Complete Proof Systems for Weighted Modal Logic

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Abstract

The weighted transition systems (WTS) considered in this paper are transition systems having both states and transitions labeled with real numbers: the state labels denote quantitative resources, while the transition labels denote costs of transitions in terms of resources. Weighted Modal Logic (WML) is a multi-modal logic that expresses qualitative and quantitative properties of WTSs. While WML has been studied in various contexts and for various application domains, no proof system has been developed for it. In this paper we solve this open problem and propose both weak-complete and strong-complete axiomatizations for WML against WTSs. We prove a series of metatheorems including the finite model property and the existence of canonical models. We show how the proof system can be used in the context of a process-algebra semantics for WML to convert a model-checking problem into a theorem-proving problem. This work emphasizes a series of similarities between WML and the probabilistic/stochastic modal logics for Markov processes and Harsanyi type spaces, such as the use of particular infinitary rules to guarantee the strong-completeness.

Keywords: Weighted transition systems, modal logic, proof systems, completeness

1. Introduction

Model-driven and component-based development (MDD) is finding its way into industrial practice, in particular within the area of embedded systems. Here a key challenge is how to handle the growing complexity of systems, while meeting requirements on correctness, predictability, performance and not least time- and cost-to-market. In this respect MDD is seen as a valuable and promising approach, as it allows early design-space exploration and verification and may be used as the basis for systematic and unambiguous testing of a final product. However, for embedded systems, verification should not only address functional properties but also a number of non-functional properties related to timing and resource constraints. Within the area of model checking,

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a number of state-machine based modeling formalisms have emerged, which allows for such quantitative aspects to be expressed. In particular the formalisms of timed automata [2], and the extensions to weighted timed automata [7, 3] allow for such constraints to be modeled and efficiently analyzed.

In several ways, the work on process calculi – pioneered by Tony Hoare [15] and Robin Milner [24] – addresses and provide principal solutions to several of the issues that are now considered within the application area of embedded systems. The desire for component-based development requires semantically well-defined notions of compositions, which preserve suitable notions of behavioral equivalence – as found in the algebraic part of a process calculus. Also, the logical part of a process calculus provides an immediate link to an unambiguous treatment of requirements to systems.

A desirable property of a process calculus is that of *adequacy* in the sense that the behavioral equivalence is identical with that of the process equivalence induced by the logic. This notion was coined in the landmark paper [14] showing that bisimilarity agrees with logical equivalence with respect to Hennessy-Milner logic. Soon after, several researchers[1] were developing proof systems and algorithms for establishing that (the behaviour of) a given process-algebraic term P satisfies a given logical property ϕ, i.e. $P \models \phi$. In particular, research was aiming at so-called *local* (or on-the-fly) methods [19] [32] [28] [11] and *compositional* methods [5, 31, 30] [18] [4] [22, 20]. In this effort, the pioneering work by Glynn Winskel was especially influential.

The additional notion of *expressiveness*[2] of a process calculus was introduced slightly later than that of *adequacy*. Here a process calculus is *expressive* if for any process algebraic term P, there exists a logical formula f_P such that $Q \models f_P$ precisely when Q is behavioural equivalent to P. In this way, equivalence checking $P \sim Q$ may be translated in to a model checking problem $Q \models f_P$, where f_P often is referred to as the characteristic property of P. Several behavioural equivalences have been shown to possess characteristic properties, e.g., [27, 10, 26]. Moreover, model checking problems $P \models \phi$ may be translated into validity problems of type $\models \psi_P \rightarrow \phi$, thus making the search for complete axiomatizations of validity the most fundamental research question. Here the work on axiomatizing the modal mu-calculus by Dexter Kozen [16] and Igor Walukiewicz [29] is a landmark result.

Motivated by the needs from embedded systems, we consider in this paper Weighted Modal Logic (WML) for weighted transition systems (WTS), allowing to specify and reason about not only the discrete behavior of a system but also its consumption of resources. However, rather than focusing on language theoretic issues, our aim is to investigate the fundamental question of axiomatization of the proposed weighted logic.

Our notion of weighted transition systems is not just a simple instance of a weighted automata [1], but we also study infinite and infinitely branching systems. We iden-

[1] including Glynn Winskel and the first author of the present paper.

[2] supposedly introduced by Amir Pnueli

tify, however, the subclass of WTSs that can be generated by finite terms of a simple Weighted Process Algebra (WPA) with only prefix and choice operations.

Weighted Modal Logic is a multi-modal logic defined for a semantics based on WTSs. It is endowed with modal operators that predicate about the values of both state and transition labels. While in a WTS we can have real labels, the modalities only encode rational values and often we are in the situation of characterizing a state or a transition using an infinite convergent sequences of rationals. Since in practice we often work with finite WTSs, we also developed a WPA-semantics for WML and we prove that WPA-processes can be in fact characterized by a WML formula. As mentioned before, this is important in applications since it can turn any model-checking problem into a validity-checking one.

In this paper we prove a series of metaproperties of WML. Firstly, we propose a weak-complete axiomatization for this logic guaranteeing that a logical formula is valid if and only if it is provable in our axiomatic system. In order to prove this result we demonstrate that WML enjoys the *finite model property* meaning that any consistent property has a finite model (WTS) and the cardinality of this model is bounded by parameters that depend on the syntactical structure of the property. In the context of a complete axiomatization, the logical characterization of WPA processes can be used to transform a model-checking problem of type $P \models \phi$ not only into a validity problem of type $\models f_P \to \phi$, but also into a theorem-proving problem of type $\vdash f_P \to \phi$ that has to be derived within the axiomatic system.

A second major achievement of this paper is providing a strong-complete extension of the aforementioned axiomatic system, which means that we can prove any consistent theory, possibly involving an infinite set of formulas. To get the strong completeness we had to consider, in addition to the infinitary version of Modus Ponens, one infinitary rule and to assume the Lindenbaum's lemma[3] as a meta-axiom. These assumptions are in line with the assumptions one needs to do to get strong completeness for other modal logics with quantitative modalities, such as the probabilistic logics defined for semantics on Markov processes or Harsanyi type spaces [33] and stochastic logics [9], [23]. In fact our infinitary rule is similar to the rule known in literature as the countable additivity rule used by Goldblatt to prove the strong completeness of logics for measurable polynomial functors on the category of measurable spaces [12].

