\epsilon'} y' \vdash (x + x') =_{\max(\epsilon,\epsilon')} (y + y')$.

In the free models $P_f(X)$, the interpretation of 0 is the empty set, and the intepretation of + is set union. The first four axioms simply express the requirement that $(+, 0)$ forms a semilattice (this is natural enough, since for sets the collection of finite subsets is the free semilattice on a set), and the final axiom schema[1] bounds the metric of unions in terms of the distances between the subsets. One then proves that the maximal metric satisfying this bound and the requirement $d(\{x\},\{y\}) \leq d(x,y)$ is the Hausdorff metric.

In fact, for a compact metric space X, this theory has even better properties — the initial model with a map from X among *complete* metric spaces is the space of closed subsets of X in the Hausdorff metric.

Example 1.2 (Interpolative barycentric algebras). An other example from [11] are the interpolative barycentric algebras proposed to axiomatize the Kantorovich and p-Wasserstein distances on probabilistic distributions. This quantitative algebraic theory is supported by the algebraic signature of barycentric algebras [16], so the signature contains the binary operators $+_e$ for each $e \in [0,1]$, and the axiomas are listed below.

- $\vdash x +_1 x' =_0 x$
- $\vdash x +_e x =_0 x$
- $\vdash x +_e x' =_0 x' +_{1-e} x$
- $\vdash (x +_e x') +_{e'} x'' =_0 x +_{ee'} (x' +_{\frac{e'-ee'}{1-ee'}} x'')$
- $x =_\varepsilon y,\ x' =_{\varepsilon'} y' \vdash (x +_e x') =_{e\varepsilon+(1-e)\varepsilon'} (y +_e y')$.

Consider the Giry monad $G : \mathsf{Met} \to \mathsf{Met}$ which carries a metric space (M,d) to the space of $(G(M,d), d^K)$ of the distributions with finite support defined on Borel algebra of (M,d), and with the Kantorovich metric d^K defined by the classic transportation problem [17]. In this context $s +_e t$ for $e \in [0,1]$ defines the e-convex combination of the distributions s and t. Thus, the first four axioms state that this is indeed a barycentric algebra, while the last one defines an upper bound between convex combination of distributions based on the pairwise upper bound of the given distributions. In [11] it is proven that the maximal metric satisfying this bound is indeed the Kantorovich metric, that the spaec $(G(M,d), d^K)$ is the free IB-algebra generated by X, and that with a small modification in the last axiom one can get also p-Wasserstein metric [17].

Moreover, for a separable X, the initial model among *complete* metric spaces with a map from X is the full space of Radon probability measures on X in the Kantorovich metric.

1.2 Operations of metric arity

The examples above demonstrate the expressiveness of quantitative equational theories. However, we also see a limitation — to construct a metric space of interest, we often have to pass to a Cauchy completion of the free algebra. For example, the space of Radon probability measures on a (separable) metric space X is the *completion* of the free IB-algebra generated by X.

On the other hand, Cauchy completion is itself a monad on Met (since it is the left adjoint to the inclusion of complete metric spaces Met). And this monad, though it is not finitary, is in fact of countable arity (in the sense of commuting with countably filtered colimits). Thus, if quantitative equational theories are supposed to be the metric space answer to traditional equational theories, we might ask whether the completion monad can't be presented as the free monad of an equational theory (with some operations of countably infinite arity, possibly). After all, this would be possible for a monad on Set of countable arity.

[1]It generates one axiom for each choice of ϵ, ϵ'.

However, this is not possible. The basic problem is that the Cauchy completion of a discrete metric space is always the space itself. So if a theory had the Cauchy completion as its free model, the free model on a discrete space X is just X. This means all the operations f in the theory must satisfy $\vdash f(x_1,\dots) =_0 x_n$ for some n. But clearly that means the inclusion of X in the free model on X is surjective for *any* metric space (it may not be an isomorphism — the theory could have the equation $\vdash x =_0 y$, for example, meaning the free model is a singleton for any nonempty space), so the free model can't possibly be the completion in general.

From an algebraic point of view, the limitation is that quantitative equational theories only have operations of *discrete arity*. We can't define an operation $f(x_1,\dots)$ which is only defined subject to a restriction on the distances between the x_i. For a monad T on Met, on the other hand, there is no particular reason $T(f) : T(X) \to T(X')$ should be surjective just because if $f : X \to X'$ is a bijection.

We might look around for a different framework to replace QETs, given this limitation. There is a completely general framework in the literature, due to Kelly and Power ([7]), for presenting an enriched monad (over essentially any base of enrichment) by generators and relations. In their approach, one gives a (possibly empty) metric space of generators of each possible arity, and sets of equations between terms in these generators. Since these equations can only identify terms, not express distances between them, we must encode the metric in the generators — we can do this because the generators of a given arity form a metric space.

However, in the examples above, since we need to impose a different bound on the distance between $x +_e x'$ and $y +_e y'$ respectively, depending on our assumed bound on the distance between x and y (and x' and y'), we are faced with a problem.

The resolution of this problem is to introduce a family of generating operations, two for each choice of (ϵ, ϵ', e), call them $f_e^{\epsilon,\epsilon'}, g_e^{\epsilon,\epsilon'}$ which are four-ary. Given arguments x, x', y, y', one of these families is supposed to compute $x +_e x'$, the other $y +_e y'$. The distance between these generators in the space of operations of arity $(\{x, x', y, y'\}, d(x,y) = \epsilon, d(x',y') = \epsilon')$ is then set to $e\varepsilon + (1-e)\varepsilon'$. Finally, we impose the equation

$$f^{\epsilon,\epsilon'}(x, x', y, y') = x + x', g^{\epsilon,\epsilon'}(x, x,', y, y') = y + y'.$$

This is clearly a much more awkward approach. We are forced to introduce a large family of auxiliary generating operations to express the desired information about distance. These operations aren't very natural — they are always equal to some term in the "proper" operations, and only exist so we can express conditional distance bounds. It seems much more natural to think of distance information as a generalization of information about which terms are equal, and thus to give it as generating "quantitative equations", as in the method of [11]. In that case, we make conditional distance bounds possible by admitting some preconditions into the theory.

The approach of this paper is to generalize the quantiative equational theories of [11], to encompass the full class of countable-arity monads in Met. This is accomplished by extending the class of signatures to allow operations with *metric arity*, which amounts to letting them be defined only subject to a restriction on the distances between the arguments. We also extend the logic to accomodate reasoning about terms in these operations.

Thus, for example, the monad which carries a metric space to its Cauchy completion is axiomatized by a theory with one operation lim, of countably infinite arity, with the metric of the arity being $d(n, m) = 1/2^{\min(n,m)}$. Thus $\lim(x_0, \dots)$ is defined if and only if $d(x_n, x_m) \le 1/2^n$ whenever $m > n$. The theory has the equations $\{x_n =_{1/2^n} x_m \mid 0 \le n < m\} \vdash \lim(x_0, \dots) =_{1/2^k} x_k$, for each $k = 0, 1, \dots$, which imply that, supposing the variables are such that lim is defined, it is in fact a limit of the sequence.

1.3 Outline of the paper

We begin in section 2 by recalling some aspects of the category theory of metric spaces, and the theory of enriched Lawvere theories as it specializes to this case.

In section 3 we proceed to introduce the main topic of the paper, Metric Equational Theories, their categories of models, and proving basic properties of these. In section 4, we construct a free-forgetful adjunction for each MET, and prove that these correspond to enriched Lawvere theories — this is the first direction of the central correspondence, corollary 4.10. In section 5, we carry out the other direction, constructing for each enriched Lawvere theory $\mathcal{T}$ over Met an MET whose category of models is equivalent to $\mathrm{Mod}(\mathcal{T})$ (proposition 5.2).

Finally, we prove a completeness theorem for METs in section 6, discuss the MET of Cauchy completeness in section 7, and consider some special classes of METs and study their corresponding monads in section 8 — in particular, we characterize the class of monads axiomatized by ordinary QETs.

1.4 Related work

As discussed above, quantitative equational theories were introduced by Mardare, Panangaden and Plotkin in [11], and further developed in [12]. Apart from the papers mentioned above, there is also [6], developing coproducts and tensors of QETs.

The work of Adámek, Dostál, and Velebil in [2] is also highly relevant in this context — they study a subclass of QETs and prove a classification theorem quite analogous to ours in the context of *ultrametric spaces*, showing that they present exactly the *strongly finitary* monads on ultrametric spaces ([2, Theorem 5.8]). However, their classification does not extend to monads with operations of metric arity.

Further afield, there is much recent work describing the category theory o Met from other points of view, for example [1], describing "Hausdorff polynomial functors" on Met and their associated monads, as well as [3], developing various species of "approximate limits" in Met.

