

A COMPLETE QUANTITATIVE DEDUCTION SYSTEM FOR THE BISIMILARITY DISTANCE ON MARKOV CHAINS

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ABSTRACT. In this paper we propose a complete axiomatization of the bisimilarity distance of Desharnais et al. for the class of finite labelled Markov chains.

Our axiomatization is given in the style of a quantitative extension of equational logic recently proposed by Mardare, Panangaden, and Plotkin (LICS 2016) that uses equality relations $t \equiv_\varepsilon s$ indexed by rationals, expressing that “ t is approximately equal to s up to an error ε ”. Notably, our quantitative deduction system extends in a natural way the equational system for probabilistic bisimilarity given by Stark and Smolka by introducing an axiom for dealing with the Kantorovich distance between probability distributions.

The axiomatization is then used to propose a metric extension of a Kleene’s style representation theorem for finite labelled Markov chains, that was proposed (in a more general coalgebraic fashion) by Silva et al. (Inf. Comput. 2011).

1. INTRODUCTION

In [Kle56], Kleene presented an algebra of *regular events* with the declared objective of “*showing that all and only regular events can be represented by [...] finite automata.*” This fundamental correspondence is known as Kleene’s representation theorem for regular languages. Kleene’s approach was essentially equational, but he did not provide a proof of completeness

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for his equational characterization. The first sound and complete axiomatization for proving equivalence of regular events is due to Salomaa [Sal66], later refined by Kozen [Koz91].

The above programme was applied by Milner [Mil84] to process behaviors and non-deterministic labelled transition system. Milner’s algebra of process behaviors consists of a prefix operator (representing the observation of an atomic event), a non-deterministic choice operator (for union of behaviors), and a recursive operator (for the definition of recursive behaviors). He proved an analogue of Kleene’s theorem, showing that process behaviors represent all and only finite labelled transition system up to bisimilarity. Milner also provided a sound and complete axiomatization for behavior expressions, with the property that two expression are provably equal iff they represent bisimilar labelled transition systems.

Stark and Smolka [SS00] extended Milner’s axiomatization to probabilistic process behaviors, providing a complete axiomatization for probabilistic bisimilarity on (generative) labelled Markov chains. The key idea was to represent probabilistic non-determinism by a probabilistic choice operator for the convex combination of probabilistic behaviors. Their equational axiomatization differs from Milner’s only by replacing the semi-lattices axioms for non-deterministic choice by the Stone’s barycentric axioms for probabilistic choice.

Similar extensions to Milner’s axiomatization have been investigated by several authors. Amongst them we recall some of the works that have been done on probabilistic systems: Bandini and Segala [BS01] on simple probabilistic automata; Mislove, Ouaknine, and Worrell [MOW04] on (fully) non-deterministic probabilistic automata; Deng and Palamidessi [DP07] axiomatizing probabilistic weak-bisimulation and behavioral equivalence on Segala and Lynch’s probabilistic automata; and Silva et al. [SBBR11, BMS13] providing a systematic way to generate sound and complete axiomatizations and Kleene’s representation theorems thereof, for a wide variety of systems represented as coalgebras.

The attractiveness towards sound and complete axiomatizations for process behaviors comes from the fact that one can reason about their equivalence in a purely syntactic fashion by means of classical logical deduction of valid equational statements.

Jou and Smolka [JS90], however, observed that for reasoning about the behavior of probabilistic systems (and more in general all type of quantitative systems) a notion of distance is preferable to that of equivalence, since the latter is not robust w.r.t. small variations of numerical values. This motivated the development of metric-based approximated semantics for probabilistic systems, initiated by Desharnais et al. [DGJP04] on labelled Markov chains and greatly developed and explored by van Breugel, Worrell and others [vBW01, vBW06, vBSW08, BBLM15b, BBLM15a]. It consists in proposing a pseudometric which measures the dissimilarities between quantitative behaviors. The pseudometric proposed by Desharnais et al. [DGJP04] on labelled Markov chains, a.k.a. *probabilistic bisimilarity distance*, is defined as the least solution of a functional operator on 1-bounded pseudometrics based on the Kantorovich distance on probability distributions.

The first proposals of sound and complete axiomatizations of behavioral distances are due to Larsen et al. [LFT11] on weighted transition systems, and D’Argenio et al. [DGL14] on probabilistic systems. These approaches, however, are rather specific and based on *ad hoc* assumptions. Recently, Mardare, Panangaden, and Plotkin [MPP16] —with the purpose of developing a general research programme for a quantitative algebraic theory of effects [PP01]—proposed the concepts of *quantitative equational theory* and *quantitative algebra* as models for these theories. The key idea behind their approach is to use “quantitative equations” of the form $t \equiv_\varepsilon s$ annotated with a rational number $\varepsilon \geq 0$ to be interpreted as “ t is approximately equal to s up to an error ε ”. Their main result is that completeness for a quantitative

theory always holds on the freely-generated algebra of terms equipped with a metric that is freely-induced by the axioms. Due to this result, they were able to prove soundness and completeness theorems for many interesting axiomatizations, such as the Hausdorff metric, the total variation metric, the p -Wasserstein metric, and the Kantorovich metric.

In this paper, we contribute to the quest of axiomatizations of behavioral metrics, by proposing a quantitative deduction system in the sense of [MPP16], that is proved to be sound and complete w.r.t. the probabilistic bisimilarity distance of Desharnais et al. on labelled Markov chains. The proposed axiomatization extends Stark and Smolka’s one [SS00] with three additional axioms —the last of these borrowed from [MPP16]— for expressing: (1) 1-boundedness of the metric; (2) non-expansivity of the prefix operator; and (3) the Kantorovich lifting of a distance to probability distributions.

The resulting axiomatization is simpler than the one presented in [DGL14] for probabilistic transition systems and it extends [DGL14] by allowing recursive behaviors.

An important detail about our axiomatization that should be mentioned since the introduction, is that the proposed quantitative deduction system do not fully adhere to the conditions required to fit within the quantitative algebraic framework of [MPP16]. Indeed, one of the conditions that we do not satisfy is soundness w.r.t. non-expansivity for the recursion operator. To overcome this problem we propose to relax the original notion of quantitative deduction system from [MPP16] by not requiring non-expansivity of the algebraic operators. In this way we could not use the general proof technique of [MPP16] to obtain the completeness theorem, and we needed to appeal to specific properties of the distance. The property used to prove completeness is ω -cocontinuity (i.e., preservation of *infima* of countable decreasing chains) of the functional operator defining the distance. Interestingly, the proof technique proposed in this paper appears to be generic on the used functional operator, provided its ω -cocontinuity. This generality is showed on a specific example, where we prove soundness and completeness for another (although similar) quantitative deduction system w.r.t. the *discounted* bisimilarity distance of Desharnais et al.

As a final result we prove a metric analogue of Kleene’s representation theorem for finite labelled Markov chains. Specifically, we show that the class of expressible behaviors correspond, up to bisimilarity, to the class of finite labelled Markov chains. Moreover, if the set of expressions is equipped with the pseudometric that is freely-induced by the axioms, this correspondence is metric invariant. Note that this establishes a stronger correspondence than the usual “equational” Kleene’s theorem.

This work is an extended version of the conference paper [BBLM16]. In comparison to [BBLM16], this paper includes new and more detailed examples, improved proofs of the soundness and completeness of the axiomatization, and two new results: (i) a sound and complete axiomatization for the discounted bisimilarity distance (Section 6) and (ii) a quantitative Kleene’s representation theorem for finite (open) Markov chains (Section 7).

Synopsis. Section 2 introduces the notation and the preliminary basic concepts. In Section 3 we recall the quantitative equational framework of [MPP16]. Section 4 presents the algebra of open Markov chains. In Section 5 we present a quantitative deduction system (Section 5.3) that is proved to be sound (Section 5.4) and complete (Section 5.5) w.r.t. the probabilistic bisimilarity distance of Desharnais et al.. Section 6 shows how to extend these results to the discounted version of the bisimilarity distance. In Section 7 we present a quantitative Kleene’s representation theorem of finite open Markov chains. Finally, in Section 8 we conclude and present suggestions for future work.

2. PRELIMINARIES AND NOTATION

For $R \subseteq X \times X$ an equivalence relation, we denote by X/R its quotient set. For two sets X and Y , we denote by $X \uplus Y$ their disjoint union and by $[X \rightarrow Y]$ (or alternatively, Y^X) the set of all functions from X to Y .

A *discrete sub-probability* on X is a function $\mu: X \rightarrow [0, 1]$, such that $\mu(X) \leq 1$, where, for $E \subseteq X$, $\mu(E) = \sum_{x \in E} \mu(x)$; it is a *probability distribution* if $\mu(X) = 1$. The support of μ is the set $\text{supp}(\mu) = \{x \in X \mid \mu(x) > 0\}$. We denote by $\Delta(X)$ and $\mathcal{D}(X)$ the set of discrete probability and finitely-supported sub-probability distributions on X , respectively.

A 1-bounded *pseudometric* on X is a function $d: X \times X \rightarrow [0, 1]$ such that, for any $x, y, z \in X$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$; d is a *metric* if, in addition, $d(x, y) = 0$ implies $x = y$. The pair (X, d) is called *(pseudo)metric space*. The kernel of a (pseudo)metric d is the set $\ker(d) = \{(x, y) \mid d(x, y) = 0\}$.

3. QUANTITATIVE ALGEBRAS AND THEIR EQUATIONAL THEORIES

We recall the notions of quantitative equational theory and quantitative algebra from [MPP16].

Let Σ be an algebraic signature of function symbols $f: n \in \Sigma$ of arity $n \in \mathbb{N}$. Fix a countable set of *metavariables* X , ranged over by $x, y, z, \dots \in X$. We denote by $\mathbb{T}(\Sigma, X)$ the set of Σ -terms freely generated over X ; terms will be ranged over by t, s, u, \dots . A *substitution of type Σ* is a function $\sigma: X \rightarrow \mathbb{T}(\Sigma, X)$ that is homomorphically extended to terms as $\sigma(f(t_1, \dots, t_n)) = f(\sigma(t_1), \dots, \sigma(t_n))$; by $\mathcal{S}(\Sigma)$ we denote the set of substitutions of type Σ .

A *quantitative equation of type Σ* is an expression of the form $t \equiv_\varepsilon s$, where $t, s \in \mathbb{T}(\Sigma, X)$ and $\varepsilon \in \mathbb{Q}_+$. Let $\mathcal{E}(\Sigma)$ denote the set of quantitative equations of type Σ . The subsets of $\mathcal{E}(\Sigma)$ will be ranged over by $\Gamma, \Theta, \Pi, \dots \subseteq \mathcal{E}(\Sigma)$.

Let $\vdash \subseteq 2^{\mathcal{E}(\Sigma)} \times \mathcal{E}(\Sigma)$ be a binary relation from the powerset of $\mathcal{E}(\Sigma)$ to $\mathcal{E}(\Sigma)$. We write $\Gamma \vdash t \equiv_\varepsilon s$ if $(\Gamma, t \equiv_\varepsilon s) \in \vdash$, and $\Gamma \not\vdash t \equiv_\varepsilon s$ otherwise; by $\vdash t \equiv_\varepsilon s$ we denote $\emptyset \vdash t \equiv_\varepsilon s$, and by $\Gamma \vdash \Theta$ we mean that $\Gamma \vdash t \equiv_\varepsilon s$, for all $t \equiv_\varepsilon s \in \Theta$. The relation \vdash is called *quantitative deduction system of type Σ* if it satisfies the following axioms and rules

- (Refl) $\vdash t \equiv_0 t$,
- (Symm) $\{t \equiv_\varepsilon s\} \vdash s \equiv_\varepsilon t$,
- (Triang) $\{t \equiv_\varepsilon u, u \equiv_{\varepsilon'} s\} \vdash t \equiv_{\varepsilon+\varepsilon'} s$,
- (Max) $\{t \equiv_\varepsilon s\} \vdash t \equiv_{\varepsilon+\varepsilon'} s$, for all $\varepsilon' > 0$,
- (Arch) $\{t \equiv_{\varepsilon'} s \mid \varepsilon' > \varepsilon\} \vdash t \equiv_\varepsilon s$,
- (NExp) $\{t_1 \equiv_\varepsilon s_1, \dots, t_n \equiv_\varepsilon s_n\} \vdash f(t_1, \dots, t_n) \equiv_\varepsilon f(s_1, \dots, s_n)$, for all $f: n \in \Sigma$,
- (Subst) If $\Gamma \vdash t \equiv_\varepsilon s$, then $\sigma(\Gamma) \vdash \sigma(t) \equiv_\varepsilon \sigma(s)$, for all $\sigma \in \mathcal{S}(\Sigma)$,
- (Cut) If $\Gamma \vdash \Theta$ and $\Theta \vdash t \equiv_\varepsilon s$, then $\Gamma \vdash t \equiv_\varepsilon s$,
- (Assum) If $t \equiv_\varepsilon s \in \Gamma$, then $\Gamma \vdash t \equiv_\varepsilon s$.

where $\sigma(\Gamma) = \{\sigma(t) \equiv_\varepsilon \sigma(s) \mid t \equiv_\varepsilon s \in \Gamma\}$.

The rules (Subst), (Cut), (Assum) are the classical deduction rules from equational logic. The axioms (Refl), (Symm), (Triang) correspond, respectively, to reflexivity, symmetry, and triangular inequality for a pseudometric; (Max) represents inclusion of neighborhoods of increasing diameter; (Arch) is the Archimedean property of the reals w.r.t. a decreasing

chain of neighborhoods with converging diameters; and (**NExp**) stands for non-expansivity of the algebraic operators $f \in \Sigma$.

A *quantitative equational theory* is a set \mathcal{U} of universally quantified *quantitative inferences*, (i.e., expressions of the form

$$\{t_1 \equiv_{\varepsilon_1} s_1, \dots, t_n \equiv_{\varepsilon_n} s_n\} \vdash t \equiv_{\varepsilon} s,$$

with a finite set of hypotheses) closed under \vdash -deducibility. A set \mathcal{A} of quantitative inferences is said to axiomatize a quantitative equational theory \mathcal{U} , if \mathcal{U} is the smallest quantitative equational theory containing \mathcal{A} . A theory \mathcal{U} is called *inconsistent* if $\vdash x \equiv_0 y \in \mathcal{U}$, for distinct metavariables $x, y \in X$, it is called *consistent* otherwise¹.

The models of quantitative equational theories are standard Σ -algebras equipped with a pseudometric, called *quantitative algebras*.

Definition 3.1 (Quantitative Algebra). A *quantitative Σ -algebra* is a tuple $\mathcal{A} = (A, \Sigma^{\mathcal{A}}, d^{\mathcal{A}})$, consisting of a pseudometric space $(A, d^{\mathcal{A}})$ and a set $\Sigma^{\mathcal{A}} = \{f^{\mathcal{A}}: A^n \rightarrow A \mid f: n \in \Sigma\}$ of *interpretations* for the function symbols in Σ , required to be non-expansive w.r.t. $d^{\mathcal{A}}$, i.e., for all $1 \leq i \leq n$ and $a_i, b_i \in A$, $d^{\mathcal{A}}(a_i, b_i) \geq d^{\mathcal{A}}(f^{\mathcal{A}}(a_1, \dots, a_n), f^{\mathcal{A}}(b_1, \dots, b_n))$.