2. Weighted Transition Systems and Weighted Process Algebra

In this section we introduce the concept of Weighted Transition System (WTS), which is a transition system that has both the nodes and the transitions labeled with real numbers such that if the transition from m to n is indexed by x, then the label of n is the sum of x and the label of m. One can interpret the label of a state as the resource available

[3]Lindenbaum's lemma states that any consistent set of formulas can be extended to a maximally-consistent one.

for possible transitions and the label of a transition as the resource consumed/produced for the transition to take place (the cost of the transition). Our intention is to remain as general as possible and for this reason we impose no restriction on the labels: they can be any real number, possibly negative.

Definition 2.1 (Weighted Transition System)**.** *A* weighted transition system *is a tuple* $\mathcal{W} = (M, \theta, l)$ *where* M *is an arbitrary set of* states*,* $\theta \subseteq M \times \mathbb{R} \times M$ *is the* transition function *and* $l : M \to \mathbb{R}$ *is a labeling function such that whenever* $(m, x, m') \in \theta$*,*

$$l(m') = l(m) + x.$$

As usual when working with transition systems, instead of $(m, x, m') \in \theta$, we write $m \stackrel{x}{\longrightarrow} m'$.

Now we introduce the concept of Weighted Bisimulation, which is an equivalence relation that relates systems with identical weighted behaviour.

Definition 2.2 (Weighted Bisimulation)**.** *Given a WTS* $\mathcal{W} = (M, \theta, l)$*, a weighted bisimulation is an equivalence relation* $R \subseteq M \times M$ *such that whenever* $(m, n) \in R$*, then*

- $l(m) = l(n)$*;*
- *if* $m \stackrel{x}{\longrightarrow} m'$*, then there exists* $n' \in M$ *s.t.* $n \stackrel{x}{\longrightarrow} n'$ *and* $(m', n') \in R$*.*
- *if* $n \stackrel{x}{\longrightarrow} n'$*, then there exists* $m' \in M$ *s.t.* $m \stackrel{x}{\longrightarrow} m'$ *and* $(m', n') \in R$*.*

If there exists a weighted bisimulation relation R *such that* $(m, n) \in R$*, we say that* m *and* n *are* bisimilar*. Thus,* weighted bisimilarity *is the largest bisimulation relation that can be defined for a given WTS.*

To give a formal support to the concept of WTS, in what follows we define a basic process algebra, named *Weighted Process Algebra* (WPA), which encodes weighted behaviours. This process algebra is defined on top of 0, which is a terminal process, and includes the basic operations of prefixing and nondeterministic choice. The prefixes are rational numbers that encode the cost of a transition.

Definition 2.3 (Weighted Process Algebra)**.** *The set* $\mathcal{P}$ *of WPA processes is inductively defined by the grammar presented below, for arbitrary* $r \in \mathbb{Q}$*.*

$$P ::= 0 \mid r.P \mid P + P.$$

Processes that are identical modulo their syntactic representation are related by the structural congruence relation. We made this clear in what follows where we also present a structural operational semantics for WPA.

Definition 2.4 (Structural Congruence)**.** *The structural congruence is an equivalence relation* $\equiv \subseteq \mathcal{P} \times \mathcal{P}$ *that satisfies the following axioms.*

(Associativity):	$(P + Q) + R \equiv P + (Q + R)$
(Commutativity):	$P + Q \equiv Q + P$
(Null):	$P + 0 \equiv P$
(Idempotence):	$P + P \equiv P$

Observe that a WPA processes can only define the skeleton of a WTS since it cannot express the resources available in a state. To overcome this, in Table 1 we define a structural operational semantics for WPA; the transitions are defined not between processes but between tuples of type $(r, P) \in \mathbb{Q} \times \mathcal{P}$, where r represents the resources available in a state and P the process enabled in that state.

(Prefix).	$\dfrac{}{(s, r.P) \xrightarrow{r} (s + r, P)}$
(Plus).	$\dfrac{(s, P) \xrightarrow{r} (s', P')}{(s, P + Q) \xrightarrow{r} (s', P')}$
(Str).	$\dfrac{P \equiv Q \qquad (s, P) \xrightarrow{r} (s', P')}{(s, Q) \xrightarrow{r} (s', P')}$

Table 1: Structural Operational Semantics for WPA

Observe that the aforementioned SOS induces indeed a WTS,

$$\mathcal{W}_{\mathcal{P}} = (\mathbb{Q} \times \mathcal{P}, \theta_{\mathcal{P}}, l_{\mathcal{P}}),$$

where the transition function $\theta_{\mathbb{P}}$ is induced by SOS and $l_{\mathcal{P}}(s, P) = s$ for any $P \in \mathcal{P}$. Structural congruent processes have always identical behaviours when they use identical resources. Notice also that since the syntax of WPA only involves rational prefixes (we do not want an uncountable language), the set of WTS that correspond to some process is quite limited including only the transition systems that are finite, finitely branching and have rational labels.

It is useful, for our future analysis, to exploit the structural congruence rules in order to define canonical forms for the (syntax of) processes, since we are only interested in processes up to structural congruence. To do this we firstly adopt the symbol $\sum$ to denote nondetermnistic choice of a set of processes; for instance, instead of $P_1+P_2+P_3$ we write $\sum_{i\in\{1,2,3\}} P_i$.

We assume that 0 is a canonic form and the process $P \in \mathcal{P}$ is in canonical for if there exists $k \in \mathbb{N}$, for each $i = 1..k$ there exists a finite set J_i of indexes and there exists a strictly increasing sequence $(x_i)_{i\in\{1,..k\}}$ of rationals such that

$$P = \sum_{i\in\{1,..k\}} \sum_{j\in J_i} x_i.P_i^j,$$

where P_i^j are processes in canonical form such that for each $i \in \{1, ..k\}$, $P_i^k \not\equiv P_i^l$ whenever $k \neq l$.

A simple induction on the structure of a process proves that any WPA-process has a canonical form.

3. Weighted Modal Logic

Returning to the weighted transition systems, in this section we propose a multi-modal logic that can encode properties of WTSs called Weighted Modal Logic (WML). Our logic is endowed, in addition to the classic boolean operators, with a class of modalities of arity 0 called *state modalities* of type $(\leq r)$ or $(\geq r)$ for $r \in \mathbb{Q}$ that predicates about the value of the label of the initial state of a WTS; and a class of modalities of arity 1, named *transition modalities*, of type $[\leq r]$ or $[\geq r]$, for $r \in \mathbb{Q}$, which refer to the transition labels.