2 ENRICHED LAWVERE THEORIES IN Met

The concept of *enriched Lawvere theory* was proposed by Power in [14]. Up to equivalence, a Lawvere theory is a category with finite products, whose objects are generated by a distinguished object under finite products. These then classify finitary monads on Set. Power states these results for the finitary case, remarking although that they generalize straightforwardly to any regular cardinal.

Given a regular cardinal κ and a κ-presentable biclosed monoidal category $\mathcal{V}$, we obtain a theory of $\mathcal{V}$-enriched Lawvere theories. These are defined to be $\mathcal{V}$-categories generated under products of cardinality less than κ and powers by κ-small objects of $\mathcal{V}$ by a distinguished object. These then classify strong (i.e enriched) κ-presentable monads on $\mathcal{V}$. The classical Lawvere theory is obtained by taking $\mathcal{V} = \mathrm{Set}$ and $\kappa = \aleph_0$.

In this paper, we will consider the case in which $\mathcal{V} = \mathrm{Met}$ and $\kappa = \aleph_1$. Since Met is not finitely presentable, we will be looking at *countable-arity* operations in general, but we will also discuss the subset of finitary monads.

We recall, in what follows, some general definitions and results about the category Met, see [15] for details. To simplify the language, in what follows we will refer to $\aleph_1$-locally presentable categories as *countably locally presentable*, $\aleph_1$-accessible monads as *countable-arity monads*, and so on.

As emphasized above, Met is the category where

(1) Objects are *extended metric spaces*, i.e., sets X equipped with a metric $d : X \times X \to [0, \infty]$;
(2) A morphism $(X, d_X) \to (Y, d_Y)$ is a *nonexpansive* map $f : X \to Y$ so that $d_Y(f(x), f(x')) \le d_X(x, x')$ for all $x, x' \in X$;
(3) Composition and identities are the ordinary function composition and the identity functions.

The *tensor product* $\otimes$ on Met is given by

$$(X, d_X) \otimes (Y, d_Y) = (X \times Y, d_{X \otimes Y})$$

where $d_{X\otimes Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$.

Met is a *symmetric monoidal category*, with associator, unitor and symmetry given as for the Cartesian symmetric monoidal structure on Set. Moreover, Met is closed as a monoidal category, with the internal hom $[X, Y]$ being given by the set of nonexpansive maps $X \to Y$ in the metric $d_{[X,Y]}(f, g) = \sup_x d_Y(f(x), g(x))$. Moreover, Met is *countably locally presentable* [10], and its countable objects are precisely those metric spaces with a countable underlying set [3].

Recall that in a category C enriched over $\mathcal{V}$, a *power* or *cotensor* of an object $X \in C$ by $V \in \mathcal{V}$ is an object X^V so that $C(A, X^V) \cong [V, C(A, X)]$ (with $[-, -]$ being the internal hom in $\mathcal{V}$, here). In the self-enrichment of $\mathcal{V}$, powers are given by the internal hom $W^V = [V, W]$. In the present case of metric spaces, we generally prefer the exponential notation Y^X for this space, which to reiterate is given by the set of nonexpansive maps $X \to Y$ in the sup-metric $d(f, g) = \sup_x d_Y(f(x), g(x))$.

Note that a category enriched in Met is simply an ordinary category C equipped with a metric on each hom-set, so that the composition operation $\circ : C(Y, Z) \otimes C(X, Y) \to C(X, Z)$ is nonexpansive. A functor $C \to \mathcal{D}$ in the enriched sense is simply a functor F between the underlying categories so that each map $F : C(X, Y) \to \mathcal{D}(FX, FY)$ is nonexpansive. In particular, being enriched is a *property* of an ordinary functor, not extra data.

We will primarily be interested in monads on Met itself — here an "enriched monad" is just an ordinary monad (T, μ, η) on the category Met where the functor T happens to be nonexpansive, and the Eilenberg-Moore category of such a monad is just the EM category in the ordinary sense, metrized with the sup-metric restricted to the subset of homomorphisms.

An enriched functor is also called *strong* (because for a closed monoidal category $\mathcal{V}$, an enriched endofunctor on $\mathcal{V}$ is equivalent to a strong endofunctor, [8, Theorem 1.3]), and we will also use the term *strong monad*.

Almost every functor we work with will be enriched in this sense, so we will often not bother to be precise about the difference.

Recall that a *pseudometric space* is the generalization of metric spaces without the requirement that $d(x, y) = 0 \Rightarrow x = y$. There is an obvious category PMet of pseudometric spaces and nonexpansive maps. Met is a full subcategory of PMet, and this inclusion is reflective — for each pseudometric space X, the reflection is given by the quotient $X/\!\sim$ where $x \sim y$ if $d(x, y) = 0$. We refer to this as the *metric quotient* of X. It will often be convenient to describe metric spaces in this way.

Definition 2.1. Let CMet denotes the full subcategory of Met of countable extended metric spaces. CMet inherits a Met enrichment. Since CMet is closed under tensor products, the opposite category $\mathrm{CMet}^{\mathrm{op}}$, has all powers by countable objects (and they are simply given by the tensor product).

Definition 2.2. A Met *Lawvere theory* is an Met-category $\mathcal{L}$ equipped with an identity-on-objects functor $\mathrm{CMet}^{\mathrm{op}} \to \mathcal{L}$ which is enriched and preserves powers by countable objects.

Remark 2.3. Note that every object X in $\mathrm{CMet}^{\mathrm{op}}$ can be written as a (countable) power of the singleton $*$ by X. Hence, if F is the functor $\mathrm{CMet}^{\mathrm{op}} \to \mathcal{L}$, we have $F(X) = F(*)^X$, and these are all the objects of $\mathcal{L}$, up to isomorphism.

Note that $\mathrm{CMet}^{\mathrm{op}}$ has products (given by coproducts in Met), and these are automatically preserved by F, since

$$F(X \sqcup Y) = F(*)^{X \sqcup Y} = F(*)^X \times F(*)^Y.$$

In what follows, we abuse the notation as usual, and denote a Met Lawvere theory $(\mathcal{L}, F : \mathrm{CMet}^{\mathrm{op}} \to \mathcal{L})$ simply by $\mathcal{L}$.

We will denote the object $F(*)$ by x, and hence we can use the notation x^X, with X a countable metric space, to denote a general object of $\mathcal{L}$.

Note that, by definition of powers, the hom-object $\mathcal{L}(x^A, x^B)$ is isomorphic to the metric space $[B, \mathcal{L}(X^A, x)]$ of maps $B \to \mathcal{L}(x^A, x)$. Hence the hom-objects of a Met Lawvere theory are determined by the spaces $\mathcal{L}(x^A, x)$.

Definition 2.4. The metric space $\mathcal{L}(x^A, x)$ is called the *space of A-ary operations* of $\mathcal{L}$.

Definition 2.5. A *model* of a Met Lawvere theory $\mathcal{L}$ is a power-preserving Met-functor $M : \mathcal{L} \to \mathrm{Met}$. The *category* of models, denoted $\mathrm{Mod}(\mathcal{L})$, is the category of such functors and natural transformations between them. Note that this carries a Met-enrichment. There is a forgetful functor $\mathrm{Mod}(\mathcal{L}) \to \mathrm{Met}$ which carries a model M to $M(x)$.

The following result is [14, Theorem 4.3] specialized to our case. Let Law denote the category of Met-Lawvere theories and CMnd the subcategory of countable-arity monads and monad transformations.

Proposition 2.6. *Given a* Met *Lawvere theory $\mathcal{L}$, the category of its models is monadic over* Met. *The monad arising from this adjunction is always of countable arity (it preserves countably filtered colimits). This construction induces a functor* $\mathrm{Law} \to \mathrm{Mnd}(\mathrm{CMet})$, *which is an equivalence of categories. Its inverse carries a monad T from* $\mathrm{Mnd}(\mathrm{CMet})$ *to the dual of its Kleisli category restricted to countable metric spaces.*

3 METRIC EQUATIONAL THEORIES

In this section we revisit the theory of quantitative equational logic with the intention of making use of metric arities and produce explicit Met-enriched Lawvere theories. To this end, we need to extend the syntax proposed in [11] to properly encode operators with metric arities.

Our generalization, which we call *metric equational theories*, will have two kinds of judgement. The structural judgements, of type $\Gamma \vdash s =_\epsilon t$, where s and t are terms, express the claim that s and t, given the set of hypotheses Γ on the variables, are within distance ϵ of each other; these are the judgements used in [11]. In addition, we will also need *formational judgments* of type $\Gamma \vdash t$ ok, asserting that in the given context, t is a well-formed term — that is, it is provably well-defined. The constructs of type t ok (for terms t) are called *formational atoms*, and together with the *quantitative equalities* of type $s =_\varepsilon t$ (for terms s and t and positive reals ε) are the building blocks for our judgements. We need to combine these types of judgment because we don't want to ever make any assertions about ill-formed terms, so we need to prove t ok before we apply certain inference rules. But we cannot know which terms are well-formed without reasoning about their distances. Of course, a judgment that a term is well-defined ultimately boils down to a judgment about distances between the subterms — as such, the theory could be rewritten without the ok judgments. However, they are a useful bookkeeping device and as such have been left in.