Morphisms of quantitative algebras are non-expansive homomorphisms.

A quantitative algebra $\mathcal{A} = (A, \Sigma^{\mathcal{A}}, d^{\mathcal{A}})$ *satisfies* the quantitative inference $\Gamma \vdash t \equiv_{\varepsilon} s$, written $\Gamma \models_{\mathcal{A}} t \equiv_{\varepsilon} s$, if for any assignment of the meta-variables $\iota: X \rightarrow A$,

$$\left(\text{for all } t' \equiv_{\varepsilon'} s' \in \Gamma, d^{\mathcal{A}}(\iota(t'), \iota(s')) \leq \varepsilon'\right) \quad \text{implies} \quad d^{\mathcal{A}}(\iota(t), \iota(s)) \leq \varepsilon,$$

where, for a term $t \in \mathbb{T}(\Sigma, X)$, $\iota(t)$ denotes the homomorphic interpretation of t in \mathcal{A} . A quantitative algebra \mathcal{A} is said to *satisfy* (or is a *model* for) the quantitative theory \mathcal{U} , if whenever $\Gamma \vdash t \equiv_{\varepsilon} s \in \mathcal{U}$, then $\Gamma \models_{\mathcal{A}} t \equiv_{\varepsilon} s$. The collection of all models of a theory \mathcal{U} of type Σ , is denoted by $\mathbb{K}(\Sigma, \mathcal{U})$.

In [MPP16] it is shown that any quantitative theory \mathcal{U} has a universal model $\mathcal{T}_{\mathcal{U}}$ (the freely generated \vdash -model) satisfying exactly those quantitative equations belonging to \mathcal{U} . Moreover, [MPP16, Theorem 5.2] proves a completeness theorem for quantitative equational theories \mathcal{U} , stating that a quantitative inference is satisfied by all the algebras satisfying \mathcal{U} if and only if it belongs to \mathcal{U} .

Theorem 3.2 (Completeness). *For any given quantitative equational theory \mathcal{U} of type Σ ,*

$$\left(\text{for any } \mathcal{A} \in \mathbb{K}(\Sigma, \mathcal{U}), \Gamma \models_{\mathcal{A}} t \equiv_{\varepsilon} s\right) \quad \text{if and only if} \quad \Gamma \vdash t \equiv_{\varepsilon} s \in \mathcal{U}.$$

Furthermore, in [MPP16] several interesting examples of quantitative equational theories have been proposed. The one we will focus on later in this paper is the so called interpolative barycentric equational theory (cf. §10 in [MPP16]).

¹Note that for an inconsistent theory \mathcal{U} , by **Subst**, we have $\vdash t \equiv_0 s \in \mathcal{U}$, for all $t, s \in \mathbb{T}(\Sigma, X)$.

4. THE ALGEBRA OF PROBABILISTIC BEHAVIORS

Recall from the introduction that the aim of the paper is to study the quantitative algebraic properties of two different behavioral pseudometrics on Markov chains, namely the probabilistic bisimilarity distance and the total variation distance. This will be done by employing the framework of quantitative algebras and their equational theories. For the moment we will focus only on the purely algebraic part, leaving to later sections the definition of the pseudometrics and the quantitative equational theories.

In this section we present the algebra of open Markov chains. Open Markov chains extend the familiar notion of discrete-time labelled Markov chain with “open” states taken from a fixed countable set \mathcal{X} of names ranged over by $X, Y, Z, \dots \in \mathcal{X}$. Names indicate states at which the behavior of the Markov chain can be extended by substitution of another Markov chain, in a way which will be made precise later.

4.1. Open Markov Chains. In what follows we fix a countable set \mathcal{L} of labels, ranged over by $a, b, c, \dots \in \mathcal{L}$. Recall that $\mathcal{D}(M)$ denotes the set of *finitely supported* discrete sub-probability distributions over a set M .

Definition 4.1 (Open Markov Chain). An *open Markov chain* $\mathcal{M} = (M, \tau)$ consists of a set M of *states* and a *transition probability function* $\tau: M \rightarrow \mathcal{D}((\mathcal{L} \times M) \uplus \mathcal{X})$.

Intuitively, if \mathcal{M} is in a state $m \in M$, then with probability $\tau(m)(a, n)$ it emits $a \in \mathcal{L}$ and moves to state $n \in M$, or it moves with probability $\tau(m)(X)$ to a name $X \in \mathcal{X}$ without emitting any label. A state $m \in M$ with probability zero of emitting any label and moving to any name, is called *terminating*. A name $X \in \mathcal{X}$ is said to be *unguarded* in a state $m \in M$, if $\tau(m)(X) > 0$. Clearly, open Markov chains where all names are guarded in each state (i.e., for all $m \in M$, $\tau(m)(\mathcal{X}) = 0$) are just standard labelled sub-probabilistic Markov chains.

A *pointed open Markov chain*, denoted by (\mathcal{M}, m) , is an open Markov chain $\mathcal{M} = (M, \tau)$ with a distinguished *initial state* $m \in M$.

Hereafter, we will use $\mathcal{M} = (M, \tau)$, $\mathcal{N} = (N, \theta)$ to range over open Markov chains and (\mathcal{M}, m) , (\mathcal{N}, n) to range over the set **OMC** of pointed open Markov chains. To ease the reading, we will often refer to the constituents of \mathcal{M} and \mathcal{N} implicitly, so that we will try to keep this notation consistent as much as possible along the paper.

For the definition of the algebra we will need to consider open Markov chains up to *probabilistic bisimilarity*. Next we recall its definition, due to Larsen and Skou [LS89]².

Definition 4.2 (Bisimulation). An equivalence relation $R \subseteq M \times M$ is a *bisimulation* on \mathcal{M} if whenever $m R m'$, then, for all $a \in \mathcal{L}$, $X \in \mathcal{X}$ and $C \in M/R$,

- (i) $\tau(m)(X) = \tau(m')(X)$,
- (ii) $\tau(m)(\{a\} \times C) = \tau(m')(\{a\} \times C)$.

Two states $m, m' \in M$ are *bisimilar* w.r.t. \mathcal{M} , written $m \sim_{\mathcal{M}} m'$, if there exists a bisimulation relation on \mathcal{M} relating them.

Intuitively, two states are bisimilar if they have the same probability of (i) moving to a name $X \in \mathcal{X}$ and (ii) emitting a label $a \in \mathcal{L}$ and moving to the same bisimilarity class.

In the following we consider bisimilarity between pointed open Markov chains. Two pointed open Markov chains (\mathcal{M}, m) , $(\mathcal{N}, n) \in \mathbf{OMC}$ are bisimilar, written $(\mathcal{M}, m) \sim (\mathcal{N}, n)$,

²The original definition of probabilistic bisimulation was given for classical labelled Markov chains. Definition 4.2 is a straightforward generalization of the same concept for open Markov chains.

if m and n are bisimilar w.r.t. the disjoint union of \mathcal{M} and \mathcal{N} (denoted by $\mathcal{M} \oplus \mathcal{N}$) defined as expected. One can readily see that $\sim \subseteq \mathbf{OMC} \times \mathbf{OMC}$ is an equivalence.

4.2. An Algebra of Open Markov Chains. Next we turn to a simple algebra of pointed Markov chains. The signature of the algebra is defined as follows,

$$\begin{aligned} \Sigma &= \{X: 0 \mid X \in \mathcal{X}\} \cup && \text{(NAMES)} \\ &\{a.(\cdot): 1 \mid a \in \mathcal{L}\} \cup && \text{(PREFIX)} \\ &\{+_e: 2 \mid e \in [0, 1]\} \cup && \text{(PROBABILISTIC CHOICE)} \\ &\{\mathbf{rec} X: 1 \mid X \in \mathcal{X}\}, && \text{(RECURSION)} \end{aligned}$$

consisting of a constant X for each name in \mathcal{X} ; a prefix $a.\cdot$ and a recursion $\mathbf{rec} X$ unary operators, for each $a \in \mathcal{L}$ and $X \in \mathcal{X}$; and a probabilistic choice $+_e$ binary operator for each $e \in [0, 1]$. For $t \in \mathbb{T}(\Sigma, M)$, $fn(t)$ denotes the set of free names in t , where the notions of *free* and *bound name* are defined in the standard way, with $\mathbf{rec} X$ acting as a binding construct. A term is *closed* if it does not contain any free variable. Throughout the paper we consider two terms as syntactically identical if they are identical up to renaming of their bound names (α -equivalence). For $t, s_1, \dots, s_n \in \mathbb{T}(\Sigma, M)$ and an n -vector $\bar{X} = (X_1, \dots, X_n)$ of distinct names, $t[\bar{s}/\bar{X}]$ denotes the simultaneous *capture avoiding substitution* of X_i in t with s_i , for $i = 1, \dots, n$. A name X is *guarded*³ in a term t if every free occurrence of X in t occurs within a context the following forms: $a.[\cdot]$, $s +_1 [\cdot]$, or $[\cdot] +_0 s$.

Since from now on we will only refer to terms constructed over the signature Σ , we will simply write $\mathbb{T}(M)$ and \mathbb{T} , in place of $\mathbb{T}(\Sigma, M)$ and $\mathbb{T}(\Sigma, \emptyset)$, respectively.

To give the interpretation of the operators in Σ , we define an operator \mathbb{U} on open Markov chains, taking \mathcal{M} to the open Markov chain $\mathbb{U}(\mathcal{M}) = (\mathbb{T}(M), \mu_{\mathcal{M}})$, where the transition probability function $\mu_{\mathcal{M}}$ is defined as the least solution (over the complete partial order of the set of all functions mapping elements in $\mathbb{T}(M)$ to a $[0, 1]$ -valued functions from $(\mathcal{L} \times \mathbb{T}(M)) \uplus \mathcal{X}$, ordered point-wise) of the recursive equation

$$\mu_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}(\mu_{\mathcal{M}}).$$

The functional operator $\mathcal{P}_{\mathcal{M}}$ is defined by structural induction on $\mathbb{T}(M)$, for arbitrary functions $\theta: \mathbb{T}(M) \rightarrow [0, 1]$, as follows:

$$\begin{aligned} \mathcal{P}_{\mathcal{M}}(\theta)(m) &= \tau(m) & \mathcal{P}_{\mathcal{M}}(\theta)(a.t) &= \mathbb{1}_{\{(a,t)\}} \\ \mathcal{P}_{\mathcal{M}}(\theta)(X) &= \mathbb{1}_{\{X\}} & \mathcal{P}_{\mathcal{M}}(\theta)(t +_e s) &= e\theta(t) + (1 - e)\theta(s) \\ & & \mathcal{P}_{\mathcal{M}}(\theta)(\mathbf{rec} X.t) &= \theta(t[\mathbf{rec} X.t/X]), \end{aligned}$$

where $\mathbb{1}_E$ denotes the characteristic function of the set E .

Notice that, requiring $\mu_{\mathcal{M}}$ to be the *least* solution is essential for ensuring it to be a proper transition probability function, *i.e.*, that for all $t \in \mathbb{T}(M)$, $\mu_{\mathcal{M}}(t) \in \mathcal{D}((\mathcal{L} \times \mathbb{T}(M)) \uplus \mathcal{X})$. We would also like point up that, for all $X \in \mathcal{X}$, the above definition renders $\mathbf{rec} X.X$ a terminating state in $\mathbb{U}(\mathcal{M})$, that is, $\mu_{\mathcal{M}}(\mathbf{rec} X.X)(\mathcal{L} \times \mathbb{T}(M)) = 0$ and $\mu_{\mathcal{M}}(\mathbf{rec} X.X)(\mathcal{X}) = 0$.

Remark 4.3. The definition of $\mu_{\mathcal{M}}$ corresponds essentially to the transition probability of the operational semantics of probabilistic processes given by Stark and Smolka in [SS00]. The only difference with their semantics is that the one above is defined over generic terms

³This notion, coincides with the one in [SS00], though our definition may seem more involved due to the fact that we allow the probabilistic choice operators $+_e$ with e ranging in the closed interval $[0, 1]$.

in $\mathbb{T}(M)$ rather than just in \mathbb{T} . Moreover, our formulation simplifies theirs by skipping the definition of a labelled transition system. For a detailed discussion about the well-definition of $\mu_{\mathcal{M}}$ we refer the interested reader to [AÉI02] and [SS00].

Definition 4.4 (Universal open Markov chain). Let \mathcal{M} be an open Markov chain. The *universal open Markov chain* w.r.t. \mathcal{M} is given by $\mathbb{U}(\mathcal{M})$.

The reason why it is called universal will be clarified soon. As for now, just notice that $\mathbb{U}(\mathcal{M}_\emptyset)$, where $\mathcal{M}_\emptyset = (\emptyset, \tau_\emptyset)$ is the open Markov chain with τ_\emptyset the empty transition function, has \mathbb{T} as the set of states and that its transition probability function corresponds to the one defined in [SS00]. To ease the notation we will denote $\mathbb{U}(\mathcal{M}_\emptyset)$ as $\mathbb{U} = (\mathbb{T}, \mu_{\mathbb{T}})$.

Finally we are ready to define the Σ -algebra of pointed open Markov chains.

For arbitrary pointed open Markov chains (\mathcal{M}, m) , (\mathcal{N}, n) and n -ary operator $f \in \Sigma$, define $f^{\text{omc}}: \mathbf{OMC}^n \rightarrow \mathbf{OMC} \in \Sigma^{\text{omc}}$ as follows:

$$\begin{aligned} X^{\text{omc}} &= (\mathbb{U}, X), \\ (a.(\mathcal{M}, m))^{\text{omc}} &= (\mathbb{U}(\mathcal{M}), a.m), \\ (\mathcal{M}, m) +_e^{\text{omc}} (\mathcal{N}, n) &= (\mathbb{U}(\mathcal{M} \oplus \mathcal{N}), m +_e n), \\ (\text{rec } X.(\mathcal{M}, m))^{\text{omc}} &= (\mathbb{U}(\mathcal{M}_{X,m}^*), \text{rec } X.m), \end{aligned}$$

where, for $\mathcal{M} = (M, \tau)$, $\mathcal{M}_{X,m}^*$ denotes the open Markov chain $(M, \tau_{X,m}^*)$ with transition function defined, for all $m' \in M$ and $E \subseteq (\mathcal{L} \times M) \uplus \mathcal{X}$, as

$$\tau_{X,m}^*(m')(E) = \tau(m')(X)\tau(m)(E \setminus \{X\}) + \tau(m')(X^c)\tau(m')(E \setminus \{X\}).$$

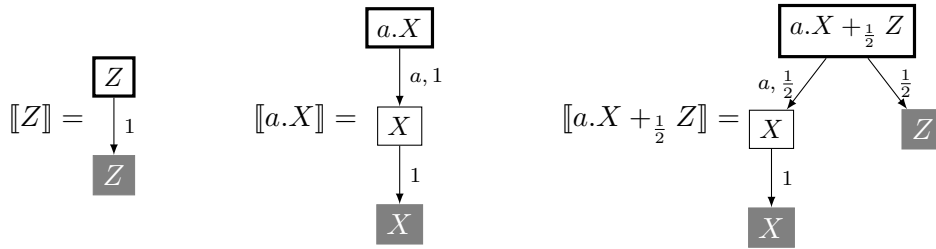
where $X^c = ((\mathcal{L} \times M) \uplus \mathcal{X}) \setminus \{X\}$. Intuitively, $\tau_{X,m}^*$ modifies τ by removing the name $X \in \mathcal{X}$ from the support of $\tau(m')$ and replacing it with the probabilistic behavior of m .