3.1. Basic definitions

Before proceeding with the formal definitions we establish some useful notations. In the rest of the paper we use $\unlhd$ and $\unrhd$ to range over the set $\{\leq, \geq\}$ such that $\{\unlhd, \unrhd\} = \{\leq, \geq\}$; these mean that $\unlhd$ can either represent $\leq$ or $\geq$ and if $\unlhd$ represents one, the $\unrhd$ denotes the other. Similarly, we use $\lhd$ and $\rhd$ to range over the set $\{<, >\}$ such that $\{\lhd, \rhd\} = \{<, >\}$.

Definition 3.1 (Syntax of WML)**.** *The formulas of WML are collected in the set $\mathcal{L}$ introduced, for arbitrary $r \in \mathbb{Q}$ and $\unlhd \in \{\leq, \geq\}$, by the following grammar.*

$$\mathcal{L}: \quad \phi := \top \mid (\unlhd r) \mid \neg\phi \mid \phi \wedge \phi \mid [\unlhd r]\phi.$$

The *WTS-semantics* of WML is given by the *satisfiability relation* defined, for an arbitrary WTS $\mathcal{W} = (M, \theta, l)$ and arbitrary $m \in M$, inductively as follows. $\mathcal{W}, m \models \top$

always,

$\mathcal{W}, m \models (\unlhd r)$ iff $l(m) \unlhd r$,

$\mathcal{W}, m \models \neg\phi$ iff it is not the case that $\mathcal{W}, m \models \phi$,

$\mathcal{W}, m \models \phi \wedge \psi$ iff $\mathcal{W}, m \models \phi$ and $\mathcal{W}, m \models \psi$,

$\mathcal{W}, m \models [\unlhd r]\phi$ iff for any $m' \in M$ and $x \in \mathbb{R}$ such that $x \unlhd r$ and $m \xrightarrow{x} m'$, $\mathcal{W}, m' \models \phi$.

Observe in the semantics of $[\unlhd r]\phi$ that, as usual in modal logics, the universal quantification is trivially satisfied if the initial state admits no transition.

In addition to the basic operators, we work with all boolean operators, including $\bot = \neg\top$. We also define De Morgan duals of $[\unlhd x]$, by $\langle \unlhd x\rangle\phi = \neg[\unlhd x]\neg\phi$; and the derived operators $(r) = (\leq r) \wedge (\geq r)$.

Observe the semantics of the derived operators:

$\mathcal{W}, m \models (r)$ iff $l(m) = r$,

$\mathcal{W}, m \models \langle r\rangle\phi$ iff there exists $m' \in M$ such that $m \xrightarrow{r} m'$ and $\mathcal{W}, m' \models \phi$.

When it is not the case that $\mathcal{W}, m \models \phi$, we write $\mathcal{W}, m \not\models \phi$. We say that a formula ϕ is *satisfiable* if there exists at least one WTS that satisfies it in some state. We say that ϕ is a *validity* if it is satisfied by any WTS in any state - in this case we write $\models \phi$. If $\Phi \subseteq \mathcal{L}$, we write $\Phi \models \phi$ if any model that satisfies all the formulas in Φ also satisfies ϕ.

Notice that while WTS can have irrational labels, WML can only express properties using rational labels in modalities. We will see in the next section that this does not restrict too much the expressive power of our language since we can express any irrational using a convergent sequence of rationals.

3.2. Expressiveness of WML for process semantics

Since SOS of WPA induces weighted transition systems, we can naturally adapt the semantics of WML to process algebra semantics. Let us define $\models \subseteq (\mathbb{R}\times\mathcal{P})\times\mathcal{L}$ inductively as follows.

$(s, P) \models \top$ always,

$(s, P) \models (\trianglelefteq r)$ iff $s \trianglelefteq r$,

$(s, P) \models \neg\phi$ iff it is not the case that $(s, P) \models \phi$,

$(s, P) \models \phi \wedge \psi$ iff $(s, P) \models \phi$ and $(s, P) \models \psi$,

$(s, P) \models [\trianglelefteq r]\phi$ iff for any $(r, P') \in \mathbb{Q} \times \mathcal{P}$ and $x \in \mathbb{Q}$ such that $x \trianglelefteq r$ and $(s, P) \xrightarrow{x} (s', P')$, $(s', P') \models \phi$.

Observe the relation between the process algebra semantics and the WTS-based semantics: for arbitrary $(s, P) \in \mathbb{Q} \times \mathcal{P}$ and $\phi \in \mathcal{L}$,

$$(s, P) \models \phi \quad \text{iff} \quad \mathcal{W}_{\mathcal{P}}, (s, P) \models \phi.$$

One of the key applications of WML for process algebra is the use of characteristic formulas. Recall that the set of WTSs associated to WPA-processes represents a strict subset of the class of WTSs. More exactly, the WTS of a process is finite, finitely branching and it can only have rational labels. For this reason we can define logical formulas that characterize WPA processes and their resources.

Definition 3.2 (Characteristic Formulas)**.** *Consider the set $C \subseteq \mathcal{L}$ of logical formulas defined inductively on the structure of tuples $(s, P) \in \mathbb{Q} \times \mathcal{P}$ as follows*

$$f_{(s,0)} = (s) \wedge ([\leq 0]\bot \wedge [\geq 0]\bot);$$

if $P = \sum_{i\in\{1,..k\}} \sum_{j\in J_i} x_i.P_i^j$ *is in canonical form then,*

$$f_{(s,P)} = (s) \wedge \Big(\bigwedge_{i\in\{1,..k\}} \bigwedge_{j\in J_i} \langle\leq x_i\rangle f_{(s+x_i,P_i^j)}\Big) \wedge \Big(\bigwedge_{i\in\{1,..k\}} \bigwedge_{j\in J_i} \langle\geq x_i\rangle f_{(s+x_i,P_i^j)}\Big)$$

$$\wedge \bigwedge_{i\in\{1,..k\}} [\leq x_i](\bigvee_{l\in\{1,..i\}} \bigvee_{j\in J_l} f_{(s+x_l,P_l^j)})$$

$$\wedge \bigwedge_{i\in\{1,..k\}} [\geq x_i](\bigvee_{l\in\{i,..k\}} \bigvee_{j\in J_l} f_{(s+x_l,P_l^j)}).$$

Since any process has a canonic representation, we can define a characteristic formula for any process. In what follows we prove that, indeed, the formulas previously defined are characteristic formulas for their corresponding processes.