From the start, we're faced with a difficulty not seen in the classical case. Since our operations have arities in metric spaces, their domain of definition depends on the distance between the arguments. This means that the set of terms under consideration is not the entire set of trees of operators as in the ordinary case, but only a subset. But, crucially, the set of well-formed terms depends not only on the context, but on the equational axioms.

While it would be possible to give mutually inductive definitions of "theory", "sequent" and "term", we find it simpler to consider the whole set of "preterms" and define the well-formed subset.

Definition 3.1. A *metric signature* (or simply *signature*) Ω is a set of operation symbols, each equipped with a *metric arity*, which is a countable cardinal (i.e either $\{0, \ldots, n-1\}$ for some natural number $n \geq 0$, or $\{0, \ldots\}$ the set of natural numbers) equipped with a metric. We write $f : N \in \Omega$ if f is an operation in Ω with arity $N = (N, d)$.

Definition 3.2. Let X be a set. $\tilde{\Omega}(X)$ is the set of terms (in the classic sense) in the signature, called *preterms*, given inductively by

(1) For each $x \in X$, there is a preterm $x \in \tilde{\Omega}(X)$.
(2) If t_i is a preterm for each $i \in N$, and $f : N \in \Omega$, there is a preterm $f(t_1, \ldots)$.

Definition 3.3. Let X be a set of variables. A *context* Γ *over* X consists of a list of equations $x =_\varepsilon y$, where x, y are variables and $\varepsilon \geq 0$ is a nonnegative real number.

Given a metric space M and an assignment $\alpha : X \to M$ of the variables, we say α satisfies an equation $x =_\varepsilon y$ if $d(\alpha(x), \alpha(y)) \leq \varepsilon$.

A variable assignment $\alpha : X \to M$ satisfies a context if it satisfies all the equations.

Definition 3.4. Fix a countably infinite set of variables X. A collection of judgments (formational and structural) is called a *deducibility relation* if it is closed under the following inference rules stated for arbitrary $x, x_i \in X$, $f : (N, d) \in \Omega$, $t, t', t_i, t_i', u_i \in \tilde{\Omega}(X)$, ϕ either a quantitative equality or a formational atom, and $\epsilon, \epsilon' \in \mathbf{R}_{\geq 0}$

$$\frac{}{\Gamma \vdash x \text{ ok}} \textbf{ Var}$$

$$\frac{\forall i : \Gamma \vdash t_i \text{ ok},\ \forall i, j, d(i,j) < \infty : \Gamma \vdash t_i =_{d(i,j)} t_j}{\Gamma \vdash f((t_i)) \text{ ok}} \textbf{ App}$$

$$\frac{x =_\epsilon y \in \Gamma}{\Gamma \vdash x =_\epsilon y} \textbf{ Assum}$$

$$\frac{\Gamma \vdash t \text{ ok}}{\Gamma \vdash t =_0 t} \textbf{ Refl}$$

$$\frac{\Gamma \vdash t' =_\epsilon t}{\Gamma \vdash t =_\epsilon t'} \textbf{ Symm}$$

$$\frac{\Gamma \vdash t =_\epsilon t', \Gamma \vdash t' =_{\epsilon'} t''}{\Gamma \vdash t =_{\epsilon + \epsilon'} t''} \textbf{ Triang}$$

$$\frac{\epsilon' < \epsilon,\ \Gamma \vdash t =_{\epsilon'} t'}{\Gamma \vdash t =_\epsilon t'} \textbf{ Max}$$

$$\frac{\Gamma \vdash t =_{\epsilon'} t\ \forall \epsilon' > \epsilon}{\Gamma \vdash t =_\epsilon t'} \textbf{ Cont}$$

$$\frac{\Gamma \vdash s_i =_{d(i,j)} s_j,\ \Gamma \vdash t_i =_{d(i,j)} t_j,\ \Gamma \vdash s_i =_\epsilon t_i \forall i, j}{\Gamma \vdash f((s_i)) =_\epsilon f((t_i))} \textbf{ Nexp}$$

$$\frac{\Gamma \vdash u_i =_{\delta_{ij}} u_j, \{x_i =_{\delta_{ij}} x_j\} \vdash s =_\epsilon t}{\Gamma \vdash s[u_i/x_i] =_\epsilon t[u_i/x_i]} \textbf{ Subst}$$

Given a set S of judgments, the least deducibility relation generated by them is denoted $\vdash_S$.

Since the arity of the operations in the theory are supposed to determine their domain of definition entirely, we don't want to have any *axioms* of the form $\Gamma \vdash t$ ok, only equations. Furthermore, when axiomatizing a theory, we need to ensure that for each axiomatic equation, the terms are well-formed. But which terms are well-formed in a given context depends on the theory, since the theory provides bounds on distance that may imply well-formedness.

This means we have a kind of cyclic dependency, which makes things more difficult than they would be without the metric arities. However, in practice, the theory is not difficult to work with. First, since well-formedness of a term depends only on the distances between its subterms, there is no actual problem of cyclical dependency — we can proceed from subterms to larger terms, forming judgments about their distances and well-definedness inductively. And second, when working with a set of equational axioms which define the theory we want to work with, it suffices to prove that those equations are well-formed — then any sequents provable from them will also be well-formed.

Definition 3.5. A collection of judgments S is called *well-formed* if, whenever $\Gamma \vdash t =_\epsilon s \in S$, we also have $\Gamma \vdash t$ ok, $\Gamma \vdash s$ ok $\in S$. A well-formed collection of judgments $\mathbb{T}$ is called a *metric equational equational theory* (or just a *theory*) if $\mathbb{T} = \vdash_{\mathbb{T}^=}$, where $\mathbb{T}^= \subseteq \mathbb{T}$ is the subset of equational judgments.

Proposition 3.6. *Let S be a collection of equational judgments. Suppose for each $\Gamma \vdash t =_\epsilon s \in S$, $\Gamma \vdash_S t$ ok, $\Gamma \vdash_S s$ ok. Then $\mathbb{T} = \vdash_S$ is a theory.*

Proof. Clearly $\mathbb{T}$ is generated by a set of equations, so it suffices to check that it's well-founded.

First, we prove that whenever $\Gamma \vdash f((t_i))$ ok $\in \mathbb{T}$ for an operation $f : (N, d)$, we have $\Gamma \vdash t_i =_{d(i,j)} t_j$ for each $i, j \in N$. To see this, let $\mathbb{T}' \subseteq \mathbb{T}$ be the subset with the same equational sequents, but only those well-formedness sequents for which this rule holds. Clearly this contains S (since it contains all equations in $\mathbb{T}$), so it suffices to prove that it's stable under the inference rules. Since it has all the equations in $\mathbb{T}$, we only have to check **Var** and **App**. The former is true by definition, the latter because the preconditions are exactly the extra condition necessary for the postcondition to be in $\mathbb{T}'$.

Now we prove that $\mathbb{T}$ is well-founded. Again, we define a subset $\mathbb{T}' \subseteq \mathbb{T}$. This time, we let it contain the same formational judgments, but only those equational judgments $\Gamma \vdash t =_\epsilon s$ where $\Gamma \vdash t$ ok, $\Gamma \vdash s$ ok $\in \mathbb{T}$. Note that this contains all of S, so again it suffices to show that it's stable under the inference rules.

(1) **Var** and **App** are now clear.
(2) **Assum** holds because $\mathbb{T}$ satisfies **Var**.
(3) **Refl** holds because the precondition is exactly what's required for the postcondition to be in $\mathbb{T}'$.
(4) **Symm** holds because, given the precondition, we must further have $\Gamma \vdash t$ ok $\in \mathbb{T}$ (and also t'). Hence, since $\mathbb{T}$ satisfies **Symm**, the postcondition must also be in $\mathbb{T}'$.
(5) **Triang**, **Max**, **Cont** all hold for essentially the same reason as **Symm**.
(6) To prove **NExp**, we have to show that, given the preconditions, $\Gamma \vdash f((s_i))$ ok $\in \mathbb{T}$. But this follows from the preconditions, the further fact that because the preconditions are in $\mathbb{T}'$ we have $\Gamma \vdash s_i$ ok $\in \mathbb{T}$ for each i, and the fact that $\mathbb{T}$ satisfies **App**.
(7) Finally, the most difficult rule is **Subst**. We will prove that, given the preconditions, $\Gamma \vdash s[u_i/x_i]$ ok. If s is a variable not among the x_i, this is obvious. If $s = x_i$, then $s[u_i/x_i] = u_i$, and we are done by assumption. Suppose $s = g((s_k))$ for some symbol $g : (N', d')$. By induction, assume $\Gamma \vdash s_k[u_i/x_i]$ ok $\in \mathbb{T}$ for each k. By assumption, $\{x_i =_{d(i,j)} x_j\} \vdash g((s_k))$ ok $\in \mathbb{T}$. But by the rule we proved above, this means that $\{x_i =_{d(i,j)} x_j\} \vdash s_k =_{d'(k,k')} s_{k'} \in \mathbb{T}$ for each $k, k' \in N'$. Now **Subst** for $\mathbb{T}$ implies that $\Gamma \vdash s_k[u_i/x_i] =_{d'(k,k')} s_{k'}[u_i/x_i]$. These claims and the inductive assumption, together with **App**, imply the desired conclusion.