Definition 4.5. The *algebra of open pointed Markov chains* is $(\mathbf{OMC}, \Sigma^{\text{omc}})$.

The (initial) semantics for terms $t \in \mathbb{T}$ to pointed open Markov chains is given via the Σ -homomorphism of algebras $\llbracket \cdot \rrbracket: \mathbb{T} \rightarrow \mathbf{OMC}$, defined by induction on terms as follows

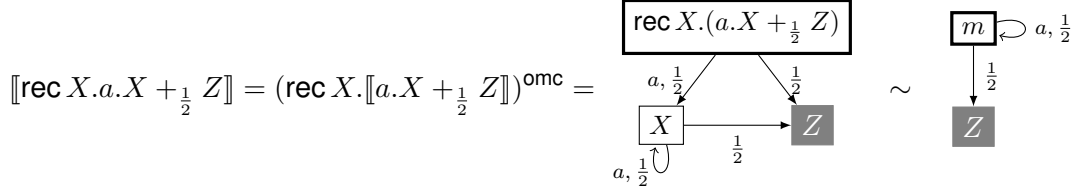
$$\begin{aligned} \llbracket X \rrbracket &= X^{\text{omc}} & \llbracket t +_e s \rrbracket &= \llbracket t \rrbracket +_e^{\text{omc}} \llbracket s \rrbracket, \\ \llbracket a.t \rrbracket &= (a.\llbracket t \rrbracket)^{\text{omc}}, & \llbracket \text{rec } X.t \rrbracket &= (\text{rec } X.\llbracket t \rrbracket)^{\text{omc}}. \end{aligned} \quad (\text{SEMANTICS})$$

Example 4.6. To clarify the semantics of Σ -terms, we show a step-by-step construction of the pointed open Markov chain $\llbracket \text{rec } X.(a.X +_{\frac{1}{2}} Z) \rrbracket$. We start by giving the semantics of the sub-terms Z , $a.X$, and $a.X +_{\frac{1}{2}} Z$.



For the sake of readability, the pointed Markov chains associated with each term are presented using the usual graphical representation. White colored nodes represent the states; gray colored ones names; and initial states are marked in bold. For clarity, only the sub-chain reachable from the initial state is shown.

The semantics of $\mathbf{rec} X.(a.X + \frac{1}{2} Z)$ is obtained from $\llbracket a.X + \frac{1}{2} Z \rrbracket$ as follows:



It is important remarking that by definition of the interpretation of the recursion operator, for each $t \in \mathbb{T}(M)$, the states $\mathbf{rec} X.t$ and X are always bisimilar in $\llbracket \mathbf{rec} X.t \rrbracket$ (*cf.* picture). Hence, the semantics of $\mathbf{rec} X.a.X + \frac{1}{2} Z$ is bisimilar to the pointed open Markov chain with initial state m depicted on the right hand side of $\llbracket \mathbf{rec} X.a.X + \frac{1}{2} Z \rrbracket$. \square

The next result states that it is totally equivalent to reason about the equivalence of the behavior of $\llbracket t \rrbracket$ and $\llbracket s \rrbracket$ by just considering bisimilarity between the corresponding states t and s in the universal open Markov chain \mathbb{U} .

Theorem 4.7 (Universality). *For all $t \in \mathbb{T}$, $\llbracket t \rrbracket \sim (\mathbb{U}, t)$.*

Proof (sketch). The proof of $\llbracket t \rrbracket \sim (\mathbb{U}, t)$ is by induction on t . The base case is trivial. The cases for the prefix and probabilistic choice operations are completely routine from the definition of the interpretations and the operator $\mathbb{U}: \mathbf{OMC} \rightarrow \mathbf{OMC}$ (in each case a bisimulation can be constructed from those given by the inductive hypothesis). The only nontrivial case is when $t = \mathbf{rec} X.t'$. The proof carries over in two steps. First one shows that $(\mathbb{U}, \mathbf{rec} X.t') \sim (\mathbf{rec} X.(\mathbb{U}, t'))^{\text{omc}}$; then, by using the inductive hypothesis $\llbracket t' \rrbracket \sim (\mathbb{U}, t')$, that $(\mathbf{rec} X.(\mathbb{U}, t'))^{\text{omc}} \sim (\mathbf{rec} X.\llbracket t' \rrbracket)^{\text{omc}}$. Since $\llbracket \mathbf{rec} X.t' \rrbracket = (\mathbf{rec} X.\llbracket t' \rrbracket)^{\text{omc}}$, by transitivity of the bisimilarity relation $\llbracket \mathbf{rec} X.t' \rrbracket \sim (\mathbb{U}, \mathbf{rec} X.t')$. \square

Remark 4.8. We already noted that the universal open Markov chain \mathbb{U} corresponds to the operational semantics of probabilistic expressions given by Stark and Smolka [SS00]. In the light of Theorem 4.7, the soundness and completeness results for axiomatic equational system w.r.t. probabilistic bisimilarity over probabilistic expressions given in [SS00], can be moved without further efforts to the class of open Markov chains of the form $\llbracket t \rrbracket$.

5. AXIOMATIZATION OF THE BISIMILARITY DISTANCE

In this section we present the probabilistic bisimilarity distance of Desharnais et al. [DGJP04], that we use to define a quantitative algebra of open Markov chains. After that, we define a quantitative deduction system that is proven sound and complete w.r.t. probabilistic bisimilarity distance.

We will see that for obtaining these results we cannot directly use the quantitative algebraic framework of Mardare, Panangaden, and Plotkin [MPP16] recalled in Section 3, because the recursive operator does not satisfy the conditions required to applying their general equational theory. The divergences from [MPP16] in our development of a quantitative equational theory for the bisimilarity distance over open Markov chains will be explained at length as soon as they are introduced.

5.1. The Probabilistic Bisimilarity Distance. The notion of probabilistic bisimilarity can be lifted to a pseudometric by means of a straightforward extension to open Markov chains of the probabilistic bisimilarity distance of Desharnais et al. [DGJP04]. For more details about its original definition and properties, we refer the interested reader to [DGJP04, vBW01].

This distance is based on the *Kantorovich (pseudo)metric* between discrete probability distributions $\mu, \nu \in \Delta(A)$ over a finite set A with underlying (pseudo)metric d , defined as

$$\mathcal{K}(d)(\mu, \nu) = \min \left\{ \int d \, d\omega \mid \omega \in \Omega(\mu, \nu) \right\}. \quad (\text{KANTOROVICH METRIC})$$

where $\Omega(\mu, \nu)$ denotes the set of *couplings* for (μ, ν) , i.e., distributions $\omega \in \Delta(A \times A)$ on the cartesian product $A \times A$ such that, for all $E \subseteq A$, $\omega(E \times A) = \mu(E)$ and $\omega(A \times E) = \nu(E)$.

Remark 5.1. The definition of $\mathcal{K}(d)$ is tailored on probability distributions, whereas later we will use it on sub-probability distributions. In these situations one usually interpret sub-probabilities μ in A as full-probabilities μ^* in A_\perp (i.e., A extended with a bottom element \perp that is assumed to be at maximum distance from all elements $a \in A$) uniquely defined as $\mu^*(E) = \mu(E)$, for all $E \subseteq A$, and $\mu^*(\perp) = 1 - \mu(A)$.

Definition 5.2 (Bisimilarity Distance). Let $\mathcal{M} = (M, \tau)$ be an open Markov chain. The *probabilistic bisimilarity pseudometric* $\mathbf{d}_{\mathcal{M}}: M \times M \rightarrow [0, 1]$ on \mathcal{M} is the least fixed-point of the following functional operator on 1-bounded pseudometrics (ordered point-wise),

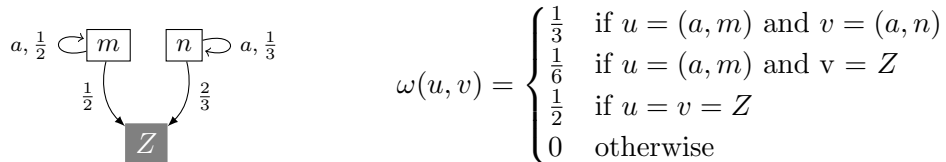
$$\Psi_{\mathcal{M}}(d)(m, m') = \mathcal{K}(\Lambda(d))(\tau^*(m), \tau^*(m')) \quad (\text{KANTOROVICH OPERATOR})$$

where $\Lambda(d)$ is the greatest 1-bounded pseudometric on $((\mathcal{L} \times M) \uplus \mathcal{X})_\perp$ such that, for all $a \in \mathcal{L}$ and $m, m' \in M$, $\Lambda(d)((a, m), (a, m')) = d(m, m')$.

The well definition of $\mathbf{d}_{\mathcal{M}}$ is ensured by monotonicity of $\Psi_{\mathcal{M}}$ (Lemma 5.4) and Knaster-Tarski fixed-point theorem once it is noticed that the set of 1-bounded pseudometrics with point-wise order $d \sqsubseteq d'$ iff $d(m, n) \leq d'(m, n)$, for all $m, n \in M$, is a complete partial order.

Hereafter, whenever \mathcal{M} is clear from the context we will simply write \mathbf{d} and Ψ in place of $\mathbf{d}_{\mathcal{M}}$ and $\Psi_{\mathcal{M}}$, respectively.

Example 5.3. To better understand how the functional operator Ψ works, we look at a simple example of computation of the probabilistic bisimilarity distance \mathbf{d} between two states. Consider the open Markov chain and coupling ω for the transition probability distribution $(\tau(m), \tau(n))$ of the states m and n given as follows:



By definition, $\mathbf{d}(m, n) = \Psi(\mathbf{d})(m, n) = \mathcal{K}(\Lambda(\mathbf{d}))(\tau^*(m), \tau^*(n))$. Since $\tau(m)$ and $\tau(n)$ are fully-probability distributions, we apply the direct definition of Kantorovich distance to get

$$\begin{aligned} \mathbf{d}(m, n) &= \mathcal{K}(\Lambda(\mathbf{d}))(\tau(m), \tau(n)) \\ &\leq \frac{1}{3} \Lambda(\mathbf{d})((a, m), (a, n)) + \frac{1}{6} \Lambda(\mathbf{d})((a, m), Z) + \frac{1}{3} \Lambda(\mathbf{d})(Z, Z). \end{aligned}$$

By definition of Λ , we have that $\Lambda(\mathbf{d})((a, m), (a, n)) = \mathbf{d}(m, n)$, $\Lambda(\mathbf{d})((a, m), Z) = 1$, and $\Lambda(\mathbf{d})(Z, Z) = 0$.

By an easy analysis of the inequality obtained above, one may readily notice that the coupling $\Omega(\tau(m), \tau(n))$ that minimizes the distance between m and n is the one maximizing the probability mass on the pairs $((a, m), (a, m))$ and (Z, Z) . Since ω is already doing so, we have that $\mathbf{d}(m, n) = \frac{1}{3}\mathbf{d}(m, n) + \frac{1}{6}$, thus the distance is $\mathbf{d}(m, n) = \frac{1}{4}$. \square

By the next lemma and Kleene fixed-point theorem, the bisimilarity distance can be alternatively characterized as $\mathbf{d} = \bigsqcup_{n \in \omega} \Psi^n(\mathbf{0})$.

Lemma 5.4. Ψ is monotonic and ω -continuous, i.e., for any countable increasing sequence $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$, it holds $\bigsqcup_{i \in \omega} \Psi(d_i) = \Psi(\bigsqcup_{i \in \omega} d_i)$.

Proof. Monotonicity of Ψ follows from the monotonicity of \mathcal{K} and Λ . ω -continuity follows from [vB12, Theorem 1] by showing that Ψ is non-expansive, i.e., for all $d, d' : M \times M \rightarrow [0, 1]$, $\|\Psi(d') - \Psi(d)\| \leq \|d' - d\|$, where $\|f\| = \sup_x |f(x)|$ is the supremum norm. It suffices to prove that for all $d \sqsubseteq d'$ and $m, m' \in M$, $\Psi(d')(m, m') - \Psi(d)(m, m') \leq \|d' - d\|$:

$$\begin{aligned} & \Psi(d')(m, m') - \Psi(d)(m, m') \\ &= \mathcal{K}(\Lambda(d'))(\tau^*(m), \tau^*(m')) - \mathcal{K}(\Lambda(d))(\tau^*(m), \tau^*(m')) \end{aligned} \quad (\text{by def. } \Psi)$$

by choosing $\omega \in \Omega(\tau^*(m), \tau^*(m'))$ such that $\mathcal{K}(\Lambda(d))(\tau^*(m), \tau^*(m')) = \int \Lambda(d) \, d\omega$,

$$\begin{aligned} &= \mathcal{K}(\Lambda(d'))(\tau^*(m), \tau^*(m')) - \int \Lambda(d) \, d\omega \\ &\leq \int \Lambda(d') \, d\omega - \int \Lambda(d) \, d\omega \quad (\text{by def. of } \mathcal{K}(\Lambda(d'))) \\ &= \int (\Lambda(d') - \Lambda(d)) \, d\omega \quad (\text{by linearity}) \end{aligned}$$

and since, for all $(\alpha, \beta) \notin E = \{((a, n), (a, n')) \mid a \in \mathcal{L}, n, n' \in M\}$, $\Lambda(d')(\alpha, \beta) = \Lambda(d)(\alpha, \beta)$,

$$\begin{aligned} &= \int_E (\Lambda(d') - \Lambda(d)) \, d\omega \\ &\leq \int_E \|d' - d\| \, d\omega \quad (\text{by def. } \Lambda) \\ &\leq \|d' - d\|. \quad (\text{by linearity and } \int_E 1 \, d\omega \leq 1) \end{aligned}$$

\square

Next we show that \mathbf{d} is indeed a lifting of the probabilistic bisimilarity to pseudometrics.

Lemma 5.5. $\mathbf{d}(m, m') = 0$ iff $m \sim m'$.