Theorem 1. *For arbitrary* $(s,P),(s',P')\in\mathbb{Q}\times\mathcal{P}$*,*

1. $(s,P)\models f_{(s,P)}$*;*
2. *if* $(s',P')\models f_{(s,P)}$*, then* $s=s'$ *and* $P\equiv P'$*;*

Proof. 1. A simple induction on (s,P).
2. We prove it by induction on the structure of (s,P).

The case $(s,0)$**:** formula (s) in the definition of $f_{(s,0)}$ guarantees that $l_{\mathcal{P}}(s,P')=s$, i.e., $s=s'$, while $[\leq 0]\bot\wedge[\geq 0]\bot$ encodes the fact that the process P' cannot do any transition, i.e., it is structural congruent to 0.

The inductive step: suppose that $P=\sum_{i\in\{1,..k\}}\sum_{j\in J_i} x_i.P_i^j$ is in the canonical form.

Since $(s',P')\models(s')$ and (s) is a conjunct of $F_{(s,P)}$, we obtain that $s=s'$ since $\not\models(s)\wedge(s')$ for $s\neq s'$.

Because $(s,P')\models\bigwedge_{i\in\{1,..k\}}\bigwedge_{j\in J_i}\langle\leq x_i\rangle f_{(s+x_i,P_i^j)}$, there exists $y_i\in\mathbb{Q}$, $y_i\leq x_i$, such that $P'\xrightarrow{y_i} P_i^j$. Because $(s+x_i,P_i^j)$ is a conjunct of $f_{(s+x_i,P_i^j)}$, we obtain that $(s,P')\models\langle\leq x_i\rangle(s+x_i)$ which proves that $x_i=y_i$.

Consequently, for each $i\in\{1,..k\}$ and each $j\in J_i$, $P'\xrightarrow{x_i}P_i^j$. To conclude that $P\equiv P'$, we have to show that these are all the transitions of P'.

$(s,P')\models\wedge\bigwedge_{i\in\{1,..k\}}[\leq x_i](\bigvee_{l\in\{1,..i\}}\bigvee_{j\in J_l} f_{(s+x_l,P_l^j)})$ proves that the previous transitions are all the transitions of P' at a cost inferior to x_k. Similarly, $(s,P')\models\wedge\bigwedge_{i\in\{1,..k\}}[\geq x_i](\bigvee_{l\in\{1,..i\}}\bigvee_{j\in J_l} f_{(s+x_l,P_l^j)})$ proves that the previous transitions are all the transitions of P' at a cost superior to x_1. Hence, the aforementioned transitions are all the transitions of P' and this proves in our context that $P\equiv P'$. ∎

The characteristic formulas are useful since they can help us converting any model-checking problem into a satisfiability problem. For instance, suppose that we want to decide whether $(s,P)\models\phi$. Since $(s,P)\models f_{(s,P)}$, our model-checking problem is equivalent to deciding whether $\models f_{(s,P)}\to\phi$. Indeed for a different model (s',P'), we have that $(s',P')\models\neg f_{(s,P)}$ and consequently $(s',P')\models f_{(s,P)}\to\phi$; and $(s,P)\models f_{(s,P)}\to\phi$ iff $(s,P)\models\phi$.

4. Metatheory for Weighted Modal Logic

In this section we develop the metatheory of weighted modal logic. We present a weak-complete and a strong-complete axiomatization, we prove the finite model property and we construct canonical models. We also show how the model-checking problem for process algebra can be turned into a theorem-proving problem.

4.1. A sound axiomatization

In the table 2 we present an axiomatic system for WML. The axioms are considered in addition to the axioms of propositional logic. To state them we use the following notation

- $\{\unlhd, \unrhd\} = \{\leq, \geq\}$;
- the labels of the modal operators ( ) and $\langle\ \rangle$ involve the operations $*$ and $\overline{*}$ defined by

$$\unlhd r * s = \begin{cases} \unlhd r + s & \text{if } \unlhd = \leq \\ \unlhd r - s & \text{if } \unlhd = \geq \end{cases} \qquad \unlhd r \overline{*} s = \begin{cases} \unlhd r + s & \text{if } \unlhd = \geq \\ \unlhd r - s & \text{if } \unlhd = \leq \end{cases}$$

- we consider the set $X = \{[\unlhd r] \mid r \in \mathbb{Q}, \unlhd \in \{\leq, \geq\}\}$ and the language X^* of finite sequences of elements in X including the empty sequence.

The axioms and rules below are stated for arbitrary $\phi \in \mathcal{L}$, $r, s \in \mathbb{Q}$ and $w \in X^*$.

(A1):	$\vdash [\unlhd r](\phi \to \psi) \to ([\unlhd r]\phi \to [\unlhd r]\psi)$
(A2):	$\vdash (\unlhd r) \vee (\unrhd r)$
(A3):	$\vdash (\unlhd r) \to (\unlhd r * s), \quad s > 0$
(A4):	$\vdash (\unlhd r) \to \neg(\unrhd r \overline{*} s), \quad s > 0$
(A5):	$\vdash \langle \unlhd r \rangle \phi \to \langle \unlhd r * s \rangle \phi, \quad s > 0$
(A6):	$\vdash (\unlhd r) \to [\unlhd s](\unlhd r + s)$
(R1):	If $\vdash \phi$, then $\vdash [\unlhd r]\phi$
(R2):	$\{w(\unlhd r) \mid \text{ for all } r \lhd s\} \vdash w(\unlhd s)$
(R3):	$\{w(\unlhd r) \mid \text{ for all } r \in \mathbb{Q}\} \vdash w\bot$

Table 2: Axioms of Weighted Logic

Axiom (A1) expresses in a condensed way the following two axioms-schema $\vdash [\leq r](\phi \to \psi) \to ([\leq r]\phi \to [\leq r]\psi)$ and $\vdash [\geq r](\phi \to \psi) \to ([\geq r]\phi \to [\geq r]\psi)$, and it expresses the fact that these modalities are normal modalities.

Axiom (A2) guarantees the uniqueness of the state label.

The instances of the axiom (A3) are $\vdash (\leq r) \to (\leq r + s)$ and $\vdash (\geq r) \to (\geq r - s)$ for $s > 0$ and they encode obvious monotonicity properties of the state modalities. Similar results are encoded by (A4) with the instances $\vdash (\leq r) \to \neg(\geq r + s)$ and $\vdash (\geq r) \to \neg(\leq r - s)$ for $s > 0$.