□

Definition 3.7. Let Γ be a context. Then $\hat{X_\Gamma}$ is the set X equipped with the pseudometric $d(x, x') = \min\{\epsilon \mid \Gamma vdashx =_\epsilon x'\}$. (Note that this minimum is attained because of **Arch**, and this is a pseudometric because of **Symm**, **Triang** and **Refl**).

X_Γ is the metric reflection of $\hat{X_\Gamma}$. Note that a nonexpansive map $X_\Gamma \to M$ is precisely a variable assignment that satisfies Γ

Definition 3.8. Let Ω be a signature. A *model* M of Ω is a metric space M equipped with, for each $f : A \in \Omega$, a map $M[f] : M^A \to M$ — recall that M^A is the metric space of nonexpansive maps $A \to M$ equipped with the supremum metric. Given models M, N, a homomorphism is a nonexpansive map $\phi : M \to N$ so that $\phi(M[f](x_1, \dots) = N[f](\phi(x_1), \phi(x_2), \dots)$. Note that if $(x_i) \in M^A$, then $(\phi(x_i)) \in N^A$ since ϕ is nonexpansive. The category of models and homomorphisms is denoted $\mathrm{Mod}(\Omega)$.

We will now define recursively what it means for M to satisfy a judgment, and the interpretation $M[t] : M^{X_\Gamma} \to M$ of every term as a function, whenever M satisfies $\Gamma \vdash t$ ok.

(1) Every model satisfied every judgment $\Gamma \vdash x$ ok for x a variable. $M[x](\alpha)$ is simply $\alpha(x)$.
(2) M satisfies $\Gamma \vdash f(t_1, \dots)$ ok if it satisfies each $\Gamma \vdash t_i$ ok and $d(M[t_i], M[t_j]) \le d_f(i, j)$ for all i, j (using the supremum metric on the function space). In this case $M[f(t_1, \dots)](\alpha) = M[f](M[t_1](\alpha), \dots)$.
(3) M satisfies $\Gamma \vdash t =_\epsilon s$ if it satisfies both $\Gamma \vdash t$ ok and $\Gamma \vdash s$ ok, and if $d(M[t], M[s]) \le \epsilon$ in the function space.

We write $\Gamma \vDash_M \phi$ if M satisfies the sequent $\Gamma \vdash \phi$.

A model of a theory $\mathbb{T}$ is a model of the signature of $\mathbb{T}$ which satisfies every sequent in $\mathbb{T}$.

The category of models of $\mathbb{T}$, is the full subcategory $\mathrm{Mod}(\mathbb{T}) \subseteq \mathrm{Mod}(\Omega)$ spanned by the models of $\mathbb{T}$.

The following proposition is readily verified:

Proposition 3.9 (Soundness). *Let M be any model of Ω. Then the relation $\vDash_M$ is well-founded and stable under the inference rules. In particular, to prove that M is a model of a theory $\mathbb{T}$ generated by some axioms S, it suffices to prove that M satisfies all the axioms.*

Remark 3.10. A theory is always defined over some signature Ω. We will often just leave the signature implicit when speaking of a theory. When $\mathbb{T}$ is a theory over Ω and f is a symbol in Ω, we will abuse notation by writing $f : N \in \mathbb{T}$, speak of "an operation in $\mathbb{T}$", and so on.

4 FREE MODELS AND MONADICITY

We now turn to the comparison of METs and Lawvere theories over Met. First, we will prove that for each MET $\mathbb{T}$, the forgetful functor admits a left adjoint (taking each metric space to a free model on it), and that this adjunction is monadic. After proving that the monads are of countable arity, it follows by the general theory of enriched Lawvere theories ([14]) that these monads (and hence the categories of models) come from enriched Lawvere theories.

First, we'll prove that all theories have initial models. Then, given a theory $\mathbb{T}$ and a metric space A, we'll construct a new theory whose models are models of $\mathbb{T}$ equipped with a map from A (that is, objects of the comma category $\mathrm{Mod}(\mathbb{T})_{A/}$). The initial models of these theories are precisely the free models of $\mathbb{T}$, and the fact that they all exist proves the existence of a left adjoint.

Proposition 4.1. *Let $\mathbb{T} = (\Omega, \mathbb{T})$ be a metric equational theory. Consider the metric space given by*

(1) Its elements are the closed terms $t \in \tilde{\Omega}(\emptyset)$ such that $\vdash_{\mathbb{T}} t$ ok, quotiented by the equivalence relation $t \sim t' \Leftrightarrow \vdash_{\mathbb{T}} t =_0 t'$.
(2) The metric is $d([t], [t']) = \min\{\epsilon \mid \vdash_{\mathbb{T}} t =_\epsilon t'\}$

Given an operation $f : A \in \Omega$ and a collection of elements $[t_i]$ satisfying $d([t_i], [t_j]) \le d_A(i, j)$ for all $i, j \in A$, $[f((t_i)_{i \in A}]$ is another well-defined element, and this gives a model of $\mathbb{T}$. This model is the initial model of $\mathbb{T}$

Proof. We will denote the space by $\mathrm{Free}^{\mathbb{T}}(\emptyset)$ (clearly the initial model is the free model on the empty space — we will construct an entire functor $\mathrm{Free}^{\mathbb{T}}$ soon).

Let t, t', s be elements and suppose $\vdash t =_0 t'$, and $\vdash t =_\epsilon s$. Then using **Triang** and **Symm**, also $\vdash t' =_\epsilon s$, and vice versa. Hence the distance is well-defined on equivalence classes. **Refl**, **Symm**, **Triang** straightforwardly imply that it's an (extended) metric. (Note that the minimum defining d is always attained, by **Arch**).

App implies that applying functions to a collection of well-formed closed terms with suitable bounds on their distance results in another well-formed closed term. **NExp** implies that this is well-defined (if we replace each input term with an equivalent one, the resulting terms are equivalent) and nonexpansive, so these operations $\mathrm{Free}^{\mathbb{T}}(\emptyset)$ of Ω.

Given some equation $\Gamma \vdash s =_\epsilon t$ in $\mathbb{T}$ using the variables $\{x_i\}$, and elements $u_i \in \mathrm{Free}^{\mathbb{T}}(\emptyset)$ so that $d([u_i], [u_j]) \le \epsilon$ whenever $x_i =_\epsilon x_j \in \Gamma$ (in other words, a variable assignment satisfying Γ), by **Subst** this variable assignment also satisfies $s =_\epsilon t$. Hence this is a model of $\mathbb{T}$.

Now suppose M is another model of $\mathbb{T}$. Note that if t is a term without variables, $M[t] : M^{X_\Gamma} \to M$ is constant (by induction on t) - let's abuse notation by writing $M[t] \in M$ for the constant value of this map. (If M is empty, of course, this definition won't make sense. But in that case $\mathbb{T}$ must have no constant symbols, and so there can't be any terms without variables). Define a map $\phi : \mathrm{Free}^{\mathbb{T}}(\emptyset) \to M$ by sending each class $[t]$ into $M[t]$. Since M satisfies $\mathbb{T}$, if $[t] = [t']$, then $d_M(M[t], M[t']) = 0$, so they are equal. Hence this is well-defined. Analogously, if $d(t, t') = \epsilon$, then $d_M(M[t], M[t']) \le \epsilon$, so this is a nonexpansive map. It's clearly a homomorphism. On the other hand, clearly the value of any homomorphism on closed terms is determined — it must go to the interpretation of that term. So this is unique. □

Definition 4.2. Let A be any metric space. Then Ω^A is the signature with one nullary operation $[a]$ for each point in A, and $\mathbb{T}(A)$ is the theory over this signature generated by the sequents $\vdash [a] =_{d(a,a')} [a']$ for each $a, a' \in A$.