Proof. We prove the two implications separately. (\Leftarrow) It suffices to show that the relation $R = \{(m, m') \mid \mathbf{d}(m, m') = 0\}$ (i.e., $\ker(\mathbf{d})$) is a bisimulation. Clearly, R is an equivalence, and also $\ker(\Lambda(d))$ is so. Assume $(m, m') \in R$. By definition of Ψ , we have that $\mathcal{K}(\Lambda(\mathbf{d}))(\tau^*(m), \tau^*(m')) = 0$. By [FPP04, Lemma 3.1], for all $\ker(\Lambda(d))$ -equivalence classes $D \subseteq ((\mathcal{L} \times M) \uplus \mathcal{X})_\perp$, $\tau^*(m)(D) = \tau^*(m')(D)$. By definition of Λ , this implies that, for all $a \in \mathcal{L}$, $X \in \mathcal{X}$ and $C \in M/R$, $\tau(m)(X) = \tau(m')(X)$ and, moreover, $\tau(m)(\{a\} \times C) = \tau(m')(\{a\} \times C)$. (\Rightarrow) Let $R \subseteq M \times M$ be a bisimulation on \mathcal{M} , and define $d_R : M \times M \rightarrow [0, 1]$ by $d_R(m, m') = 0$ if $(m, m') \in R$ and $d_R(m, m') = 1$ otherwise. We show that $\Psi(d_R) \sqsubseteq d_R$. If $(m, m') \notin R$, then $d_R(m, m') = 1 \geq \Psi(d_R)(m, m')$. If $(m, m') \in R$, then for all $a \in \mathcal{L}$, $X \in \mathcal{X}$ and $C \in M/R$, $\tau(m)(X) = \tau(m')(X)$, $\tau(m)(\{a\} \times C) = \tau(m')(\{a\} \times C)$. This implies that for all $\ker(\Lambda(d_R))$ -equivalence class $D \subseteq ((\mathcal{L} \times M) \uplus \mathcal{X})_\perp$, $\tau^*(m)(D) = \tau^*(m')(D)$. By [FPP04, Lemma 3.1], we have $\mathcal{K}(\Lambda(d_R))(\tau^*(m), \tau^*(m')) = 0$. This implies that $\Psi(d_R) \sqsubseteq d_R$. Since \sim is a bisimulation, $\Psi(d_\sim) \sqsubseteq d_\sim$, so that, by Tarski's fixed point theorem, $\mathbf{d} \sqsubseteq d_\sim$. By definition of d_\sim and $\mathbf{d} \sqsubseteq d_\sim$, $m \sim m'$ implies $\mathbf{d}(m, m') = 0$. \square

The bisimilarity distance can alternatively be obtained as $\mathbf{d} = \bigcap_{k \in \omega} \tilde{\Psi}^k(\mathbf{1})$, *i.e.*, as the ω -limit of the decreasing sequence $\mathbf{1} \supseteq \tilde{\Psi}(\mathbf{1}) \supseteq \tilde{\Psi}^2(\mathbf{1}) \supseteq \dots$ of the operator

$$\tilde{\Psi}(d)(m, m') = \begin{cases} 0 & \text{if } m \sim m', \\ \Psi(d)(m, m') & \text{otherwise.} \end{cases}$$

Lemma 5.6. *$\tilde{\Psi}$ is monotone and ω -cocontinuous, *i.e.*, for any countable decreasing sequence $d_0 \supseteq d_1 \supseteq d_2 \supseteq \dots$, it holds $\bigcap_{i \in \omega} \Psi(d_i) = \Psi(\bigcap_{i \in \omega} d_i)$. Moreover, $\mathbf{d} = \bigcap_{k \in \omega} \tilde{\Psi}^k(\mathbf{1})$.*

Proof. Monotonicity and ω -cocontinuity follow similarly to Lemma 5.4 and [vB12, Theorem 1]. By ω -cocontinuity $\bigcap_{k \in \omega} \tilde{\Psi}^k(\mathbf{1})$ is a fixed point. By Lemma 5.5 and $\mathbf{d} = \Psi(\mathbf{d})$, also \mathbf{d} is a fixed point of $\tilde{\Psi}$. We show that they coincide by proving that $\tilde{\Psi}$ has a unique fixed point.

Assume that $\tilde{\Psi}$ has two fixed points d and d' such that $d \sqsubset d'$. Define $R \subseteq M \times M$ as $m R m'$ iff $d'(m, m') - d(m, m') = \|d' - d\|$. By the assumption made on d and d' we have that $\|d' - d\| > 0$ and $R \cap \sim = \emptyset$. Consider arbitrary $m, m' \in M$ such that $m R m'$, then

$$\begin{aligned} \|d' - d\| &= d'(m, m') - d(m, m') \\ &= \tilde{\Psi}(d')(m, m') - \tilde{\Psi}(d)(m, m') && \text{(by } d = \tilde{\Psi}(d) \text{ and } d' = \tilde{\Psi}(d')) \\ &= \Psi(d')(m, m') - \Psi(d)(m, m') && \text{(by } m \not\sim m' \text{ and def. } \tilde{\Psi}) \\ &\leq \int_E (\Lambda(d') - \Lambda(d)) \, d\omega, && \text{(as proved in Lemma 5.4)} \end{aligned}$$

where we recall that $E = \{((a, n), (a, n')) \mid a \in \mathcal{L}, n, n' \in M\}$.

Observe that $(\Lambda(d') - \Lambda(d))((a, n), (a, n')) = d'(n) - d(n') \leq \|d' - d\|$, for all $n, n' \in M$ and $a \in \mathcal{L}$. Since $\|d' - d\| > 0$ the inequality $\|d' - d\| \leq \int_E (\Lambda(d') - \Lambda(d)) \, d\omega \leq \|d' - d\|$ holds only if the support of ω is included in $E_R = \{((a, n), (a, n')) \mid a \in \mathcal{L} \text{ and } n R n'\}$. Since the argument holds for arbitrary $m, m' \in M$ such that $m R m'$, we have that R is a bisimulation, which is in contradiction with the initial assumptions. \square

5.2. A Quantitative Algebra of Open Markov Chains. We turn the algebra of pointed open Markov chains ($\mathbf{OMC}, \Sigma^{\text{omc}}$) given in Section 4.2 into a “quantitative algebra” by endowing it with the probabilistic bisimilarity pseudometric of Desharnais et al.. Our definition, however, does not comply with the non-expansivity conditions that are required for the interpretations of the algebraic operators by Definition 3.1. Indeed we show that the recursion operator fails to be non-expansive.

We conclude the section by extending the universality result of Theorem 4.7 to the quantitative setting.

In Definition 5.2, the bisimilarity pseudometric $\mathbf{d}_{\mathcal{M}}$ is defined over the states of a given open Markov chain \mathcal{M} . This can be extended to distance $\mathbf{d}_{\mathbf{OMC}}: \mathbf{OMC} \times \mathbf{OMC} \rightarrow [0, 1]$ over the set \mathbf{OMC} of open Markov chains by simply computing the bisimilarity distance between the initial states on the disjoint union of the two open Markov chains, *i.e.*

$$\mathbf{d}_{\mathbf{OMC}}((\mathcal{M}, m), (\mathcal{N}, n)) = \mathbf{d}_{\mathcal{M} \oplus \mathcal{N}}(m, n).$$

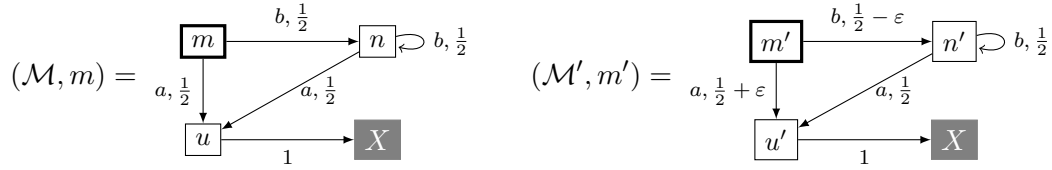
Definition 5.7. The *quantitative algebra of open Markov chains* is $(\mathbf{OMC}, \Sigma^{\text{omc}}, \mathbf{d}_{\mathbf{OMC}})$.

It is important to remark that the algebraic structure we just defined *is not* a quantitative algebra in the sense of [MPP16] (*cf.* Definition 3.1), because as we show in Example 5.9 the interpretation of the recursion operator fails to be non-expansive.

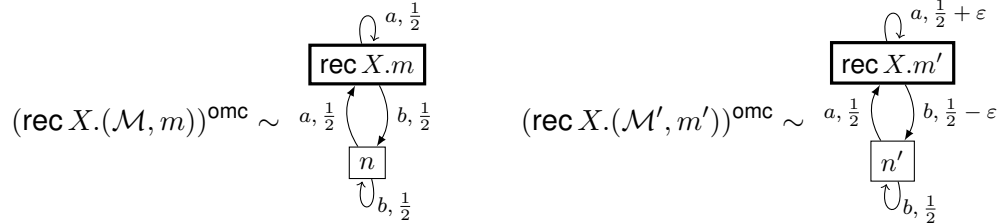
Remark 5.8. The use of the term “quantitative algebra” in Definition 5.7 is clearly an abuse of terminology. Perhaps, we should call such structures “*relaxed quantitative algebras*” to emphasize the fact that the interpretations of the operators do not need to be non-expansive. For the sake of readability, however, we decided to omit the adjective “relaxed” since, as it will be showed in Sections 5.4 and 5.5, this detail will not cause troubles for the soundness and completeness results.

Example 5.9 (Recursion is not non-expansive!). We show an example where the recursion operator $\text{rec } X$ fails to be non-expansive w.r.t. the bisimilarity distance.

Let $0 < \varepsilon < \frac{1}{2}$ and consider the two pointed open Markov chains depicted below.



By a straightforward computation one can readily show that the bisimilarity distance between (\mathcal{M}, m) and (\mathcal{M}', m') is $\mathbf{d}((\mathcal{M}, m), (\mathcal{M}', m')) = \varepsilon$. Let now consider the application of the operator $(\text{rec } X)^{\text{omc}}$ on these two pointed open Markov chains. It turns out that the resulting chains have the following behavior:



By an easy calculation, independently of the value $\varepsilon \in (0, \frac{1}{2})$, we obtain that

$$\mathbf{d}((\text{rec } X.(\mathcal{M}, m))^{\text{omc}}, (\text{rec } X.(\mathcal{M}', m'))^{\text{omc}}) = 1.$$

Since $\varepsilon < 1$, $\mathbf{d}((\mathcal{M}, m), (\mathcal{M}', m')) \not\geq \mathbf{d}((\text{rec } X.(\mathcal{M}, m))^{\text{omc}}, (\text{rec } X.(\mathcal{M}', m'))^{\text{omc}})$. This proves that $(\text{rec } X)^{\text{omc}}$ fails to be non-expansive w.r.t. the bisimilarity distance. \square

By Theorem 4.7, we know that one can equivalently reason about bisimilarity between the semantics of terms by simply considering bisimilarity for the corresponding terms as states in the universal open Markov chain \mathbb{U} . The next result states that the situation is similar when one needs to compute the distance between the semantics of terms.

Theorem 5.10 (Quantitative universality). *Let $t, s \in \mathbb{T}$. Then $\mathbf{d}_{\text{OMC}}(\llbracket t \rrbracket, \llbracket s \rrbracket) = \mathbf{d}_{\mathbb{U}}(t, s)$.*

Proof. The equality $\mathbf{d}_{\text{OMC}}(\llbracket t \rrbracket, \llbracket s \rrbracket) = \mathbf{d}_{\mathbb{U}}(t, s)$ follows by Lemma 5.5 and Theorem 4.7.

$$\begin{aligned} \mathbf{d}_{\text{OMC}}(\llbracket t \rrbracket, \llbracket s \rrbracket) &= \mathbf{d}(\llbracket t \rrbracket, \llbracket s \rrbracket) && \text{(def. } \mathbf{d}) \\ &\leq \mathbf{d}(\llbracket t \rrbracket, (\mathbb{U}, t)) + \mathbf{d}((\mathbb{U}, t), (\mathbb{U}, s)) + \mathbf{d}((\mathbb{U}, s), \llbracket s \rrbracket) && \text{(triangular ineq.)} \\ &= \mathbf{d}((\mathbb{U}, t), (\mathbb{U}, s)) && \text{(Theorem 4.7 \& Lemma 5.5)} \\ &= \mathbf{d}_{\mathbb{U}}(t, s). && \text{(def. } \mathbf{d}) \end{aligned}$$

By a similar argument we also have $\mathbf{d}_{\text{OMC}}(\llbracket t \rrbracket, \llbracket s \rrbracket) \geq \mathbf{d}_{\mathbb{U}}(t, s)$, hence the thesis. \square

5.3. A Quantitative Deduction System. Now we present a quantitative deduction system which will be later shown to be sound and complete w.r.t. the probabilistic bisimilarity distance of Desharnais et al.. The deduction system we propose will not be a quantitative deduction system in the sense of [MPP16], because it does not satisfies the **(NExp)** axiom of non-expansivity for the signature operators (*cf.* Section 3).

The quantitative deduction system $\vdash \subseteq 2^{\mathcal{E}(\Sigma)} \times \mathcal{E}(\Sigma)$ of type Σ that we consider satisfies the axioms **(Refl)**, **(Symm)**, **(Triang)**, **(Max)**, **(Arch)** and rules **(Subst)**, **(Cut)** **(Assum)** from Section 3 and the following additional axioms

- (B1) $\vdash t +_1 s \equiv_0 t$,
- (B2) $\vdash t +_e t \equiv_0 t$,
- (SC) $\vdash t +_e s \equiv_0 s +_{1-e} t$,
- (SA) $\vdash (t +_e s) +_{e'} u \equiv_0 t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$, for $e, e' \in [0, 1)$,
- (Unfold) $\vdash \text{rec } X.t \equiv_0 t[\text{rec } X.t/X]$,
- (Unguard) $\vdash \text{rec } X.(t +_e X) \equiv_0 \text{rec } X.t$,
- (Fix) $\{s \equiv_0 t[s/X]\} \vdash s \equiv_0 \text{rec } X.t$, for X guarded in t ,
- (Cong) $\{t \equiv_0 s\} \vdash \text{rec } X.t \equiv_0 \text{rec } X.s$,
- (Top) $\vdash t \equiv_1 s$,
- (Pref) $\{t \equiv_\varepsilon s\} \vdash a.t \equiv_\varepsilon a.s$,
- (IB) $\{t \equiv_\varepsilon s, t' \equiv_{\varepsilon'} s'\} \vdash t +_e t' \equiv_{\varepsilon''} s +_e s'$, for $\varepsilon'' \geq e\varepsilon + (1-e)\varepsilon'$.

Note that the axiom **(NExp)** is not included in the definition.

Remark 5.11. The use of the term “quantitative deduction system” is again an abuse of terminology. To emphasize the fact that the axiom **(NExp)** is not required to be satisfied, we should perhaps have used the word “*relaxed quantitative deduction system*” to mark the difference w.r.t. the quantitative algebraic framework of Mardare, Panangaden, and Plotkin [MPP16]. However the omission of the adjective “relaxed” will not cause troubles in the further development of the paper.