Axiom (A5) states the monotonicity of the transition modalities and can be instantiated by $\vdash \langle \leq r\rangle\phi \to \langle \leq r+s\rangle\phi$ and $\vdash \langle \geq r\rangle\phi \to \langle \geq r-s\rangle\phi$.

Axiom (A6) is probably the most important axiom and it states the relation between state modalities and transition modalities. Its instance $\vdash (\leq r) \to [\leq s](\leq r+s)$ states that if the initial state of a transition is labeled with "at least r" and the transition is label with "at least s", then the final state has the label "at least $r+s$"; its other instance $\vdash (\geq r) \to [\geq s](\geq r+s)$ encode the dual property for "at most".

Rule (R1) complements the normality condition of the transition modalities stated in (A1).

Rules (R2) and (R3) are infinitary rules that encode the archimedean properties of rationals. (R2) states that if a state label is at least (at most) equal to r for any $r < s$, then it is at least (at most) equal to s. The prefix $w \in X^*$ lifts the Archimedeean property to arbitrary prefixing using box modalities. (R3) rules out the possibility that a state label is infinite.

As usual, we say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the given axioms and rules. We say that ϕ is *consistent*, if $\phi \to \bot$ is not provable. Given a set Φ of formulas, we say that Φ proves ϕ, $\Phi \vdash \phi$, if from the formulas of Φ and the axioms one can prove ϕ. Φ is *consistent* if it is not the case that $\Phi \vdash \bot$; Φ is *finite-consistent* if any finite subset of it is consistent. For a sublanguage $\Lambda \subseteq \mathcal{L}$, we say that Φ is Λ-*maximally consistent* if Φ is consistent and no formula of Λ can be added to Φ without making it inconsistent.

The axiomatic system can now be used to derive theorems as the ones collected in the following proposition.

Proposition 1. *For arbitrary $\phi \in \mathcal{L}$ and $r, s \in \mathbb{Q}$ the following statements are provable from the axioms in table 2.*

1. $\vdash (r) \to \neg(s)$ *for $r \neq s$;*
2. $\vdash [\leq r]\phi \to [\leq r-s]\phi$ *for any $s > 0$;*
3. $\vdash [\geq r]\phi \to [\geq r+s]\phi$ *for any $s > 0$.*

We conclude this section stating the soundness of the axiomatic system.

Theorem 2 (Soundness)**.** *The axiomatic system in Table 2 is sound with respect to the semantics based on WTSs, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\vdash \phi \text{ implies } \models \phi.$$

Proof. As usual, the soundness is proved by verifying that each axiom is sound and that the rules preserve soundness. These can be trivially verified. ∎

4.2. Weak-Completeness for Weighted Modal Logic

In what follows we prove that the axiomatic system in Table 2 is not only sound, but also weak-complete for the WTS-semantics, meaning that all the validities can be proved. In order to prove this, it is sufficient to prove that any consistent formula of $\mathcal{L}$ has a model.

Consider an arbitrary formula $\phi \in \mathcal{L}$ and let $R_\phi \subseteq \mathbb{Q}$ be the set of all $r \in \mathbb{Q}$ such that r is in the label of some state or transition modality $(\unlhd r)$, $\langle \unlhd r \rangle$ or $[\unlhd r]$ that appears in the syntax of ϕ. Obviously, R_ϕ is finite.

- The *granularity of* $\phi \in \mathcal{L}$, denoted by $gr(\phi)$ is the least common denominator of the elements of R_ϕ.
- The *modal depth of* ϕ, denoted by $md(\phi)$, is defined inductively by

$$md(\phi) = \begin{cases} 0, & \text{if } \phi = \top \text{ or } \phi = (\unlhd r) \\ md(\psi), & \text{if } \phi = \neg\psi \\ max\{md(\psi), md(\psi')\}, & \text{if } \phi = \psi \wedge \psi' \\ md(\psi) + 1, & \text{if } \phi = \langle \unlhd r \rangle \psi \end{cases}$$

- The *boundary of* ϕ is the interval $I(\phi) = [min(\phi), max(\phi)]$, where

$$max(\phi) = \max\{max(R_\phi) \cdot md(\phi), 0\}, \quad min(\phi) = \min\{0, min(R_\phi) \cdot md(\phi)\}.$$

In what follows we fix a consistent formula $\rho \in \mathcal{L}$ and we construct a model for it. Let

$$\mathcal{L}[\rho] = \{\phi \in \mathcal{L} \mid gr(\phi) \leq gr(\rho), md(\phi) \leq md(\rho), I(\phi) \subseteq I(\rho)\}.$$

Let Ω be the set of $\mathcal{L}$-maximal consistent sets of formulas and $\Omega[\rho]$ the set of $\mathcal{L}[\rho]$-maximal consistent sets of formulas. Observe that by construction $\mathcal{L}[\rho]$ is finite modulo logical equivalence, implying that $\Omega[\rho]$ is finite.

Using Rasiowa-Sikorski theorem [13, 17], we know that for each $\Lambda \in \Omega[\rho]$ there exists $\Gamma \in \Omega$ such that $\Lambda \subseteq \Gamma$. In fact, for one Λ there might exist more than one Γ with this property.

Suppose that for each $\Lambda \in \Omega[\rho]$ we chose one $\Gamma \in \Omega$ such that $\Lambda \subseteq \Gamma$; to identify it, we denote this Γ by Λ^+. Let $\Omega^+[\rho] = \{\Lambda^+ \in \Omega \mid \Lambda \in \Omega[\rho]\}$.

In what follows we will construct a WTS $\mathcal{W}_\rho = (S_\rho, \theta_\rho, l_\rho)$ such that $S_\rho = \Omega^+[\rho]$. To do this, we have to properly define the transition function and the labeling function.

In order to define the transition function, we need to prove the following lemma that will be repeatedly used in the sequent proofs.