Recall that, given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object $A \in \mathcal{D}$, the comma category $\mathcal{C}_{A/}$ has objects pairs $(X \in \mathcal{C}, f : A \to FC)$, and morphisms from $(X, f) \to (Y, g)$ given by $\phi : X \to Y \in \mathcal{C}$ so that $gF(\phi) = f$ (i.e so that the obvious triangle in $\mathcal{D}$ commutes).

Proposition 4.3. *As categories over* EMet, $\mathrm{Mod}(\mathbb{T}(A)) = \mathsf{Met}_{/A}$.

Proof. To give a space M the structure of a model of $\mathbb{T}(A)$ is to choose for each $a \in A$ a point $M[a] \in M$, satisfying $d(M[a], M[a']) \le d_A(a, a')$. This is equivalent to a map $A \to M$ which is nonexpansive. A homomorphism is a nonexpansive map $f : M \to N$ so that

$f(M[a]) = N[a]$. This is just the definition of a map in the slice category. ◻

Definition 4.4. Let $\mathbb{T}, \mathbb{T}'$ be two theories. Then $\mathbb{T} \sqcup \mathbb{T}'$ is the theory with signature $\Omega^{\mathbb{T}} \sqcup \Omega^{\mathbb{T}'}$, and generated by the union of the sequents in $\mathbb{T}$ and $\mathbb{T}'$.

Proposition 4.5.

$$\mathrm{Mod}(\mathbb{T} \sqcup \mathbb{T}') \cong \mathrm{Mod}(\mathbb{T}) \times_{\mathrm{Met}} \mathrm{Mod}(\mathbb{T}').$$

In other words, to give a space the structure of a model of $\mathbb{T} \sqcup \mathbb{T}'$ is simply to give it independently the structure of a model of each theory, and a homomorphism is just a function which is a homomorphism for each theory separately.

Proof. This is essentially true by definition, since (by construction) the operations of $\mathbb{T}$ and $\mathbb{T}'$ don't overlap within $\mathbb{T} \sqcup \mathbb{T}'$, and it suffices to satisfy the axioms of each theory independently (since the disjoint union theory is just generated by these). ◻

Applying the preceding proposition to the characterization of $\mathrm{Mod}(\mathbb{T}(A))$, we obtain the following:

Corollary 4.6. $\mathrm{Mod}(\mathbb{T} \sqcup \mathbb{T}(A)) \cong \mathrm{Mod}(\mathbb{T})_{/A}$.

Since a left adjoint exists if and only if each comma category has an initial object (and is then given by these initial objects, see eg [9, Theorem IV.1.1]), we obtain:

Corollary 4.7. $\mathrm{Mod}(\mathbb{T}) \to \mathrm{Met}$ *admits a left adjoint, which we denote* $\mathrm{Free}^{\mathbb{T}}(A)$. *Concretely,* $\mathrm{Free}^{\mathbb{T}}(A)$ *is given by terms t using the operations of $\mathbb{T}$ and a further constant symbol for each element of A, quotiented by provable equality, with $d(t, t')$ being the smallest ϵ so that $t =_\epsilon t'$ is provable using $\mathbb{T}$ and the further axioms $\vdash [a] =_{d(a,a')} [a']$ for every pair of elements in A.*

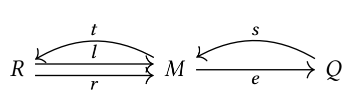
We now turn to the proof of monadicity — this is really the key ingredient in the comparison with Lawvere theories. Of course, the existence of the left adjoint $\mathrm{Free}^{\mathbb{T}}$ (along with the proof that the associated monad has countable arity, see below) already gives us a Lawvere theory — the question is whether its models are the same as the models of $\mathbb{T}$.

Proposition 4.8. *The adjunction* $\mathrm{Mod}(\mathbb{T}) \to \mathrm{Met}$ *is monadic.*

Proof. We will apply Beck's monadicity theorem (see eg [9, Theorem 7.1]. We must prove that the forgetful functor has a left adjoint, that both categories are finitely complete, and that the forgetful functor creates coequalizers for those pairs which have split coequalizers in Met.

The first condition we already proved, it's easy to see that the forgetful functor creates limits, taking care of the completeness.

So let's look at the last condition. Let $(l, r) : R \to M$ be a pair of homomorphisms in $\mathrm{Mod}(\mathbb{T})$ A split coequalizer of the underlying metric spaces of this diagram is a diagram like this:



So that $es = 1_Q, se = rt, lt = 1_M$. Note that in this case,

$$d_Q(q, q') = d_Q(es(q), es(q')) \le d_M(s(q), s(q')) \le d_Q(q, q'),$$

so $d_Q(q, q') = d_M(s(q), s(q'))$, and e is surjective with $e(m) = e(m')$ if and only if $rt(m) = rt(m')$.

This implies that Q as a set is the coequalizer of $l, r : R \to M$, and for each point $m \in M$, we can pick out the pair $t(m) \in R$ which identifies m and $s(e(m))$.

We must show that we can equip Q with the structure of an $\mathbb{T}$-model, so that e becomes a homomorphism and so that Q acquires the universal property of coequalizing (l, r).

First, let's define $Q[f]$ for each operation symbol $f \in \mathbb{T}$. Since s is distance nonexpanding, we can certainly define $Q[f](q_1, \dots)$ to be $e(M[f](s(q_1), \dots))$.

Now, suppose $\Gamma \vdash t =_\epsilon s$ is an equation in $\mathbb{T}$. Fix a variable assignment $\alpha : X \to Q$ satisfying Γ. Then postcomposing with s gives an assignment in M which also satisfies Γ (since this is nonexpansive). It's apparent by structural induction on t that $Q[t](\alpha)$, the value of t under α, equals $e(M[t](s\alpha)$. Since M is a model and e is nonexpansive, the equation also holds for Q. Hence Q is a model.

Now, if $(m_1, \dots)$ is an element of M^N (where N is the arity of f), $(t(m_1), t(m_2), \dots)$ gives an element of R^N, so that $R[f](t(m_1), \dots)$ must identify $M[f](m_1, \dots)$ and $M[f](se(m_1), \dots)$. This means $Q[f](e(m_1), \dots) = e(M[f](se(m_1), \dots)) = e(M[f](m_1, \dots))$ (since e equalizes l and r), so that e is a homomorphism for the $\mathbb{T}$-structures on M and Q.

Now suppose given a homomorphism $\phi : M \to A$ which equalizes l and r, for some other model A. We may attempt to define $Q \to A$ as the composite $Q \xrightarrow{s} M \to A$. This will be nonexpansive, and it's straightforward to see that it'll be a homomorphism. On the other hand, if $\hat{\phi} : Q \to A$ is any homomorphism so that $\hat{\phi} e = \phi$, we have $\hat{\phi} = \hat{\phi} er = \phi r$, so this is in fact the unique such morphism. This concludes the proof of monadicity. ◻

Proposition 4.9. *The functor* $\mathrm{Free}^{\mathbb{T}} : \mathrm{Met} \to \mathrm{Met}$ *is countable-arity, i.e it commutes with countably filtered colimits.*

Proof. Let $(X_i)_{i \in I}$ be a diagram in Met over a countably filtered index category I. We must prove

$$\mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i) \to \mathrm{Free}^{\mathbb{T}}(\mathrm{colim}_i\, X_i)$$

is an isomorphism. Let $f : A = (A, d)$ be an operation symbol in $\mathbb{T}$. Note that countable powers commute with countably filtered colimits in Met (this is essentially what it means that the countable metric spaces are the countably small objects in Met), so

$$(\mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i))^A \cong \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i)^A$$

Hence we can equip the colimit with the structure of a model of the signature of $\mathbb{T}$, by defining the interpretation of f to be the map

$$(\mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i))^A \cong \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i)^A \to \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i),$$

where the last map just applies f in each part of the colimit. This amounts to, given a sequence t_a, finding X_i so that they're all present in $\mathrm{Free}^{\mathbb{T}}(X_i)$ (and at the right distance) and just taking $f((t_a))$.

Given an equation $\Gamma \vdash t =_\epsilon s$ in $\mathbb{T}$, let $\alpha : X \to \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i)$ be some assignment validating Γ. Then there's some i so that this factors as a map $X \to \mathrm{Free}^{\mathbb{T}}(X_i)$ which also validates Γ. Up to equivalence, the intepretation of t and s under this assignment in the colimit is just their interpretation under this assignment in $\mathrm{Free}^{\mathbb{T}}(X_i)$. Since this free model satisfies the equation (being a model), and the inclusion of this in the colimit is nonexpansive, the colimit also satisfies the equation. Hence it's a model of $\mathbb{T}$.