(B1), (B2), (SC), (SA) are the axioms of *barycentric algebras* (a.k.a. *abstract convex sets*) due to M. H. Stone [Sto49], used here to axiomatize the convex set of probability distributions. (SC) stands for *skew commutativity* and (SA) for *skew associativity*. Barycentric algebras are *entropic* in the sense that all operations $+_e$ are affine maps, that is, for all $e, d \in [0, 1]$ we have the entropic identity

$$\vdash (t +_e s) +_d (t' +_e s') \equiv_0 (t +_d t') +_e (s +_d s'). \quad (\text{Entr})$$

If $t = s$, by (B2) the above reduces to the distributivity law

$$\vdash u +_d (t' +_e s') \equiv_0 (u +_d t') +_e (u +_d s'). \quad (\text{Distr})$$

The entropic identity (Entr) can be verified by direct deduction. However, deductions in barycentric algebras are tedious as skew associativity (SA) is difficult to apply. For a simpler proof of (Entr) we refer the reader to [KP17, Lemma 2.3].

The axioms (Unfold), (Unguard), (Fix), (Cong) are the recursion axioms of Milner [Mil84], used here to axiomatize the coinductive behavior of open Markov chains. (Unfold) and (Fix) state that, whenever X is guarded in a term t , $\text{rec } X.t$ is *the unique solution* of the recursive

equation $s \equiv_0 t[s/X]$. The axiom **(Unguard)** deals with unguarded recursive behavior, and **(Cong)** states the congruential properties of the recursion operator.

As opposed to the axioms described so far, which are essentially equational, the last three are the only truly quantitative one characterizing the quantitative deduction system we have just introduced. **(Top)** states that the distance between terms is bounded by 1; **(Pref)** is the non-expansivity for the prefix operator; and **(IB)** is the *interpolative barycentric axiom* of Mardare, Panangaden, and Plotkin introduced in [MPP16] for axiomatizing the Kantorovich distance between finitely-supported probability distributions (*cf.* §10 in [MPP16]).

Note that for $\varepsilon = \varepsilon'$, **(IB)** reduces to non-expansivity for the operator $+_e$:

$$\{t \equiv_\varepsilon s, t' \equiv_\varepsilon s'\} \vdash t +_e t' \equiv_\varepsilon s +_e s',$$

and non-expansivity always entails congruence for the operator. Indeed, for $\varepsilon = \varepsilon' = 0$ in **(Pref)** and **(IB)** we obtain

$$\{t \equiv_0 s\} \vdash a.t \equiv_0 a.s, \tag{Pref-0}$$

$$\{t \equiv_0 s, t' \equiv_0 s'\} \vdash t +_e t' \equiv_0 s +_e s', \tag{IB-0}$$

corresponding to the congruence for the prefix and probabilistic choice operators, respectively.

It is important to remark that the quantitative deduction system presented above subsumes the equational deduction system of Stark and Smolka [SS00] that axiomatizes probabilistic bisimilarity.

We conclude this section by recalling some historical notes from [KP17, WMM⁺98] about the axioms of barycentric algebras and recursion.

Remark 5.12 (Historical notes). (1) The first axiomatization of convex sets can be traced back to M. H. Stone [Sto49]. Independently, Kneser [Kne52] gave a similar axiomatization. Stone’s and Kneser’s axioms were not restricted to convex sets arising in vector spaces over the reals but, by requiring an additional cancellation axiom, they axiomatized convex sets *embeddable* in vector spaces over linearly ordered skewed fields. W. D. Neumann [Neu70] seems to be the first to have looked at a truly equational theory of convex sets. He remarked that barycentric algebras may be very different from convex sets in vector spaces. Indeed \vee -semilattices are an example of barycentric algebra by interpreting $+_e$ as \vee , for all $0 < e < 1$, and $+_1$ and $+_0$ as left and right projections, respectively.

The axioms **(B1)**, **(B2)**, **(SC)**, **(SA)** that we use in this work are due to Świrszcz [Ś74] and have been reproduced by Romanowska and Smith [RS02], who actually introduced the terminology *barycentric algebra* for an abstract convex set.

(2) Early attempts to prove equational properties of recursive definitions in various specific contexts include the work of de Bakker [dB71], Manna and Vuillemin [MV72], and Kahn [Kah74]. Since then, the general study of recursion equations has been pursued under several guises: as recursive applicative program schemes [CKV74], μ -calculus [BÉ94], and perhaps most notably as the iteration theories [BÉ93] of Bloom and Ésik.

The axioms **(Unfold)**, **(Unguard)**, **(Fix)**, **(Cong)** first appeared in Milner [Mil84], who was certainly aware of de Bakker’s work (*cf.* [Mil75]). The same axioms have been used by Stark and Smolka in their development of an equational deduction system for probabilistic bisimilarity [SS00]. An equivalent axiomatization to that in [SS00] appeared in [AÉI02], where recursion is extended to vector terms and the recursive axioms were replaced by the

axioms of iteration algebras, a.k.a. Conway equations [BÉ93],

$$\vdash \mathbf{rec} X.t[s/X] \equiv_0 t[\mathbf{rec} X.s[t/X]/X] \quad (\text{Composition Identity})$$

$$\vdash \mathbf{rec} X.t[X/Y] \equiv_0 \mathbf{rec} X.\mathbf{rec} Y.t \quad (\text{Diagonal Identity})$$

capturing the equational properties of the fixed point operations in a purely equational way. Note that the composition identity reduces to **(Unfold)** when taking $s = X$.

5.4. Soundness. In this section we show the soundness of our quantitative deduction system w.r.t. the bisimilarity distance between pointed open Markov chains.

Recall that, by Theorem 5.10, it is totally equivalent to reason about the distance between $\llbracket t \rrbracket$ and $\llbracket s \rrbracket$ by just considering the bisimilarity distance between the states corresponding to the terms t and s in the universal open Markov chain \mathbb{U} . Hence hereafter, whenever we refer to the distance between terms in \mathbb{T} we will use $\mathbf{d}_{\mathbb{U}}$, often simply denoted as \mathbf{d} . Similarly, $\models_{\text{OMC}} t \equiv_{\varepsilon} s$ is equivalent to $\models_{\mathbb{U}} t \equiv_{\varepsilon} s$, and it will be denoted just by $\models t \equiv_{\varepsilon} s$.

Theorem 5.13 (Soundness). *For arbitrary $t, s \in \mathbb{T}$, if $\vdash t \equiv_{\varepsilon} s$ then $\models t \equiv_{\varepsilon} s$.*

Proof. We must show that each axiom and deduction rule of inference is valid. The axioms **(Refl)**, **(Symm)**, **(Triang)**, **(Max)**, and **(Arch)** are sound since \mathbf{d} is a pseudometric (Lemma 5.5). The soundness of the classical logical deduction rules **(Subst)**, **(Cut)**, and **(Assum)** is immediate. By Lemma 5.5, the kernel of \mathbf{d} is \sim . Hence the axioms of barycentric algebras **(B1)**, **(B2)**, **(SC)**, and **(SA)** along with **(Unfold)**, **(Unguard)**, **(Cong)**, and **(Fix)** follow directly by the soundness theorem proven in [SS00] (*cf.* Remark 4.8).

The soundness for the axiom **(Top)** is immediate consequence of the fact that \mathbf{d} is 1-bounded. To prove the soundness of **(Pref)** it suffices to show that $\mathbf{d}(t, s) \geq \mathbf{d}(a.t, a.s)$.

$$\begin{aligned} \mathbf{d}(a.t, a.s) &= \mathcal{K}(\Lambda(\mathbf{d}))(\mu_{\mathbb{T}}^*(a.t), \mu_{\mathbb{T}}^*(a.s)) && (\mathbf{d} \text{ fixed-point \& def. } \Psi) \\ &= \mathcal{K}(\Lambda(\mathbf{d}))(\mathbb{1}_{\{(a,t)\}}, \mathbb{1}_{\{(a,s)\}}) && (\text{def. } \mu_{\mathbb{T}} \text{ \& } \mathcal{P}_{\mathbb{U}}) \\ &= \Lambda(\mathbf{d})((a, t), (a, s)) && (\text{def. } \mathcal{K}) \\ &= \mathbf{d}(t, s). && (\text{def. } \Lambda) \end{aligned}$$

The soundness of **(IB)** follows by proving $e\mathbf{d}(t, s) + (1 - e)\mathbf{d}(t', s') \geq \mathbf{d}(t +_e t', s +_e s')$.

$$\begin{aligned} e\mathbf{d}(t, s) + (1 - e)\mathbf{d}(t', s') &= e\Psi(\mathbf{d})(t, s) + (1 - e)\Psi(\mathbf{d})(t', s') && (\mathbf{d} \text{ fixed point}) \\ &= e\mathcal{K}(\Lambda(\mathbf{d}))(\mu_{\mathbb{T}}^*(t), \mu_{\mathbb{T}}^*(s)) + (1 - e)\mathcal{K}(\Lambda(\mathbf{d}))(\mu_{\mathbb{T}}^*(t'), \mu_{\mathbb{T}}^*(s')) && (\text{def. } \Psi) \end{aligned}$$

then, for $\omega \in \Omega(\mu_{\mathbb{T}}^*(t), \mu_{\mathbb{T}}^*(s))$ and $\omega' \in \Omega(\mu_{\mathbb{T}}^*(t'), \mu_{\mathbb{T}}^*(s'))$ optimal couplings for $\mathcal{K}(\Lambda(\mathbf{d}))$, and by noticing that $e\omega + (1 - e)\omega' \in \Omega(e\mu_{\mathbb{T}}^*(t) + (1 - e)\mu_{\mathbb{T}}^*(t'), e\mu_{\mathbb{T}}^*(s) + (1 - e)\mu_{\mathbb{T}}^*(s'))$ we have

$$\begin{aligned} &= e \int \Lambda(\mathbf{d}) \, d\omega + (1 - e) \int \Lambda(\mathbf{d}) \, d\omega' \\ &= \int \Lambda(\mathbf{d}) \, d(e\omega + (1 - e)\omega') && (\text{linearity}) \\ &\geq \mathcal{K}(\Lambda(\mathbf{d}))(\mu_{\mathbb{T}}^*(t) + (1 - e)\mu_{\mathbb{T}}^*(t'), \mu_{\mathbb{T}}^*(s) + (1 - e)\mu_{\mathbb{T}}^*(s')) && (\text{def. } \mathcal{K} \text{ and above}) \\ &= \mathcal{K}(\Lambda(\mathbf{d}))(\mathcal{P}_{\mathbb{U}}(\mu_{\mathbb{T}})^*(t +_e t'), \mathcal{P}_{\mathbb{U}}(\mu_{\mathbb{T}})^*(s +_e s')) && (\text{def. } \mathcal{P}_{\mathbb{U}}) \\ &= \mathcal{K}(\Lambda(\mathbf{d}))(\mu_{\mathbb{T}}^*(t +_e t'), \mu_{\mathbb{T}}^*(s +_e s')) && (\mu_{\mathbb{T}} \text{ fixed-point}) \\ &= \mathbf{d}(t +_e t', s +_e s') && (\text{def. } \Psi \text{ \& } \mathbf{d} \text{ fixed-point}) \end{aligned}$$

The above concludes the proof. □

5.5. Completeness. This section is devoted to proving the completeness of our quantitative deduction system w.r.t. the bisimilarity distance between pointed open Markov chains.

For the sake of readability it will be convenient to introduce the following notation for *formal sums of terms* (or *convex combinations of terms*). For $n \geq 1$, $t_1, \dots, t_n \in \mathbb{T}$ terms, and $e_1, \dots, e_n \in [0, 1]$ positive reals such that $\sum_{i=1}^n e_i = 1$, we define

$$\sum_{i=1}^n e_i \cdot t_i = \begin{cases} t_1 & \text{if } e_1 = 1 \\ t_1 +_{e_1} \left(\sum_{i=2}^n \frac{e_i}{1-e_1} \cdot t_i \right) & \text{otherwise.} \end{cases}$$

Following the pattern of [Mil84, SS00], the completeness theorem hinges on a couple of important transformations. The first of these is the standard de Bekič-Scott construction of solutions of simultaneous recursive definitions. References for this theorem may be found in de Bakker [dB71], who seems the first to use it to support a proof rule for “program equivalence”. This is embodied in the next theorem, which is [Mil84, Theorem 5.7].

Theorem 5.14 (Unique Solution of Equations). *Let $\bar{X} = (X_1, \dots, X_k)$ and $\bar{Y} = (Y_1, \dots, Y_h)$ be distinct names, and $\bar{t} = (t_1, \dots, t_k)$ terms with free names in (\bar{X}, \bar{Y}) in which each X_i is guarded. Then there exist terms $\bar{s} = (s_1, \dots, s_k)$ with free names in \bar{Y} such that*

$$\vdash s_i \equiv_0 t[\bar{s}/\bar{X}], \quad \text{for all } i \leq k.$$

Moreover, if for some terms $\bar{u} = (u_1, \dots, u_k)$ with free variables in \bar{Y} , $\vdash u_i \equiv_0 t[\bar{u}/\bar{X}]$, for all $i \leq k$, then $\vdash s_i \equiv_0 u_i$, for all $i \leq k$.

The second transformation provides a deducible normal form for terms. This result is embodied in the following theorem, which is [SS00, Theorem 5.9]⁴.

Theorem 5.15 (Equational Characterization). *For any term t , with free names in \bar{Y} , there exist terms t_1, \dots, t_k with free names in \bar{Y} , such that $\vdash t \equiv_0 t_1$ and*

$$\vdash t_i \equiv_0 \sum_{j=1}^{h(i)} p_{ij} \cdot s_{ij} + \sum_{j=1}^{l(i)} q_{ij} \cdot Y_{g(i,j)}, \quad \text{for all } i \leq k,$$

where the terms s_{ij} and names $Y_{g(i,j)}$ are enumerated without repetitions, and s_{ij} is either *rec* $X.X$ or has the form $a_{ij} \cdot t_{f(i,j)}$.

The last lemma relates the proposed deduction system with the Kantorovich distance between probability distributions. So far this is the only transformation embodying the use of the interpolative barycentric axiom (**IB**) to deduce quantitative information on terms.

Lemma 5.16. *Let d be a 1-bounded pseudometric over \mathbb{T} and $\mu, \nu \in \Delta(\mathbb{T})$ probability measures with supports $\text{supp}(\mu) = \{t_1, \dots, t_k\}$ and $\text{supp}(\nu) = \{s_1, \dots, s_r\}$. Then*

$$\{t_i \equiv_\varepsilon s_u \mid \varepsilon \geq d(t_i, s_u)\} \vdash \sum_{i=1}^k \mu(t_i) \cdot t_i \equiv_{\varepsilon'} \sum_{u=1}^r \nu(s_u) \cdot s_u, \quad \text{for all } \varepsilon' \geq \mathcal{K}(d)(\mu, \nu).$$

⁴The formulation given here is slightly simpler than the original one in [SS00], since our deduction system satisfies the axiom (**B1**), which is not included in the equational deduction system of [SS00].