Lemma 1. *For arbitrary* $\Gamma, \Gamma' \in \Omega$ *and arbitrary* $r, s \in \mathbb{Q}$ *with* $s > 0$*,*

1. *If* $[[\leq r]\phi \in \Gamma$ *implies* $\phi \in \Gamma']$*, then* $[[\leq r + s]\phi \in \Gamma$ *implies* $\phi \in \Gamma']$*;*
2. *If* $[[\geq r]\phi \in \Gamma$ *implies* $\phi \in \Gamma']$*, then* $[[\geq r - s]\phi \in \Gamma$ *implies* $\phi \in \Gamma']$*;*
3. *If* $x = \inf\{u \in \mathbb{Q} \mid [\leq u]\phi \in \Gamma$ *implies* $\phi \in \Gamma'\}$ *and* $r \geq x$*, then*

$$[\leq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma';$$

4. *If* $x = \sup\{u \in \mathbb{Q} \mid [\geq u]\phi \in \Gamma$ *implies* $\phi \in \Gamma'\}$ *and* $r \leq x$*, then*

$$[\geq r]\phi \in \Gamma \textit{ implies } \phi \in \Gamma'.$$

Proof. 1. From Proposition 4.2 we know that $[\leq r + s]\phi \in \Gamma$ implies $[\leq r]\phi \in \Gamma$ which implies further $\phi \in \Gamma'$.

2. In the same way as the previous case.

3. It is a direct consequence of one using the fact that x is an infimum and similarly, 4. is a consequence of 2. ∎

Notice in the previous lemma in the cases 3 and 4 that x might be irrational and consequently it cannot appear as an index of a modality, but it can be the limit of some increasing or decreasing sequence of rationals.

Now we are ready to define the transition function $\theta_\rho \subseteq \Omega^+[\rho] \times \mathbb{R} \times \Omega^+[\rho]$. As before, for arbitrary $\Gamma, \Gamma' \in \Omega^+[\rho]$ and $x \in \mathbb{R}$, instead of writing $(\Gamma, x, \Gamma') \in \theta_\rho$, we write $\Gamma \xrightarrow{x} \Gamma'$.

If $\Gamma, \Gamma' \in \Omega^+[\rho]$ are such that

- $\sup\{r \in \mathbb{Q} \mid [\geq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} = \inf\{r \in \mathbb{Q} \mid [\leq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\}$,
- $x = \sup\{r \in \mathbb{Q} \mid [\geq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} \in \mathbb{R}$,

then we define $\Gamma \xrightarrow{x} \Gamma'$.

It remains now to define the labeling function l_ρ. To do this we prove the following lemma.

Lemma 2. *For arbitrary* $\Gamma \in \Omega$*,*

$$\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}.$$

Proof. We prove that $\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}$ and $\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$ are both non-empty. Axiom (A2) guarantees that for any $r \in \mathbb{Q}$ we have either $(\geq r) \in \Gamma$, or $(\leq r) \in \Gamma$. Suppose that there exists an $r \in \mathbb{Q}$ such that $(\geq r) \in \Gamma$. Then the first set is non-empty. Suppose that the second is empty, then the same axiom (A2) implies that for any $r \in \mathbb{Q}$, $(\geq r) \in \Gamma$. Using (R4) we derive further that $\bot \in \Gamma$ – this contradicts the consistency of Γ. Consequently, the second set cannot be empty. Similarly can be proved that if we assume that the second set is nonempty we obtain that the first is non-empty either.

Since the two sets are non-empty, the sup and inf exist. We prove that both are reals. Suppose that $\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} = +\infty$. Then applying (A3) we obtain that $\mathbb{Q} \subseteq \{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}$ and further (R4) proves the inconsistency of Γ - impossible! Hence, $\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} \in \mathbb{R}$. Similarly one can prove that $\inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\} \in \mathbb{R}$.

Let $\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} = x$ and $\inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\} = y$. We need to prove that $x = y$.

Suppose that $x < y$. Then, there exists $r \in \mathbb{Q}$ such that $x < r < y$. Since $x < r$, $(\geq r) \notin \Gamma$ and (A2) guarantees that $(\leq r) \in \Gamma$. But this contradicts the fact that $r \leq y$.

Suppose that $y < x$. Then, there exists $r_1, r_2 \in \mathbb{Q}$ such that $y < r_1 < r_2 < x$. Since $r_i > y$, using (A3) we obtain that $(\leq r_i) \in \Gamma$ for $i = 1, 2$ and similarly, $r_i < x$ implies $(\geq r_i) \in \Gamma$ for $i = 1, 2$. This means that $(r_1), (r_2) \in \Gamma$ and using Proposition .1, since $r_1 \neq r_2$, we obtain that Γ is inconsistent - contradiction!

Consequently, $x = y$. ∎

This result allows us to define, for arbitrary $\Gamma \in \Omega^+[\rho]$,

$$l_\rho(\Gamma) = \sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}.$$

Theorem 3. *The tuple $\mathcal{W}_\rho = (\Omega^+[\rho], \theta_\rho, l_\rho)$ defined above is a weighted transition system.*

Proof. To prove this we only need to show that if for arbitrary $\Gamma, \Gamma' \in \Omega^+[\rho]$ we have $\Gamma \xrightarrow{x} \Gamma'$, then $l_\rho(\Gamma') = l_\rho(\Gamma) + x$.

Let $y = l_\rho(\Gamma)$ and $y' = l_\rho(\Gamma')$.

Consider arbitrary $r_x, r_y \in \mathbb{Q}$ with $r_x \leq x$ and $r_y \leq y$. Due to (A3), we have $(\geq r_y) \in \Gamma$. Then, using (A6), $[\geq r_x](\geq r_x + r_y) \in \Gamma$ and since $x = \sup\{r \in \mathbb{Q} \mid [\geq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} \geq r_x$, Lemma 1 guarantees further that $(\geq r_x + r_y) \in \Gamma'$. Consequently, $l_\rho(\Gamma') \geq r_x + r_y$ for arbitrary $r_x \leq x$ and $r_y \leq y$. Hence, $l_\rho(\Gamma') \geq x + y$.

Following a similar argument, one can prove that we also have $l_\rho(\Gamma') \leq x + y$ and consequently, $l_\rho(\Gamma') = x + y$. ∎

Having this WTS constructed, we prove that actually there exists a state in this WTS such that its WTS is a model of ρ. To do this, we need to prove first a few additional lemmas.