The maps $X_i \to \mathrm{Free}^{\mathbb{T}}(X_i)$ induce a map

$$\mathrm{colim}_i\, X_i \to \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i),$$

which since the latter is a model induces a map

$$\phi : \mathrm{Free}^{\mathbb{T}}(\mathrm{colim}_i\, X_i) \to \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i)$$

By initiality, the composite

$$\mathrm{Free}^{\mathbb{T}}(\mathrm{colim}_i\, X_i) \to \mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i) \to \mathrm{Free}^{\mathbb{T}}(\mathrm{colim}_i\, X_i)$$

is the identity. This proves ϕ is an isometry. To see it's surjective, consider some element in $\mathrm{colim}_i\, \mathrm{Free}^{\mathbb{T}}(X_i)$. It's represented by some $t \in \mathrm{Free}^{\mathbb{T}}(X_i)$ for some i, i.e some term using the operations of $\mathbb{T}$ and constant symbols from X_i. Let t' be a term obtained from t by replacing each $[x]$ with $[\bar{x}]$, where $\bar{x} \in \mathrm{colim}_i\, X_i$ is the equivalence class of x. Then up to the equivalence relation in the colimit, $\phi(t') = t$.

This finishes the proof □

Corollary 4.10. *For any equational theory $\mathbb{T}$, there is a* Met*-Lawvere theory with an equivalent category of models.*

5 EQUATIONAL THEORIES FROM LAWVERE THEORIES

Definition 5.1. Let $\mathcal{T}$ be an Met Lawvere theory. Fix a choice of distinct variables $x_1, \ldots$. Given a metric cardinal C, we let $\Gamma(C)$ be the context $\{x_i =_{d_C(i,j)} x_j \mid i, j \in C\}$. Then we can define a theory $\mathbb{T}(\mathcal{T})$, as follows:

(1) The signature Ω^T has a symbol $[f] : C$ for every countable metric cardinal C and every $f \in \mathcal{T}(x^C, x)$.

(2) Given $f, f' \in \mathcal{T}(x^C, x)$, there is an axiom

$$\Gamma(C) \vdash [f](x_1, \ldots) =_{d_{T(C)}(f,f')} [f'](x_1, \ldots),$$

where $x_1, \ldots x$

(3) For $i \in C$, let $\pi_i : x^C \to x$ be the projection to the ith component. Then we have an axiom

$$\Gamma(C) \vdash [\pi_i](x_1, \ldots) =_0 x_i$$

(4) Given $f : x^C \to x$ and a tuple $(g_1, \ldots) \in \mathcal{T}(x^D, x)^C$, we can compose these using the isomorphism $\mathcal{T}(x^D, x)^C = \mathcal{T}(x^D, x^C)$. Denote this composition as $f \circ (g_1, \ldots)$. Then we have an axiom

$$\Gamma(D) \vdash [f]([g_1](x_1, \ldots), [g_2](x_1, \ldots), \ldots) = [f \circ (g_1, \ldots)](x_1, \ldots)$$

Proposition 5.2. $\mathrm{Mod}(\mathbb{T}(\mathcal{T})) \cong \mathrm{Mod}(\mathcal{T})$ *as categories over* Met*. In particular, a strong monad on* Met *comes from a metric equational theory if and only if it is of countable arity.*

Proof. Let M be a model of $\mathcal{T}$. Then we can equip $M(x)$ with the structure of a model of $\mathbb{T}(\mathcal{T})$ as follows: For each operation $f \in \mathcal{T}(x^C, x)$, we define $M(x)[f] : M(x)^C \to M(x)$ simply as $M(f)$, using the canonical isomorphism $M(x)^C \cong M(x^C)$. Since M is assumed to be an enriched functor, the map $\mathcal{T}(x^C, x) \to \mathrm{Met}(M(x)^C, M(x))$ is nonexpansive, and hence this structure satisfies the axioms from part 2. of the definition. Since M preserves powers, it carries the projections to the projections, so it satisfies part 3. And since M is functorial, it preserves the composition, which means it satisfies part 4.

Hence this is a model of $\mathbb{T}(\mathcal{T})$. If $\phi : M \to N$ is a natural transformation, the induced map $\phi_{x^C} : M(x)^C \to N(x)^C$ is given by the C-fold power of ϕ_x. Since ϕ is natural, this implies that ϕ_x is a $\mathbb{T}(\mathcal{T})$-homomorphism, so this construction defines a functor, which we want to show is an isomorphism. Since $\mathrm{Mod}(\mathcal{T})$ is equivalently the Eilenberg-Moore category of the associated monad, the forgetful functor $\mathrm{Mod}(\mathcal{T}) \to \mathrm{Met}$ is an isometry on morphisms. This implies that the functor $\mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathbb{T})$ we've just constructed is an isometry on morphisms as well, so we have to show that it's full and essentially surjective. For fullness, it's clear that being a homomorphism between $M(x)$ and $N(x)$ requires commuting with all the operations in $\mathcal{T}$, which is just what it means to be a natural transformation. For essential surjectivity, let M be some model of $\mathbb{T}(\mathcal{T})$. Define $M(x^C) = M^C$, and given a family $(f_a \in \mathcal{T}(x^C, x))_{a \in A}$ representing a morphism $f : x^C \to x^A$, let $M(f)(m_1, \ldots) = (M[f_a](m_1, \ldots))_{a \in A}$. By axiom 2., if these operations satisfy the distance bounds $d(f_a, f_{a'}) \le d(a, a')$, then so will the resulting map $M(x)^C \to M(x)^A$, and by axiom 3. and 4., this is functorial and preserves powers, so it defines a model. Clearly this model goes to M. □

6 COMPLETENESS

In this section, we will prove the following completeness theorem for our theory.

Theorem 6.1. *Let $\mathbb{T}$ be a theory, and let $\Gamma \vdash \phi$ be a sequent. Suppose every model of $\mathbb{T}$ satisfies this sequent. Then $\Gamma \vdash_{\mathbb{T}} \phi$.*

We will need the following characterization of $\mathbb{T} \sqcup \mathbb{T}(A)$ for A a countable metric space:

Lemma 6.2. *Let A be any metric space. Consider a sequent $\Gamma \vdash \phi$ in the signature of $\mathbb{T} \sqcup \mathbb{T}(A)$. We form the sequent $\Gamma^A \vdash \phi^A$ in the signature of $\mathbb{T}$ by the following procedure:*

(1) If necessary, relabel out some of the variables so that there is an infinite set of unused variables.

(2) Observe that only countably many of the constant symbols $[a], a \in A$ can occur in ϕ. For each of these, choose a distinct variable x_a not used in the sequent. Let ϕ^A be ϕ with each occurrence of $[a]$ replaced by x_a.

(3) Let Γ^A be $\Gamma \cup \{x_a =_{d_A(a,a')} x_{a'} \mid a, a' \in A, d(a, a') < \infty\}$.

Note that some arbitrary choices are involved in defining Γ^A and ϕ^A. Regardless of the choices, $\Gamma^A \vdash_{\mathbb{T}} \phi^A$ if and only if $\Gamma \vdash_{\mathbb{T} \sqcup \mathbb{T}(A)} \phi$.

Proof. First note that because of **Subst**, the arbitrary choices don't affect the provability of $\Gamma^A \vdash \phi^A$. By using **Subst** and **Assum**, it's easy to see the forwards direction:

$$\Gamma^A \vdash_{\mathbb{T}} \phi^A \Rightarrow \Gamma \vdash_{\mathbb{T} \sqcup \mathbb{T}(A)} \phi$$

On the other hand, consider the set of sequents in the signature of $\mathbb{T} \sqcup \mathbb{T}(A)$ so that the left-hand side holds. Clearly this set contains both $\mathbb{T}$ and $\mathbb{T}(A)$, so it suffices to show it's stable under the inference rules.

(1) **Var** and **Assum** are immediate, since $x^A = x$.

(2) To see it's stable under **App**, suppose first that f is an operation in $\mathbb{T}$. Note that $\mathbb{T}$ satisfies **App**, so given the assumption

that $\Gamma^A \vdash_{\mathbb{T}} t_i^A$ ok and so on, we find that $\Gamma^A \vdash_{\mathbb{T}} f(t_i^A)$ ok. But clearly $f(t_i^A) = f(t_i)^A$, so this is just what we wanted. On the other hand, if $f = [a]$ is a nullary symbol, this is automatically true.

An analogous argument proves **Refl**, **Symm**, **Triang**, **Arch**, and **NExp**.

(3) For **Subst**, we must be a bit more careful.

Let $x_i, \delta_{ij}, u_i, s, t, \Gamma$ be as in the assumptions of the inference rule, where the terms are terms over $\mathbb{T} \sqcup \mathbb{T}(A)$. We assume we've further chosen our x_a among variables not occuring in these terms. We are assuming that $\Gamma^A \vdash u_i^A =_{\delta_{ij}} u_j^A$ and $\{x_i =_{\delta_{ij}} x_j \mid i,j\}^A \vdash s^A =_\epsilon t^A$.