Proof. We proceed by well-founded induction on the strict preorder $(\mu, \nu) \prec (\mu', \nu')$ iff either $\text{supp}(\mu) \subset \text{supp}(\mu')$ or $\text{supp}(\nu) \subset \text{supp}(\nu')$.

(Base case: $\text{supp}(\mu) = \{t_1\}$ and $\text{supp}(\mu') = \{s_1\}$). In this case $\mu = \mathbb{1}_{\{t_1\}}$ and $\nu = \mathbb{1}_{\{s_1\}}$. Thus the proof follows by **(Assum)**, by noticing that $\mathcal{K}(d)(\mu, \nu) = d(t_1, s_1)$.

(Inductive step: $\text{supp}(\mu) = \{t_1, \dots, t_k\}$ and $\text{supp}(\nu) = \{s_1, \dots, s_r\}$) Assume without loss of generality that $k > 1$ (if $k = 1$, then $r > 1$ and we proceed dually). The proof is structured as follows. We find suitable $e \in (0, 1)$ and $\mu_1, \mu_2, \nu_1, \nu_2 \in \Delta(\mathbb{T})$, such that

- (1) $(\mu_1, \nu_1) \prec (\mu, \nu)$ and $(\mu_2, \nu_2) \prec (\mu, \nu)$;
- (2) $\mathcal{K}(d)(\mu, \nu) = e\mathcal{K}(d)(\mu_1, \nu_1) + (1 - e)\mathcal{K}(d)(\mu_2, \nu_2)$;
- (3) and the following are deducible

$$\vdash \sum_{i=1}^k \mu(t_i) \cdot t_i \equiv_0 \left(\sum_{i=1}^k \mu_1(t_i) \cdot t_i \right) + e \left(\sum_{i=1}^k \mu_2(t_i) \cdot t_i \right), \quad (5.1)$$

$$\vdash \sum_{u=1}^r \nu(s_u) \cdot s_u \equiv_0 \left(\sum_{u=1}^r \nu_1(s_u) \cdot s_u \right) + e \left(\sum_{u=1}^r \nu_2(s_u) \cdot s_u \right). \quad (5.2)$$

By (1) and the inductive hypothesis, we have that, for $j \in \{1, 2\}$

$$\{t_i \equiv_\varepsilon s_u \mid \varepsilon \geq d(t_i, s_u)\} \vdash \sum_{i=1}^k \mu_j(t_i) \cdot t_i \equiv_{\varepsilon'} \sum_{u=1}^r \nu_j(s_u) \cdot s_u, \quad \text{for all } \varepsilon' \geq \mathcal{K}(d)(\mu_j, \nu_j).$$

From the above, (3), and **(IB)** we deduce

$$\{t_i \equiv_\varepsilon s_u \mid \varepsilon \geq d(t_i, s_u)\} \vdash \sum_{i=1}^k \mu(t_i) \cdot t_i \equiv_{\varepsilon'} \sum_{u=1}^r \nu(s_u) \cdot s_u, \quad \text{for all } \varepsilon' \geq \kappa.$$

where $\kappa = e\mathcal{K}(d)(\mu_1, \nu_1) + (1 - e)\mathcal{K}(d)(\mu_2, \nu_2)$. Then, the proof follows from (2).

In the following we provide the definitions for $e \in (0, 1)$ and $\mu_1, \mu_2, \nu_1, \nu_2 \in \Delta(\mathbb{T})$, then in turn we prove (1), (2), and (3). Let $e = \mu(t_1)$. Note that since $\text{supp}(\mu) = \{t_1, \dots, t_k\}$ and $k > 1$, we have that $\mu(t_1) \in (0, 1)$. Let $\tilde{\omega} \in \Omega(\mu, \nu)$ be the minimal coupling for $\mathcal{K}(d)(\mu, \nu)$, *i.e.*, the one realizing the following equality (*cf.* the definition of $\mathcal{K}(d)$)

$$\mathcal{K}(d)(\mu, \nu) = \sum_{i \leq k, u \leq r} d(t_i, s_u) \cdot \tilde{\omega}(t_i, s_u), \quad (5.3)$$

and, for $2 \leq i \leq k$, $1 \leq u \leq r$, define

$$\mu_1(t_1) = 1, \quad \mu_2(t_i) = \frac{\mu(t_i)}{\mu(t_1)}, \quad \nu_1(s_u) = \frac{\tilde{\omega}(t_1, s_u)}{\mu(t_1)}, \quad \nu_2(s_u) = \frac{\nu(s_u) - \tilde{\omega}(t_1, s_u)}{1 - \mu(t_1)}.$$

Note that $\text{supp}(\mu_1) = \{t_1\}$, $\text{supp}(\mu_2) = \{t_2, \dots, t_k\}$ and $\text{supp}(\nu_1) = \text{supp}(\nu_2) = \text{supp}(\nu)$. It is easy to show that, since $\tilde{\omega} \in \Omega(\mu, \nu)$, the above are well-defined probability distributions.

(1) It follows by directly by definition of \prec and $\text{supp}(\mu_1) = \{t_1\}$, $\text{supp}(\mu_2) = \{t_2, \dots, t_k\}$, and $\text{supp}(\mu) = \{t_1, \dots, t_k\}$.

(2) For $2 \leq i \leq k$ and $1 \leq u \leq r$, define

$$\tilde{\omega}_1(t_1, s_u) = \frac{\tilde{\omega}(t_1, s_u)}{\mu(t_1)} \quad \tilde{\omega}_2(t_i, s_u) = \frac{\tilde{\omega}(t_i, s_u)}{1 - \mu(t_1)}.$$

By a straightforward calculation, from the fact that $\tilde{\omega} \in \Omega(\mu, \nu)$, one can prove that $\tilde{\omega}_1 \in \Omega(\mu_1, \nu_1)$ and $\tilde{\omega}_2 \in \Omega(\mu_2, \nu_2)$. (note that this also implies that $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are well-defined probability distributions on \mathbb{T}). From this we get the following inequality:

$$\begin{aligned} \mathcal{K}(d)(\mu_1, \nu_1) &= \sum_{i,u} d(t_i, s_u) \cdot \tilde{\omega}(t_i, s_u) && \text{(by Equation 5.3)} \\ &= \mu(t_1) \left(\sum_{i,u} d(t_i, s_u) \cdot \tilde{\omega}_1(t_i, s_u) \right) + (1 - \mu(t_1)) \left(\sum_{i,u} d(t_i, s_u) \cdot \tilde{\omega}_2(t_i, s_u) \right) \\ &\geq \mu(t_1) \mathcal{K}(d)(\mu_1, \nu_1) + (1 - \mu(t_1)) \mathcal{K}(d)(\mu_2, \nu_2) \end{aligned}$$

Note that, by the above calculation we can deduce that $\tilde{\omega}_j$ is the minimal coupling for $\mathcal{K}(d)(\mu_j, \nu_j)$, for each $j \in \{1, 2\}$. Indeed, if they were not minimal, the last inequality above could be replaced by $<$, thus producing a contradiction. Therefore, (2) holds.

(3) We start by showing (5.1). Since $\mu(t_1) \in (0, 1)$, the formal sum on the left-hand side of (5.1) is syntactically equivalent to

$$\sum_{i=1}^k \mu(t_i) \cdot t_i = t_1 +_{\mu(t_1)} \left(\sum_{i=2}^k \frac{\mu(t_i)}{\mu(t_1)} \cdot t_i \right), \quad (5.4)$$

By (B1), (SC), and the definitions of μ_1, μ_2 we easily obtain

$$\vdash t_1 \equiv_0 \sum_{i=1}^k \mu_1(t_i) \cdot t_i, \quad \vdash \sum_{i=2}^k \frac{\mu(t_i)}{\mu(t_1)} \cdot t_i \equiv_0 \sum_{i=1}^k \mu_2(t_i) \cdot t_i.$$

Thus (5.1) follows from the deductions above by applying (IB-0) to (5.4). Next we prove (5.2) by showing that for any coupling $\omega \in \Omega(\mu, \nu)$ the following is deducible:

$$\vdash \sum_{u=1}^r \nu(s_u) \cdot s_u \equiv_0 \left(\sum_{u=1}^r \frac{\omega(t_1, s_u)}{\mu(t_1)} \cdot s_u \right) +_{\mu(t_1)} \left(\sum_{u=1}^r \frac{\nu(s_u) - \omega(t_1, s_u)}{1 - \mu(t_1)} \cdot s_u \right). \quad (5.5)$$

We do this by induction on the size of the support of ν . (Base case: $\text{supp}(\nu) = \{s_1\}$). Then, $\nu(s_1) = 1$ and $\omega(t_1, s_1) = \mu(t_1)$, so (5.5) reduces to (B2). (Inductive step: $r > 1$ and $\text{supp}(\nu) = \{s_1, \dots, s_r\}$). Then $\nu(s_1) \in (0, 1)$. Thus, the formal sum on the left-hand side of (5.5) is syntactically equivalent to

$$\sum_{u=1}^r \nu(s_u) \cdot s_u = s_1 +_{\nu(s_1)} \left(\sum_{u=2}^r \nu'(s_u) \cdot s_u \right), \quad (5.6)$$

where $\nu'(s_u) = \frac{\nu(s_u)}{1 - \nu(s_1)}$, for $2 \leq u \leq r$. Note that $\omega'(t_i, s_u) = \frac{\omega(t_i, s_u)}{1 - \nu(s_1)}$, for $1 \leq i \leq k$ and $2 \leq u \leq r$, is a coupling in $\Omega(\mu, \nu')$ and that $\text{supp}(\nu') = \{s_2, \dots, s_r\}$. Thus, by inductive hypothesis on ν' we obtain

$$\begin{aligned} \vdash \sum_{u=2}^r \nu'(s_u) \cdot s_u &\equiv_0 \left(\sum_{u=2}^r \frac{\omega'(t_1, s_u)}{\mu(t_1)} \cdot s_u \right) +_{\mu(t_1)} \left(\sum_{u=2}^r \frac{\nu'(s_u) - \omega'(t_1, s_u)}{1 - \mu(t_1)} \cdot s_u \right) \\ &= \left(\sum_{u=2}^r \frac{\omega(t_1, s_u)}{\mu(t_1)(1 - \nu(s_1))} \cdot s_u \right) +_{\mu(t_1)} \left(\sum_{u=2}^r \frac{\nu(s_u) - \omega(t_1, s_u)}{(1 - \mu(t_1))(1 - \nu(s_1))} \cdot s_u \right) \end{aligned}$$

From this deduction and (5.6), by (Dist), we obtain (5.5). □

Now we are ready to prove the main result of this section. The proof of completeness can be roughly sketched as follows. Given $t, s \in \mathbb{T}$ such that $\mathbf{d}(t, s) \leq \varepsilon$, to prove $\vdash t \equiv_\varepsilon s$ we first apply Theorem 5.15 to get their deducible equational normal forms as formal sums. Then, for each $\alpha \in \omega$, by applying **(Top)** for the case $\alpha = 0$, and Lemma 5.16 and **(Pref)** for $\alpha > 0$, we deduce $\vdash t \equiv_\varepsilon s$, for all $\varepsilon \geq \tilde{\Psi}^\alpha(\mathbf{1})(t, s)$. Then, by **(Max)** and **(Arch)**, $\vdash t \equiv_\varepsilon s$ follows by noticing that $\mathbf{d}(t, s) = \prod_{\alpha \in \omega} \tilde{\Psi}^\alpha(\mathbf{1})(t, s)$ (Lemma 5.6).

Theorem 5.17 (Completeness). *For arbitrary $t, s \in \mathbb{T}$, if $\models t \equiv_\varepsilon s$, then $\vdash t \equiv_\varepsilon s$.*

Proof. Let $t, s \in \mathbb{T}$ and $\varepsilon \in \mathbb{Q}_+$. We have to show that if $\mathbf{d}(t, s) \leq \varepsilon$ then $\vdash t \equiv_\varepsilon s$. The case $\varepsilon \geq 1$ trivially follows by **(Top)** and **(Max)**. Let $\varepsilon < 1$. By Theorem 5.15, there exist terms t_1, \dots, t_k and s_1, \dots, s_r with free names in \overline{X} and \overline{Y} , respectively, such that $\vdash t \equiv_0 t_1$, $\vdash s \equiv_0 s_1$, and

$$\vdash t_i \equiv_0 \sum_{j=1}^{h(i)} p_{ij} \cdot t'_{ij}, \quad \text{for all } i \leq k, \quad (5.7)$$

$$\vdash s_u \equiv_0 \sum_{v=1}^{n(u)} q_{uv} \cdot s'_{uv}, \quad \text{for all } u \leq r, \quad (5.8)$$

where the terms t'_{ij} (resp. s'_{uv}) are enumerated without repetitions, and t'_{ij} (resp. s'_{uv}) have either the form $a_{ij}.t_{f(i,j)}$ (resp. $b_{uv}.s_{z(u,v)}$), or $X_{g(i,j)}$ (resp. $Y_{w(u,v)}$), or **rec** $Z.Z$.

If we can prove that for all $\alpha \in \omega$,

$$\vdash t_i \equiv_\varepsilon s_u, \quad \text{for all } i \leq k, u \leq r, \text{ and } \varepsilon \geq \tilde{\Psi}^\alpha(\mathbf{1})(t_i, s_u), \quad (5.9)$$

then, by Lemma 5.6 and **(Arch)**, we deduce $\vdash t_i \equiv_\varepsilon s_u$, for all $\varepsilon \geq \mathbf{d}(t_i, s_u)$. Since $\vdash t \equiv_0 t_1$, $\vdash s \equiv_0 s_1$, by **(Triang)**, we deduce $\vdash t \equiv_\varepsilon s$, for all $\varepsilon \geq \mathbf{d}(t, s)$, concluding the thesis.

The remainder of the proof is devoted to prove (5.9). We do it by induction on $\alpha \in \omega$.

(Base case: $\alpha = 0$) $\tilde{\Psi}^0(\mathbf{1})(t_i, s_u) = \mathbf{1}(t_i, s_u)$. Since $\mathbf{1}(t_i, s_u) = 0$ whenever $t_i = s_u$ and $\mathbf{1}(t_i, s_u) = 1$ if $t_i \neq s_u$, then (5.9) follows by the axioms **(Refl)**, **(Top)** and **(Max)**.

(Inductive step: $\alpha > 0$). Recall that, by definition of $\tilde{\Psi}$, we have the following:

$$\tilde{\Psi}^\alpha(\mathbf{1})(t_i, s_u) = \tilde{\Psi}(\tilde{\Psi}^{\alpha-1})(t_i, s_u) = \begin{cases} 0 & \text{if } t_i \sim_{\cup} s_u, \\ \Psi(\tilde{\Psi}^{\alpha-1})(t_i, s_u) & \text{otherwise.} \end{cases}$$

We consider the cases $t_i \sim_{\cup} s_u$ and $t_i \not\sim_{\cup} s_u$ separately.