Lemma 3. *Let $\Phi \subseteq \mathcal{L}$ be a consistent set of formulas. For arbitrary $\phi \in \mathcal{L}$ and $r \in \mathbb{Q}$, if $[\trianglelefteq r]\phi \notin \overline{\Phi}$, then the set $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]\psi \in \overline{\Phi}\} \cup \{\neg\phi\}$ is consistent.*

Proof. Let $A = \{\psi \in \mathcal{L} \mid [\trianglelefteq r]\psi \in \overline{\Phi}\}$. Suppose $A \cup \{\neg\phi\}$ is inconsistent, then there exists a set $F = \{f_i \in A \mid i \in I\}$ of formulas such that $F \vdash \phi$. Let $[\trianglelefteq r]F = \{[\trianglelefteq r]f \mid f \in F\}$. If I is finite, (R1) guarantees that $[\trianglelefteq r]F \vdash [\trianglelefteq r]\phi$. Otherwise, $F \vdash \phi$ is an instance of (R2) or (R3); in both cases $[\trianglelefteq r]F \vdash [\trianglelefteq r]\phi$ is an instance of the same rule.

Consequently, $[\trianglelefteq r]F \vdash [\trianglelefteq r]\phi$. Since $F \subseteq A$, $[\trianglelefteq r]F \subseteq \overline{\Phi}$. Hence, $[\trianglelefteq r]\phi \in \overline{\Phi}$ contradicting the assumption of consistency of Φ. ∎

Corollary 1. *For any* $\Gamma \in \Omega^+[\rho]$, $\phi \in \mathcal{L}[\rho]$ *and* $r \in \mathbb{Q}$,

$$[\trianglelefteq r]\phi \in \Gamma \textit{ iff } \forall \Gamma' \in \Omega^+[\rho] \textit{ s.t. } \Gamma \xrightarrow{x} \Gamma' \textit{ with } x \trianglelefteq r, \phi \in \Gamma'.$$

Proof. ($\Longrightarrow$) This implication derives directly from the definition of the labeled transitions on $\mathcal{W}_\rho$.
($\Longleftarrow$) Let $B = \{\Gamma' \in \Omega^+[\rho] \mid \Gamma \xrightarrow{x} \Gamma', x \trianglelefteq r\}$ and $\bigcap B = \bigcap_{\Gamma' \in B} \Gamma'$. Observe that since all the elements of $\Omega^+[\rho]$ are maximal consistent sets, $\psi \notin \bigcap B$ iff there exists $\Gamma' \in \Omega^+[\rho]$ such that $\Gamma \xrightarrow{x} \Gamma'$ and $\neg\psi \in \Gamma'$.

Let $\phi \in \bigcap B$ and suppose that $[\trianglelefteq r]\phi \notin \Gamma$. Applying Lemma 3, we obtain that the set $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]\psi \in \Gamma\} \cup \{\neg\phi\}$ is consistent. Since $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]\psi \in \Gamma\} \subseteq \bigcap B$ and $\neg\phi \in \mathcal{L}[\rho]$, it must exist $\Gamma'' \in \Omega^+[\rho]$ such that $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]\psi \in \Gamma\} \cup \{\neg\phi\} \subseteq \Gamma''$. Since $\Gamma'' \supseteq \{\psi \in \mathcal{L} \mid [\trianglelefteq r]\psi \in \Gamma\}$, we obtain that $\Gamma'' \in B$. On the other hand, $\Gamma'' \ni \neg\phi$ - contradiction!

Hence, $[\trianglelefteq r]\phi \in \Gamma$.

∎

Now we are ready to prove the Truth Lemma.
Lemma 4 (Truth Lemma)**.** *If* $\rho \in \mathcal{L}$ *is consistent, then for arbitrary* $\psi \in \mathcal{L}[\rho]$ *and* $\Gamma \in \Omega^+[\rho]$,

$$\psi \in \Gamma \quad \textit{iff} \quad \mathcal{W}_\rho, \Gamma \models \psi.$$

Proof. Induction on ψ.

The case $\psi = (\leq r)$**:** Since $l_\rho(\Gamma) = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$, $(\leq r) \in \Gamma$ is equivalent to $r \geq l_\rho(\Gamma)$, which is further equivalent to $\mathcal{W}_\rho, \Gamma \models (\leq r)$.

The case $\psi = (\geq r)$**:** It is proved similarly to the previous case.

The case $\psi = [\leq r]\phi$**:** ($\Longrightarrow$) Suppose that $[\leq r]\phi \in \Gamma$. If there exists no $\Gamma' \in \Omega[\rho]$ such that $\Gamma \xrightarrow{x} \Gamma'$ with $x \leq r$, then trivially $\mathcal{W}_\rho, \Gamma \models [\leq r]\phi$. Otherwise, consider an arbitrary $\Gamma' \in \Omega[\rho]$ such that $\Gamma \xrightarrow{x} \Gamma'$ with $x \leq r$. Since $x = \inf\{u \in \mathbb{Q} \mid [\leq u]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} \leq r$, applying Lemma 1, we obtain that $[\leq r]\phi \in \Gamma$ implies $\phi \in \Gamma'$; and using the inductive hypothesis for ϕ, we obtain $\mathcal{W}_\rho, \Gamma' \models \phi$. All these prove that $\mathcal{W}_\rho, \Gamma \models [\leq r]\phi$.
($\Longleftarrow$) Suppose that $\mathcal{W}_\rho, \Gamma \models [\leq r]\phi$. Then, for any $\Gamma' \in \Omega^+[\rho]$ such that $\Gamma \xrightarrow{x} \Gamma'$ for some $x \geq r$, $\mathcal{W}_\rho, \Gamma' \models \phi$. Using the inductive hypothesis we obtain that for any $\Gamma' \in \Omega^+[\rho]$ such that $\Gamma \xrightarrow{x} \Gamma'$ for some $x \geq r$, $\Gamma' \ni \phi$. Using Corollary 1 we obtain $[\leq r]\phi \in \Gamma$.

The case $\psi = [\geq r]\phi$**:** It is proved similarly to the case $\psi = [\leq r]\phi$. ∎

The Truth Lemma proves that $\mathcal{L}$ with the axiomatization presented in Table 2 enjoys the finite model property.

Theorem 4 (Finite model property)**.** *For any $\mathcal{L}$-consistent formula ϕ, there exists a finite WTS $\mathcal{W} = (S, \theta, l)$ of cardinality bound by the structure of ϕ and a state $s \in S$ such that $\mathcal{W}, s \models \phi$.*

Proof. The result derives from the Truth Lemma, since the consistency of $\phi \in \mathcal{L}$ guarantees that there exists a $\mathcal{L}$-maximally consistent set $\Gamma \in \Omega[\phi]$ such that $\phi \in \Gamma$. But then, from the truth lemma, $\mathcal{W}_\phi, \Gamma \models \phi$.