Now, $\{x_i =_{\delta_{ij}} x_j\}^A = \{x_i =_{\delta_{ij}} x_j\} \cup \{x_a =_{d(a,a')} x_{a'}\}$. Note that $\Gamma^A \vdash x_a =_{d(a,a')} x_{a'}$ for all a, a'. So by expanding to the set of variables containing both the x_i and the x_a, and taking $u_a = x_a$, we can apply **Subst** for $\mathbb{T}$ to prove that $\Gamma^A \vdash s^A[u_i^A/x_i] =_\epsilon t^A[u_i^A/x_i]$. Now we just observe that $s^A[u_i^A/x_i] = s[u_i/x_i]^A$ and we're done.

□

Proof of the theorem. Consider the tautological variable assignment, given by the identity function $X \to X_\Gamma$. By composing with the unit, we get a variable assignment $X \to X_\Gamma \to \mathrm{Free}^{\mathbb{T}}(X_\Gamma)$. Note that by construction this satisfies the hypotheses of Γ. Hence by assumption, it must satisfy ϕ. But this means that $\vdash_{\mathbb{T}\sqcup\mathbb{T}(X_\Gamma)} \phi[[x]/x]$, which by the lemma means that $\Gamma(X_\Gamma) \vdash_{\mathbb{T}} \phi$, which clearly implies $\Gamma \vdash_{\mathbb{T}} \phi$. □

7 CAUCHY COMPLETION

Many important metric spaces are given as completions of more simply defined subspaces. Thus for example the reals are the completion of the rationals, the $L^p(X,\mu)$ spaces are the completion of the continuous functions on X (in the L^p-metric), and so on.

As we discussed in the introduction, [11], the space of probability measures on X is constructed as the completion of $\mathrm{Free}^{\mathbb{T}}(X)$ for a certain theory $\mathbb{T}$, and the space of closed subsets is the completion of the free semilattice on X (assuming X is compact).

Completion already interacts well with quantitative equational theories, because the operation of completion preserves (finite) products. This means that if $\mathbb{T}$ is a theory (with finitary operations) and M is a model, the completion $\overline{M}$ has a canonical model structure making $M \to \overline{M}$ a homomorphism, and this is universal among homomorphisms from M to complete models. So $\overline{\mathrm{Free}^{\mathbb{T}}(X)}$ has a good universal property.

However, with our expanded theory, we can do even better. The monad $\overline{(-)} : \mathrm{Met} \to \mathrm{Met}$ which carries a metric space to its completion is of countable arity, and so it is represented by a metric equational theory. There are obviously many distinct axiomatizations of this monad — we give one example here:

Definition 7.1. Let N be the natural numbers equipped with the metric $d(n,m) = \frac{1}{2^{\min(n,m)}}$ for $n \neq m$, 0 if they're equal. Let $\mathbb{T}^{\mathrm{comp}}$ be a theory with one operation lim of arity N, and the axioms $\{x_n =_{d(n,m)} x_m \mid n,m \in N\} \vdash \lim(x_1,\dots) =_{1/2^n} x_n$.

Proposition 7.2. *Models of $\mathbb{T}^{\mathrm{comp}}$ are precisely complete metric spaces, $M[\lim] : M^N \to M$ always carries a sequence to a limit of that sequence, and every nonexpansive map between models is a homomorphism. In particular, $\mathrm{Mod}(\mathbb{T}^{\mathrm{comp}})$ is equivalent to the full subcategory of complete metric spaces, and the free model on X is the completion.*

Proof. Let X be a complete metric space. Given a sequence $(x_1,\dots x_n) \in X^N$, note that $d(x_n,x_m) \le 1/2^n$ if $m > n$, so this is a Cauchy sequence. By continuity of the metric, $d(x_n, \lim_i x_i) = \lim_i d(x_n,x_i)$, which is less than or equal $1/2^n$ from a certain point, so $d(x_n,\lim_i x_i) \le 1/2^n$. Thus defining $X[\lim](x_1,\dots) = \lim_i x_i$ makes X a model. On the other hand, any number l satisfying $d(l,x_n) \le 1/2^n$ for all n must clearly be the limit, so this is the only way to make X a model. Since nonexpansive maps are continuous, any nonexpansive map $X \to Y$ between complete metric spaces is a homomorphism between their associated models.

On the other hand, let M be a model of $\mathbb{T}^{\mathrm{comp}}$. Let $a_1,\dots a_n$ be a Cauchy sequence. Then we can find n_1 so that $d(a_{n_1},a_m) \le 1/2$ for all $m > n_1$, $n_2 > n_1$ so that $d(a_{n_2},a_m) \le 1/2^2$ for $m > n_2$, and so on, since it is a Cauchy sequence. Now the subsequence $a_{n_1}, a_{n_2},\dots$ is an element of M^N. Clearly $M[\lim](a_{n_1},\dots)$ is a limit of this subsequence, hence a limit of the original sequence (since it is Cauchy). Hence any model of $\mathbb{T}^{\mathrm{comp}}$ is complete. By the preceding, it $M[\lim]$ must be given by taking limits, and any nonexpansive map between models must be a homomorphism.

This concludes the proof. □

Corollary 7.3. *Let $\mathbb{T}$ be any theory. Then $\mathrm{Mod}(\mathbb{T} \sqcup \mathbb{T}^{\mathrm{comp}})$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathbb{T})$ spanned by the complete models.*

Thus for example, by taking the disjoint union of $\mathbb{T}^{\mathrm{comp}}$ with the theory of p-interpolative barycentric algebras from [11, section 10], we get a theory whose free model on a separable metric space X is the space of Borel probability measures on X in the p-Wasserstein metric.

8 QUANTITATIVE EQUATIONAL THEORIES AS METRIC EQUATIONAL THEORIES

Given a signature where all the arities are discrete, every sequent $\vdash t$ ok is trivially provable by repeated application of **App** and **Var**. In this case, a metric equational theory over this signature is simply a quantitative equational theory, the notion of model is the same, etc.

It is interesting to ask which monads $T : \mathrm{Met} \to \mathrm{Met}$ are axiomatizable by quantitative equational theories. Using our equivalence between metric equational theories and Lawvere theories over Met, we can answer this question.

Proposition 8.1. *A countable-arity monad T is axiomatizable by a quantitative equational theory $\mathbb{T}$, if and only if T preserves surjections (i.e. $T(f) : TX \to TY$ is a surjection whenever $f : X \to Y$ is a surjection.*

Proof. First, let $\mathbb{T}$ be any quantitative equational theory, viewed as a metric equational theory. That is, it is an MET whose operations all have discrete arities. Clearly, for any metric space A, also $\mathbb{T}(A)$ and hence $\mathbb{T} \sqcup \mathbb{T}(A)$ have this property. Then $\vdash_{\mathbb{T}\sqcup\mathbb{T}(A)} t$ ok for *any* term t. Let $f : A \to B$ be a surjection of metric spaces. The induced map $\mathrm{Free}^{\mathbb{T}}(A) \to \mathrm{Free}^{\mathbb{T}}(B)$ is given by replacing each constant symbol $[a]$ in a term t in the theory $\mathbb{T} \sqcup \mathbb{T}(A)$ with $[f(a)]$. Since f is surjective, given a term t in $\mathbb{T} \sqcup \mathbb{T}(B)$, we can always find t' which would be mapped to it by this procedure. And since all terms are well-formed in $\mathbb{T} \sqcup \mathbb{T}(A)$, this t' represents an element of $\mathrm{Free}^{\mathbb{T}}(A)$ which is therefore in the preimage of t. So the map $\mathrm{Free}^{\mathbb{T}}(f)$ is surjective.

Conversely, suppose T is of countable arity and preserves surjections. Given a metric space A, let A^d be the underlying set of A equipped with the discrete metric $d(a,b) = \infty$. Note that there is a surjection (in fact a bijection) $A^d \to A$.

T can be axiomatized by $\mathbb{T}(\mathcal{T}(T))$, but this contains a number of operation symbols of non-discrete arity. For each operation $f : x^A \to x$ where A is not discrete, we can choose a factorization over $x^A \to x^{A^d}$, because the precomposition map $\mathcal{T}(T)(x^{A^d}, x) \to \mathcal{T}(x^A, x)$ is isomorphic to $T(A^d) \to T(A)$ and hence surjective. Choose such a factorization $\bar{f} : x^{A^d} \to x$ for each f. Note that $\Gamma(A) \vdash_{\mathbb{T}(\mathcal{T}(T))} [f](x_1, \dots) = [\bar{f}](x_1, \dots)$.