Assume $t_i \sim_{\cup} s_u$. Since our deduction system includes the one of Stark and Smolka, whenever $t_i \sim_{\cup} s_u$, by completeness w.r.t. \sim_{\cup} ([SS00, Theorem 3]), we obtain $\vdash t_i \equiv_0 s_u$. By **(Max)**, $\vdash t_i \equiv_\varepsilon s_u$, for all $\varepsilon \geq \tilde{\Psi}^{\alpha+1}(\mathbf{1})(t_i, s_u) = 0$.

Assume $t_i \not\sim_{\cup} s_u$. Let H, G be the formal sums on the right-hand side of (5.7), (5.8), respectively. Then, by definition of $\mu_{\mathbb{T}}^*$, we have that, for $x, y \in (\mathcal{L} \times \mathbb{T}) \uplus \mathcal{X}$,

$$\mu_{\mathbb{T}}^*(H)(x) = \begin{cases} p_{ij} & \text{if } \gamma(x) = t'_{ij}, j \leq h(i) \\ 0 & \text{otherwise,} \end{cases} \quad \mu_{\mathbb{T}}^*(G)(y) = \begin{cases} q_{uv} & \text{if } \gamma(y) = s'_{uv}, v \leq n(u) \\ 0 & \text{otherwise.} \end{cases}$$

where γ is the mapping such that for all $t \in \mathbb{T}$, $a \in \mathcal{L}$, and $X \in \mathcal{X}$,

$$\gamma((a, t)) = a.t, \quad \gamma(X) = X, \quad \gamma(\perp) = \mathbf{rec} Z.Z.$$

If we can prove that, for all $i \leq k$, $j \leq h(i)$, $u \leq r$, and $v \leq n(u)$,

$$\vdash t'_{ij} \equiv_\varepsilon s'_{uv}, \quad \text{for all } \varepsilon \geq \Lambda(\tilde{\Psi}^{\alpha-1}(\mathbf{1}))(\gamma(t'_{ij}), \gamma(s'_{uv})), \quad (5.10)$$

then, by Lemma 5.16, we deduce

$$\vdash H \equiv_\varepsilon G, \quad \text{for all } \varepsilon \geq \mathcal{K}(\Lambda(\tilde{\Psi}^{\alpha-1}(\mathbf{1})))(\mu_{\mathbb{T}}^*(H), \mu_{\mathbb{T}}^*(G)). \quad (5.11)$$

Note that, by $t_i \not\sim_{\cup} s_u$, (5.7), (5.8), and soundness of \equiv_0 w.r.t. \sim_{\cup} in [SS00], we have that $t_i \sim_{\cup} H$, $s_u \sim_{\cup} G$ and $H \not\sim_{\cup} G$. Therefore, by definition of Ψ and triangular inequality

$$\mathcal{K}(\Lambda(\tilde{\Psi}^{\alpha-1}(\mathbf{1}))) (\mu_{\mathbb{T}}^*(H), \mu_{\mathbb{T}}^*(G)) = \tilde{\Psi}^{\alpha}(\mathbf{1})(H, G) = \tilde{\Psi}^{\alpha}(\mathbf{1})(t_i, s_u).$$

From (5.11), (5.7), (5.8), (**Triang**), and the equality above we conclude (5.9).

Next we prove (5.10). The only interesting case is $t'_{i,j} = a.t_{f(i,j)}$ and $s'_{u,v} = a.s_{z(u,v)}$ —the others follow by using (**Refl**), if $t'_{i,j} = s'_{u,v}$, (**Top**) otherwise, and then (**Max**). By definition of γ and Λ , we have $\Lambda(\tilde{\Psi}^{\alpha-1}(\mathbf{1}))(\gamma(t'_{i,j}), \gamma(s'_{uv})) = \tilde{\Psi}^{\alpha-1}(\mathbf{1})(t_{f(i,j)}, s_{z(u,v)})$. Now note that, by inductive hypothesis on $\alpha - 1$, the following is deducible:

$$\vdash t_{f(i,j)} \equiv_{\varepsilon} s_{z(u,v)}, \quad \text{for all } \varepsilon \geq \tilde{\Psi}^{\alpha}(\mathbf{1})(t_{f(i,j)}, s_{z(u,v)}),$$

Therefore, (5.10) follows by the above and (**Pref**). □

We conclude the section by showing a concrete example of deduction of the bisimilarity distance between two terms.

Example 5.18. Consider the terms $t = \text{rec } X.(a.X + \frac{1}{2} Z)$ and $s = \text{rec } Y.(a.Y + \frac{1}{3} Z)$. Similarly to Example 4.6, their pointed open Markov chain semantics are



As shown in Example 5.3 their bisimilarity distance is $\mathbf{d}(t, s) = \frac{1}{4}$, hence $\models t \equiv_{\frac{1}{4}} s$.

Next we show how $\vdash t \equiv_{\frac{1}{4}} s$ can be deduced by applying the axioms and rules of the quantitative deduction system proposed in Section 5.3. For the sake of readability, the classical logical deduction rules (**Subst**), (**Cut**), (**Assum**) will be used implicitly, as well as (**Refl**), (**Symm**), and (**Triang**). By (**Unfold**) and (**Fix**) we can deduce

$$\vdash t \equiv_0 a.t + \frac{1}{2} Z \quad \text{and} \quad \vdash s \equiv_0 a.s + \frac{1}{3} Z. \quad (5.12)$$

Note that, t and s are now in the equational normal form of Theorem 5.15. The next step consists in applying (**IB**) to get information about the distance between the two probabilistic sums on the right-hand sides of the quantitative equations in (5.12). To do so, we first need to “rearrange” the sums in a way such that (**IB**) can actually be applied:

$$\begin{array}{lcl} & \vdash a.s + \frac{1}{3} Z \equiv_0 Z + \frac{2}{3} a.s & \text{(SC)} \\ \vdash a.t + \frac{1}{2} Z \equiv_0 (a.t + \frac{1}{3} a.t) + \frac{1}{2} Z & \equiv_0 (Z + \frac{1}{4} Z) + \frac{2}{3} a.s & \text{(B2)} \\ \equiv_0 a.t + \frac{1}{6} (a.t + \frac{2}{5} Z) & \equiv_0 Z + \frac{1}{6} (Z + \frac{3}{5} a.s) & \text{(SA)} \\ & \equiv_0 Z + \frac{1}{6} (a.s + \frac{2}{5} Z) & \text{(SC)}. \end{array} \quad (5.13)$$

By (**Top**), we deduce $\vdash a.t \equiv_1 Z$ and, by (**Refl**), $\vdash Z \equiv_0 Z$. Hence, by (**Pref**) and applying (**IB**) twice on $a.t + \frac{1}{6} (a.t + \frac{2}{5} Z)$ and $Z + \frac{1}{6} (a.s + \frac{2}{5} Z)$, we obtain the quantitative inference

$$\{t \equiv_{\varepsilon} s\} \vdash a.t + \frac{1}{6} (a.t + \frac{2}{5} Z) \equiv_{\frac{1}{3}\varepsilon + \frac{1}{6}} Z + \frac{1}{6} (a.s + \frac{2}{5} Z) \quad (5.14)$$

Combining (5.12), (5.13), (5.14) we deduce the following

$$\{t \equiv_{\varepsilon} s\} \vdash t \equiv_{\frac{1}{3}\varepsilon + \frac{1}{6}} s. \quad (5.15)$$

The above quantitative inference along with $(\mathbf{Top}) \vdash t \equiv_1 s$, can be thought of as a greatest fixed-point operator, taking an over approximation $\varepsilon \leq 1$ of the distance between t and s and refining it to $\frac{1}{3}\varepsilon + \frac{1}{6}$. This interpretation is not bizarre, because it is exactly how the functional operator $\tilde{\Psi}(d)$ operates on t, s on a 1-bounded pseudometric such that $d(t, s) = \varepsilon$.

The deduction of the distance $\mathbf{d}(t, s) = \frac{1}{4}$ follows by proving that for all $1 \geq \delta > \frac{1}{4}$ we can deduce (in a finite number of steps!) $\vdash t \equiv_\delta s$ and then applying (\mathbf{Arch}) .

The case $\delta = 1$ follows by (\mathbf{Top}) . As for $1 > \delta > \frac{1}{4}$, notice that $T: [0, 1] \rightarrow [0, 1]$ defined as $T(\varepsilon) = \frac{1}{3}\varepsilon + \frac{1}{6}$ is a $\frac{1}{3}$ -Lipschitz continuous map. Hence, by Banach fixed-point theorem, T has a unique fixed point, namely $\frac{1}{4}$, and the following inequality holds for $q = \frac{1}{3}$

$$T^n(1) - \frac{1}{4} \leq \frac{q^n}{1-q} (T^0(1) - T^1(1)).$$

The above reduces to $T^n(1) \leq \frac{1}{2} \cdot (\frac{1}{3})^{n-1}$. Hence by applying (5.15) n -times, starting from $(\mathbf{Top}) \vdash t \equiv_1 s$, for some integer $n \leq \log_3(\frac{1}{2\delta}) + 1$, we deduce $\vdash t \equiv_{T^n(1)} s$ and we know that $T^n(1) \leq \delta$. Then, by (\mathbf{Max}) , we deduce $\vdash t \equiv_\delta s$, *i.e.*, the required deduction. \square

6. AXIOMATIZATION OF THE DISCOUNTED BISIMILARITY DISTANCE

Next we describe how the deductive system in Section 5.3 can be adapted to obtain soundness and completeness theorems w.r.t. the discounted bisimilarity distance of Deshainais et al.

The distance \mathbf{d} that we considered so far is a special case (a.k.a. *undiscounted* bisimilarity distance) of the original definition by Deshainais et al. [DGJP04], which was parametric on a discount factor $\lambda \in (0, 1]$. An equivalent definition of this distance (due to van Breugel and Worrell [vBW01]) adapted to the case of open Markov chains is the following.

Definition 6.1 (Discounted Bisimilarity Distance). For $\lambda \in (0, 1]$, the λ -discounted probabilistic bisimilarity pseudometric $\mathbf{d}_{\mathcal{M}}^\lambda: M \times M \rightarrow [0, 1]$ on \mathcal{M} is the least fixed-point of the following functional operator on 1-bounded pseudometrics (ordered point-wise),

$$\Psi_{\mathcal{M}}^\lambda(d)(m, m') = \mathcal{K}(\Lambda^\lambda(d))(\tau^*(m), \tau^*(m')) \quad (\lambda\text{-KANTOROVICH OPERATOR})$$

where $\Lambda^\lambda(d)$ is the greatest 1-bounded pseudometric on $((\mathcal{L} \times M) \uplus \mathcal{X})_\perp$ such that, for all $a \in \mathcal{L}$ and $t, s \in \mathbb{T}$, $\Lambda(d)((a, t), (a, s)) = \lambda \cdot d(t, s)$.

Clearly, for $\lambda = 1$ the above reduces to Definition 5.2, hence $\mathbf{d} = \mathbf{d}^1$. In [CvBW12, Theorem 6], Chen et al. noticed that when $\lambda < 1$, Ψ^λ is a λ -Lipschitz continuous operator, *i.e.*, for all $d, d': M \times M \rightarrow [0, 1]$, $\|\Psi^\lambda(d') - \Psi^\lambda(d)\| \leq \lambda \|d' - d\|$, where $\|f\| = \sup_x |f(x)|$ is the supremum norm. So, by Banach fixed-point theorem \mathbf{d}^λ is the *unique* fixed point of Ψ^λ . For the same reason $\mathbf{d}^\lambda = \prod_{\alpha \in \omega} (\Psi^\lambda)^\alpha(\mathbf{1})$. Moreover, the following also holds.

Lemma 6.2. *For any $\lambda < 1$, $\mathbf{d}^\lambda(m, m') = 0$ iff $m \sim m'$.*

6.1. A Quantitative Deduction System for the Discounted Case. In this section we provide a quantitative deduction system that is proved to be sound a complete w.r.t. the λ -discounted probabilistic bisimilarity distance.

The quantitative deduction system $\vdash_\lambda \subseteq 2^{\mathcal{E}(\Sigma)} \times \mathcal{E}(\Sigma)$ that we propose contains the one presented in Section 5.3, where we add the following axiom

$$(\lambda\text{-Pref}) \quad \{t \equiv_\varepsilon s\} \vdash_\lambda a.t \equiv_{\varepsilon'} a.s, \text{ for } \varepsilon' \geq \lambda\varepsilon.$$

Notice that, when $\lambda < 1$, $(\lambda\text{-Pref})$ and **(Max)** imply **(Pref)** —hence one may remove **(Pref)** from the definition, since is redundant.

The proof of soundness and completeness follow essentially in the same way of Theorems 5.13 and 5.17. In the reminder of the section we only highlight the parts where some adjustments are needed.

Notice that due to Lemma 6.2 a similar result to Theorem 5.10 holds also in the discounted case, *i.e.*, for all $t, s \in \mathbb{T}$, $\mathbf{d}_{\text{OMC}}^\lambda(\llbracket t \rrbracket, \llbracket s \rrbracket) = \mathbf{d}_{\mathbb{T}}^\lambda(t, s)$. So that, as done previously, we will use $\mathbf{d}_{\text{OMC}}^\lambda$ and $\mathbf{d}_{\mathbb{T}}^\lambda$, interchangeably, often simply denoted as \mathbf{d}^λ . Similarly, $\models_\lambda t \equiv_\varepsilon s$ will stand for $\mathbf{d}^\lambda \leq \varepsilon$.

Theorem 6.3 (λ -Soundness). *For arbitrary $t, s \in \mathbb{T}$, if $\vdash_\lambda t \equiv_\varepsilon s$ then $\models_\lambda t \equiv_\varepsilon s$.*

Proof. The proof follows as Theorem 5.13. We only need to check the soundness of $(\lambda\text{-Pref})$. To do so it suffices to show that $\mathbf{d}^\lambda(t, s) \geq \lambda \cdot \mathbf{d}^\lambda(a.t, a.s)$.

$$\begin{aligned} \mathbf{d}^\lambda(a.t, a.s) &= \mathcal{K}(\Lambda^\lambda(\mathbf{d}^\lambda))(\mu_{\mathbb{T}}^*(a.t), \mu_{\mathbb{T}}^*(a.s)) && (\mathbf{d}^\lambda \text{ fixed-point \& def. } \Psi^\lambda) \\ &= \mathcal{K}(\Lambda^\lambda(\mathbf{d}^\lambda))(\mathbf{1}_{\{(a,t)\}}, \mathbf{1}_{\{(a,s)\}}) && (\text{def. } \mu_{\mathbb{T}} \text{ \& } \mathcal{P}_{\mathbb{T}}) \\ &= \Lambda^\lambda(\mathbf{d}^\lambda)((a, t), (a, s)) && (\text{def. } \mathcal{K}) \\ &= \lambda \cdot \mathbf{d}^\lambda(t, s). && (\text{def. } \Lambda^\lambda) \end{aligned}$$

□

For the proof of completeness note that, since the quantitative deduction system \vdash_λ is a conservative extension of \vdash , we have that Theorems 5.14, 5.15, and Lemma 5.16 still hold. Moreover, for $\lambda < 1$, the proof of completeness is somehow simplified, since we do not need to introduce a new operator $\tilde{\Psi}^\lambda$ to obtain the convergence to the λ -discounted bisimilarity distance from above. Indeed, as noticed before $\mathbf{d}^\lambda = \prod_{\alpha \in \omega} (\Psi^\lambda)^\alpha(\mathbf{1})$.

Theorem 6.4 (λ -Completeness). *For arbitrary $t, s \in \mathbb{T}$, if $\models_\lambda t \equiv_\varepsilon s$, then $\vdash_\lambda t \equiv_\varepsilon s$.*

Proof. The proof follows as in Theorem 5.17. The only modifications need in the proof of Theorem 5.17 to getting the required result are (ii) syntactically replace \vdash with \vdash_λ ; (ii) replacing $\tilde{\Psi}$ with Ψ^λ in the proof of (5.9); and (iii) applying $(\lambda\text{-Pref})$ in place of **(Pref)** in the proof of (5.10). □

7. A QUANTITATIVE KLEENE'S THEOREM FOR OPEN MARKOV CHAINS

In this last section we give a “quantitative Kleene’s theorem” for pointed open Markov chains. Specifically, we show that any (finite) pointed open Markov chains (\mathcal{M}, m) can be represented up to bisimilarity as a Σ -term $t_{(\mathcal{M}, n)}$ and, *vice versa*, for any Σ -term t , there exist a (finite) pointed open Markov chain bisimilar to $\llbracket t \rrbracket$. This establishes a representability

theorem for finite open Markov chains similar to the celebrated Kleene's theorem [Kle56], stating the correspondence between *regular expressions* and *deterministic finite automata* (DFAs) up to language equivalence.

Even more interestingly, we show that by endowing the set of Σ -terms with the pseudo-metric freely-generated by the quantitative deduction system presented in Section 5.3 (in a way which will be made precise later) we get that the correspondence stated above is *metric invariant*. We think of this result as ‘quantitative extension’ of a Kleene's representation theorem for finite open Markov chains.

7.1. Representability. We show that the class of expressible open Markov chains corresponds up to bisimilarity to the class of finite open Markov chains⁵.

A pointed Markov chain (\mathcal{M}, m) is said *expressible* if there exists a term $t \in \mathbb{T}$ such that $\llbracket t \rrbracket \sim (\mathcal{M}, m)$. The next result is a corollary of Theorems 4.7, 5.14, and 5.13.

Corollary 7.1. *If (\mathcal{M}, m) is finite then it is expressible.*

Proof. We have to show that there exists $t \in \mathbb{T}$ such that $\llbracket t \rrbracket \sim (\mathcal{M}, m)$. Since the set of states $M = \{m_1, \dots, m_k\}$ is finite and, for each $m_i \in M$, $\tau(m_i)$ is finitely supported, then the sets of unguarded names $\{Y_1^i, \dots, Y_{h(i)}^i\} = \text{supp}(\tau(m_i)) \cap \mathcal{X}$ and labelled transitions $\{\alpha_1^i, \dots, \alpha_{l(i)}^i\} = \text{supp}(m_i) \cap (\mathcal{L} \times M)$ of m_i are finite. Let us associate with each α_j^i a name X_j^i , for all $i \leq k$ and $j \leq l(i)$. For each $i \leq k$, we define the terms

$$t_i = \sum_{j=1}^{l(i)} \tau(m_i)(\alpha_j^i) \cdot a_j^i \cdot X_j^i + \sum_{j=1}^{h(i)} \tau(m_i)(Y_j^i) \cdot Y_j^i,$$

where $\alpha_j^i = (a_j^i, m_j^i)$, for all $i \leq k$ and $j \leq l(i)$. By Theorem 5.14, for $i \leq k$, there exists terms $\overline{s^i} = (s_1^i, \dots, s_{l(i)}^i)$ such that $\vdash s_i \equiv_0 t_i[\overline{s^i}/\overline{X^i}]$, so that by soundness (Theorem 5.13), $\llbracket s_i \rrbracket \sim \llbracket t_i[\overline{s^i}/\overline{X^i}] \rrbracket$. Hence, by Theorem 4.7, we have $(\mathbb{U}, s_i) \sim (\mathbb{U}, t_i[\overline{s^i}/\overline{X^i}])$.

Let $\overline{m^i} = (m_1^i, \dots, m_{l(i)}^i)$ and $\overline{X^i} = (X_1^i, \dots, X_{l(i)}^i)$, for $i \leq k$. It is a routine check to prove that the smallest equivalence relation R_i containing $\{(m_i, t_i[\overline{m^i}/\overline{X^i}]) \mid i \leq k\}$ is a bisimulation for (\mathcal{M}, m_i) and $(\mathbb{U}(\mathcal{M}), t_i[\overline{m^i}/\overline{X^i}])$, hence $(\mathcal{M}, m_i) \sim (\mathbb{U}(\mathcal{M}), t_i[\overline{m^i}/\overline{X^i}])$. Similarly, one can prove $(\mathbb{U}(\mathcal{M}), t_i[\overline{m^i}/\overline{X^i}]) \sim (\mathbb{U}, t_i[\overline{s^i}/\overline{X^i}])$ by taking the smallest equivalence relation containing $\{(t_i[\overline{m^i}/\overline{X^i}], t_i[\overline{s^i}/\overline{X^i}]) \mid i \leq k\}$ and $\{(m_j^i, s_j^i) \mid i \leq k, j \leq l(i)\}$. By transitivity of \sim , $(\mathcal{M}, m_i) \sim \llbracket s_i \rrbracket$, for all $i \leq k$, hence (\mathcal{M}, m) is expressible. \square

The converse (up to bisimilarity) of the above result can also be proved, and it follows as a corollary of Theorems 4.7, 5.13, and 5.15.

Corollary 7.2. *If (\mathcal{M}, n) is expressible then it is finite up-to-bisimilarity.*

Proof. Let $t \in \mathbb{T}$. We have to show that there exists $(\mathcal{M}, m) \in \mathbf{OMC}$ with a finite set of states such that $\llbracket t \rrbracket \sim (\mathcal{M}, m)$. From Theorem 5.15, there exist t_1, \dots, t_k with free names in \overline{Y} , such that $\vdash t \equiv_0 t_1$ and

$$\vdash t_i \equiv_0 \sum_{j=1}^{h(i)} p_{ij} \cdot s_{ij} + \sum_{j=1}^{l(i)} q_{ij} \cdot Y_{g(i,j)}, \quad \text{for all } i \leq k,$$

where the terms s_{ij} and names $Y_{g(i,j)}$ are enumerated without repetitions, and s_{ij} is either $\text{rec } X.X$ or has the form $a_{ij}.t_{f(i,j)}$. Let Z_1, \dots, Z_k be fresh names distinct from \overline{Y} , and

⁵The results in this section can be alternatively obtained as in [SBBR11] by observing that open Markov chains are coalgebras of a quantitative functor.

define t'_i as the term obtained by replacing in the right end side of the equation above each occurrence of t_i with Z_i . Then, clearly $\vdash t_i \equiv_0 t'_i[\bar{t}/\bar{Z}]$. By soundness (Theorem 5.13), we have that $\llbracket t_i \rrbracket \sim \llbracket t'_i[\bar{t}/\bar{Z}] \rrbracket$, so that, by Theorem 4.7, $(\mathbb{U}, t_i) \sim (\mathbb{U}, t'_i[\bar{t}/\bar{Z}])$.

Define $\mathcal{M} = (M, \tau)$ by setting $M = \{t_1, \dots, t_k\}$, $m = t_1$, and, for all $i \leq k$, taking as $\tau(t_i)$ the smallest sub-probability distribution on $(\mathcal{L} \times M) \uplus \mathcal{X}$ such that $\tau(t_i)((a_{ij}, t_{f(i,j)})) = p_{ij}$ and $\tau(t_i)(Y_{g(i,e)}) = q_{ie}$, for all $i \leq k$, $j \leq h(i)$, and $e \leq l(i)$. Notice that since the equation above is without repetitions, τ is well defined. Moreover, $1 - \tau(m_i)((\mathcal{L} \times M) \uplus \mathcal{X}) = p_{iw}$ whenever there exists $w \leq h(i)$ such that $s_{iw} = \mathbf{rec} X.X$. It is not difficult to prove that $(\mathcal{M}, t_i) \sim (\mathbb{U}, t'_i[\bar{t}/\bar{Z}])$ (take the smallest equivalence relation containing the pairs $(t_i, t'_i[\bar{t}/\bar{Z}])$, for $i \leq k$), so that by transitivity of \sim , $(\mathcal{M}, t_i) \sim \llbracket t_i \rrbracket$, for all $i \leq k$. By $\vdash t \equiv_0 t_1$ and Theorem 5.13, we also have $\llbracket t \rrbracket \sim \llbracket t_1 \rrbracket$, thus $\llbracket t \rrbracket \sim (\mathcal{M}, m)$. \square

7.2. A Quantitative Kleene's Theorem. We provide a metric analogue to Kleene's representation theorem for finite pointed open Markov chains.

In [MPP16], Mardare et al. gave a construction for the free model of a quantitative theory of a generic quantitative deduction system. Here we present their definition only for the specific case of the quantitative deduction system presented in Section 5.3.

Definition 7.3 (Initial \vdash -model). The *initial \vdash -model* is defined as the quantitative algebra $(\mathbb{T}, \Sigma, d_{\mathbb{T}})$, where (\mathbb{T}, Σ) is the initial algebra of initial Σ -terms and $d_{\mathbb{T}}: \mathbb{T} \times \mathbb{T} \rightarrow [0, 1]$ is the 1-bounded pseudometric defined, for arbitrary terms $t, s \in \mathbb{T}$, as $d_{\mathbb{T}}(t, s) = \inf \{\varepsilon \mid \vdash t \equiv_{\varepsilon} s\}$.

Note that by **(Ref1)**, **(Symm)**, **(Triang)**, **(Top)** it is easy to prove that $d_{\mathbb{T}}$ is a well-defined 1-bounded pseudometric. Moreover, $(\mathbb{T}, \Sigma, d_{\mathbb{T}})$ is clearly a sound model for \vdash .

The attractiveness of the above model, as opposed to an operational one looking directly at the operational semantics of terms, is that it is purely syntactical. Indeed, one can reason about its properties by just proving statements about the distance between terms using classical logical deduction in the system \vdash .

Next we show that there is a strong correspondence between the initial \vdash -model and the quantitative algebra of finite pointed open Markov chains.

Theorem 7.4 (Quantitative Kleene's Theorem).

- (i) For every pair $(\mathcal{M}, m), (\mathcal{N}, n)$ of finite pointed open Markov chains, there exist terms $t, s \in \mathbb{T}$ such that $\llbracket t \rrbracket \sim (\mathcal{M}, m)$, $\llbracket s \rrbracket \sim (\mathcal{N}, n)$, and $\mathbf{d}((\mathcal{M}, m), (\mathcal{N}, n)) = d_{\mathbb{T}}(t, s)$;
- (ii) for every pair $t, s \in \mathbb{T}$, there exist finite pointed open Markov chains $(\mathcal{M}, m), (\mathcal{N}, n)$, such that $\llbracket t \rrbracket \sim (\mathcal{M}, m)$, $\llbracket s \rrbracket \sim (\mathcal{N}, n)$, and $\mathbf{d}((\mathcal{M}, m), (\mathcal{N}, n)) = d_{\mathbb{T}}(t, s)$.

Proof. Before proving (i) and (ii), we show that, for all $t, s \in \mathbb{T}$

$$d_{\mathbb{T}}(t, s) = \mathbf{d}(\llbracket t \rrbracket, \llbracket s \rrbracket). \tag{7.1}$$

The equality follows trivially by definition of $d_{\mathbb{T}}$ and the soundness and completeness theorems (Theorems 5.13 and 5.13). Indeed,

$$d_{\mathbb{T}}(t, s) = \inf \{\varepsilon \mid \vdash t \equiv_{\varepsilon} s\} = \inf \{\varepsilon \mid \models t \equiv_{\varepsilon} s\} = \inf \{\varepsilon \mid \mathbf{d}(\llbracket t \rrbracket, \llbracket s \rrbracket) \leq \varepsilon\} = \mathbf{d}(\llbracket t \rrbracket, \llbracket s \rrbracket).$$

(i) Given $(\mathcal{M}, m), (\mathcal{N}, n)$ finite pointed open Markov chains, we construct the Σ -terms t, s as in Corollary 7.1, obtaining that $\llbracket t \rrbracket \sim (\mathcal{M}, m)$, $\llbracket s \rrbracket \sim (\mathcal{N}, n)$. Now the results follows by (7.1) and Theorem 5.10.

(ii) Given $t, s \in \mathbb{T}$, we construct the finite pointed Markov chains $(\mathcal{M}, m), (\mathcal{N}, n)$ as in Corollary 7.2, obtaining that $\llbracket t \rrbracket \sim (\mathcal{M}, m), \llbracket s \rrbracket \sim (\mathcal{N}, n)$. Again the results follows by (7.1) and Theorem 5.10. \square

Remark 7.5 (The discounted case). Note that, by using the quantitative deduction system \vdash_λ presented in Section 6, adjusting the proof of Theorem 7.4 in the obvious way, a quantitative Kleene’s theorem can be obtained also for the λ -discounted bisimilarity distance.

8. CONCLUSIONS AND FUTURE WORK

In this paper we proposed a sound and complete axiomatization for the bisimilarity distance of Desharnais et al., later extended to its discounted variant. The axiomatic system proposed comes as a natural generalization of Stark and Smolka’s one [SS00] for probabilistic bisimilarity, to which we added only three extra axioms, namely **(Top)**, **(Pref)**, **(IB)** —along those required by the quantitative equational framework.

Although the use of the recursion operator does not fit the general framework of Mardare et al. [MPP16], we were able to prove completeness in a way that we believe is general enough to accommodate the axiomatization of other behavioral distances for probabilistic systems, such as the total variation distance. In the light of our result it would be nice to see how much the work in [MPP16] truly bases on the non-expansivity assumption for the algebraic operators. As as possible future work, it would be interesting to extend the general quantitative framework to algebraic operators that are only required to be Lipschitz-continuous (indeed, our proof uses the fact that the functional fixed point operator defining the recursion is q -Lipschitz continuous for some $q < 1$).

Another appealing direction of future work is to apply our results on quantitative systems described as coalgebras in a way similar to one proposed in [SBBR11, BMS13]. By pursuing this direction we would be able to obtain metric axiomatization for a wide variety of quantitative systems, including weighted transition systems, Segala’s systems, stratified systems, Pnueli-Zuck systems, etc.

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