Let us abbreviate $\mathbb{T} = \mathbb{T}(\mathcal{T}(T))$ Now consider a theory $\mathbb{T}'$ defined as follows:

(1) The operation symbols are the *discrete* operations of $\mathbb{T}(\mathcal{T}(T))$.
(2) Whenever $\Gamma \vdash s =_\epsilon t \in \mathbb{T}$, we let $\Gamma \vdash \bar{s} =_\epsilon \bar{t} \in \mathbb{T}'$, where $\bar{s}, \bar{t}$ denote the result of replacing each occurrence of an operation symbol $[f]$ in s and t with $[\bar{f}]$.
(3) For any term, $\Gamma \vdash t$ ok is in $\mathbb{T}'$.

This set of sequents is clearly stable under the inference rules (since $\mathbb{T}$ is). The signature of $\mathbb{T}'$ is a subset of the signature of $\mathbb{T}$, so any term in the former is also a term in the latter. Note that if $\Gamma \vdash_{\mathbb{T}'} s =_\epsilon t$, then there are some terms s', t' in $\mathbb{T}$ so that $\Gamma \vdash_{\mathbb{T}} s' =_\epsilon t'$ and $\bar{s'} = s, \bar{t'} = t$. But by repeatedly using the equation $\Gamma(A) \vdash [f](x_1, \dots) = [\bar{f}](x_1, \dots)$, (where A is the arity of f), we find that we must have $\Gamma \vdash s' = s$ and similarly for $t' = t$, and hence we must also have $\Gamma \vdash_{\mathbb{T}} s =_\epsilon t$.

Hence we get a natural forgetful functor $U : \mathrm{Mod}(\mathbb{T}) \to \mathrm{Mod}(\mathbb{T}')$ over Met. It suffices to show this is an equivalence of categories. It is clearly faithful, because the composite $\mathrm{Mod}(\mathbb{T}) \to \mathrm{Mod}(\mathbb{T}') \to \mathrm{Met}$ is faithful. Given two models $M, N \in \mathrm{Mod}(\mathbb{T})$, and a homomorphism $\phi : UM \to UN$, note that

$$\phi(M[f](x_1, \dots)) = \phi(M[\bar{f}](x_1, \dots))$$

$$= N[\bar{f}](\phi(x_1), \dots) = N[f](\phi(x_1), \dots),$$

so ϕ is already a homomorphism $M \to N$. Hence the forgetful functor is full.

Finally, given a model M' of $\mathbb{T}'$, defining M with the same underlying metric space and $M[f] = M'[\bar{f}]$ gives a model of $\mathbb{T}$ with $UM = M'$. Hence the functor is an equivalence of categories, finishing the proof. □

We can also ask which monads correspond to theories with only finitary operations. Since Met is not locally finitely presentable, there is no clean correspondence between finitary monads and finitary Lawvere theories. And for similar reasons, many plausible characterizations of the monads presented by "finitary METs" fail. We will give a few counterexamples to demonstrate the problem.

Example 8.2. Let X_i be the set consisting of two points a and b at distance $1 + 1/i$, considered as a diagram indexed by the category $(\mathbb{N}, \leq)$ (with identities as the structure morphisms). Then $\mathrm{colim}_i X_i$ consists of two points at distance 1.

(1) Consider the theory $\mathbb{T}_1$ with one binary operation symbol $f : (\{x,y\}, d(x,y) = 1)$. Then $\mathrm{Free}^{\mathbb{T}_1}(X_i) \cong X_i$, but $\mathrm{Free}^{\mathbb{T}_1}(\mathrm{colim}_i X_i)$ has three points, a, b and $f(a,b)$.
(2) Consider the theory $\mathbb{T}_2$ with no operations, and one equation $x =_1 y \vdash x =_0 y$. Then $\mathrm{Free}^{\mathbb{T}_2}(X_i) \cong X_i$, but

$$\mathrm{Free}^{\mathbb{T}_2}(\mathrm{colim}_i X_i) = \{*\}.$$

Thus, neither of these theories axiomatize monads which are finitary, in the sense that they commute with (finitely) filtered colimits.

The problem in both cases is that quantitative equations - whether as preconditions for the application of an operation, or preconditions for another equation - are not "finitary", do not commute with finitely filtered colimits. An analogous problem prevents them from being "strongly finitary" in the sense studied in [2].

This problem seems to depend in an essential way on the *discontinuity* of these conditions. The equation $x =_0 y$ appears "suddenly" once $x =_1 y$. Thus, for example, we can consider the "theory of contractions", having one unary symbol s and the equations $x =_{2\epsilon} y \vdash sx =_\epsilon sy$ for every ϵ. This is finitary in the sense that the associated monad (which is simply $X \mapsto X \times \mathbb{N}$ equipped with the metric $d((x,n),(y,n)) = d(x,y)2^{-n}, d((x,n),(y,m) = 0$ if $n \neq m$) commutes with filtered colimits.

In [2], recognizing essentially this problem, the notion of *strongly finitary functor* Met $\to$ Met is studied. These are functors F which equal the enriched left Kan extension of their restriction to the subcategory of finite and discrete metric spaces. This intuitively corresponds to allowing only finite and discrete-arity operations, and allowing only axioms of the form $\vdash t =_\epsilon s$. When restricted to ultrametric spaces, this is indeed the case - strongly finitary monads are exactly those presented by QETs of this form ([2, Theorem 5.8]).

However, this correspondence does *not* hold, as the following example shows:

Example 8.3. Let $\mathbb{T}$ be the theory with one binary operation f, two unary operations g, g', and the axiom $g =_1 g'$. Then $\mathrm{Free}^{\mathbb{T}} :$ Met $\to$ Met is not strongly finitary.

To see this, recall that the left Kan extension under consideration is given by the coend formula

$$\mathrm{Lan}_{i:\mathsf{Fin}\to\mathsf{Met}}\,\mathrm{Free}^{\mathbb{T}}(X) = \int^F X^F \otimes \mathrm{Free}^{\mathbb{T}}(F)$$

This means the question is whether the map

$$\int^F X^F \otimes \mathrm{Free}^{\mathbb{T}}(F) \to \mathrm{Free}^{\mathbb{T}}(X),$$

which carries a pair $(\alpha : F \to X, t \in \mathrm{Free}^{\mathbb{T}}(F))$ to $\mathrm{Free}^{\mathbb{T}}(\alpha)(t)$, is an isomorphism for all X.

Let $X = \{x_1, x_2, x_3\}$ with $d(x_1, x_2) = 1, d(-, x_3) = \infty$. Consider the two terms $f([x_1], g([x_3])), f([x_2], g'([x_3])) \in \mathrm{Free}^{\mathbb{T}}(X)$. Applying the axioms $\vdash [x_1] =_1 [x_2]$ and $\vdash g(y) =_1 g'(y)$, and nonexpansiveness, clearly these have distance at most 1.

Now consider the points

$$(\{a,b\}, \alpha, [f([a], g([b]))]), (\{a,b\}, \beta, [f([a], g'([b]))])$$

in the coend, where $\alpha(a) = x_1, \alpha(b) = x_3, \alpha'(a) = x_2, \alpha'(a) = x_3$. Clearly $d_{X^{\{a,b\}}}(\alpha, \beta) = 1$, and also

$$d([f([a], g([b]))], [f([a], g'([b]))]) = 1.$$

Hence the distance of these points in the tensor product is 2. (It is not too difficult to see that the distance in the coend, which is a quotient of the coproduct of all these tensor products, is not less than 2).

9 CONCLUSIONS AND FUTURE WORK

Approximation is fundamental in mathematics and computer science and motivates the extension of universal algebra from the category Set to the category Met. There are two current approaches - Quantitative Equational Theories and Enriched Lawvere Theories but neither is a complete answer. QETs produce sound and complete systems for a notion of algebraic structure but that notion is too weak to cover key examples such as Cauchy Completion. On the other hand, Enriched Lawvere Theories provide the right theoretical framework but are not accompanied by sound and complete proofs systems needed to establish distances between terms in specific theories. This paper offers the best of both worlds by taking the best of each approach, producing what we call Metric Equational Theories. The fundamental innovation is the inclusion of metric arities for operators (motivated from the Enriched Lawvere Theory framework) within METs.

There are a number of directions of future work and we highlight some here. Firstly, Enriched Lawvere Theories don't give sound and complete proof systems for the the equations of a theory. But in the case of metric spaces we showed such systems exist. Can we find conditions under which such systems exist for a broad class of Enriched Lawvere Theories? Secondly, going beyond equational theories we might ask what stronger systems look like. For example, equational theories correspond to finite product theories, but there is a natural notion of finite limit theory. The question is thus how do we extend this paper to develop finite limit theories for metric spaces? Thirdly, and from a different perspective, algebraic theories underpin effectful programming languages. So how can we use the work contained in this paper to create effectful programming language constructs for approximate computation?

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