∎

The finite model property proves the Weak-Completeness.

Theorem 5 (Weak Completeness)**.** *The logic $\mathcal{L}$ with the axiomatization presented in Table 2 is weak-complete with respect to the WTS-semantics, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\models \phi \quad \textit{implies} \quad \vdash \phi.$$

Proof. The proof is standard for logics that enjoy the finite model property: [$\models \phi$ implies $\vdash \phi$] is equivalent to [$\nvdash \phi$ implies $\not\models \phi$], that is equivalent to [the consistency of $\neg\phi$ implies the existence of a WTS $\mathcal{W} = (S, \theta, l)$ and a state $s \in S$ such that $\mathcal{W}, s \models \neg\phi$] and this is guaranteed by the finite model property. ∎

Observe that for this logic the finite model property does not imply the decidability of validity since there are countably many finite WTSs of a given finite cardinality; they might have different labels.

An important application of our complete proof system is that it allows us to translate any model-checking problem for WPA into a theorem-proving problem, and this open the perspective of useful applications.

Theorem 6. *For arbitrary $(s, P) \in \mathbb{Q} \times \mathcal{P}$ and arbitrary $\phi \in \mathcal{L}$,*

$$(s, P) \models \phi \quad \textit{iff} \quad \vdash f_{(s,P)} \rightarrow \phi.$$

Proof. We have previously seen that $(s, P) \models \phi$ iff $\models f_{(s,P)} \rightarrow \phi$. Using the completeness theorem, $\models f_{(s,P)} \rightarrow \phi$ iff $\vdash f_{(s,P)} \rightarrow \phi$ and this concludes our proof. ∎

4.3. Strong Completeness for Weighted Modal Logic

Observe that the Weighted Modal Logic is not compact: any finite subset of formulas of the set

$$\{(\leq r) \mid \text{ for all } r < s\} \cup \{(\leq s)\}$$

is consistent, but the entire set is not consistent since using rule (R2) we can prove $\bot$. Similarly, the set

$$\{(\trianglelefteq r) \mid \text{ for all } r \in \mathbb{Q}\}$$

is inconsistent, as rule (R3) proves $\bot$, but each finite subset of it is consistent.

This observation proves that the strong completeness of the Weighted Modal Logic is not implied by the weak completeness.

In what follows we prove that by adding two extra rules to the axiomatic system in Table 2 and by assuming Lindenbaum's lemma[4] as a (meta)-axiom, we obtain a strong-complete axiomatic system for Weighted Modal Logic. This means that we can prove that any consistent theory (set of logical formulas from which we cannot prove $\bot$) has at least one model.

These extra rules are stated in Table 3, for arbitrary $\Phi, \Phi', \Psi \subseteq \mathcal{L}$ and $r \in \mathbb{Q}_0$, where $\Phi \vdash \Psi$ states that using the formulas in Φ we can prove all the formulas in Ψ; and $[\trianglelefteq r]\Phi = \{[\trianglelefteq r]\phi \mid \phi \in \Phi\}$.

(R4):	If $\Phi \vdash \Phi'$ and $\Phi' \vdash \Psi$, then $\Phi \vdash \Psi$
(R5):	If Φ is closed under conjunction and $\Phi \vdash \Psi$, then $[\trianglelefteq r]\Phi \vdash [\trianglelefteq r]\Psi$

Table 3: Axioms of Weighted Logic

These rules are infinitary since, Φ, Φ', Ψ can be infinite sets. (R4) is an infinitary version of Modus Ponens. (R5) is an infinitary extension of (R1), since (R1) can be obtained by instantiating (R5) with $\Phi = \{\top\}$ and $\Psi = \{\phi\}$. Regarding the assumption of the Lindenbaum's lemma as a meta-axiom, this is in line with the quantitative modal logics, such as the probabilistic logic for Markov processes and Harsanyi type spaces [33], [34] and the Markovian logics [23]. This choice is extensively discussed in [12] and it is essential for getting the completeness proof.

In the rest of this section we prove the strong completeness and to do this we will show that each consistent set of formulas has a model.

As for the weak completeness, we consider a consistent set $\Phi \subseteq \mathcal{L}$ and we prove that it has a model. The construction follows the general line used also for the weak completeness with the only difference that now we do not build a finite model, but the entire canonical model, i.e., a WTS $\mathcal{W}_\mathcal{L} = (S_\mathcal{L}, \theta_\mathcal{L}, l_\mathcal{L})$ such that its support-set $S_\mathcal{L}$ is the set of all $\mathcal{L}$-maximally consistent sets of formulas.

Let $S_\mathcal{L}$ be the set of $\mathcal{L}$-maximally consistent sets of formulas, where the consistency is defined with respect to the axioms in tables 2 and 3.

As for the other case, we define the transition $\Gamma \xrightarrow{x} \Gamma'$ between two maximally consistent sets $\Gamma, \Gamma' \in S_\mathcal{L}$ whenever

- $\sup\{r \in \mathbb{Q} \mid [\geq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} = \inf\{r \in \mathbb{Q} \mid [\leq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\}$ and

[4]Lindenbaum's lemma states that any consistent set of formulas can be extended to a maximally-consistent set.

- $x = \inf\{r \in \mathbb{Q} \mid [\leq r]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} \in \mathbb{R}$.

Lemma 5. *For arbitrary* $\Gamma \in S_{\mathcal{L}}$,

$$\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}.$$

Proof. Since any $\mathcal{L}$-maximally consistent set in $S_{\mathcal{L}}$ is also a maximally consistent set if we define the consistency with respect to the axioms in Table 2 only, this result is a direct consequence of Lemma 2. ∎

This result allows us to define, for arbitrary $\Gamma \in S_{\mathcal{L}}$,

$$l_{\mathcal{L}}(\Gamma) = \sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}.$$

Theorem 7 (Canonical model)**.** *The tuple* $\mathcal{W}_{\mathcal{L}} = (S_{\mathcal{L}}, \theta_{\mathcal{L}}, l_{\mathcal{L}})$ *defined above is a weighted transition system.*

Proof. To prove this we only need to show that if for arbitrary $\Gamma, \Gamma' \in \Omega^{+}[\rho]$ we have $\Gamma \xrightarrow{x} \Gamma'$, then $l_\rho(\Gamma') = l_\rho(\Gamma) + x$. But we have already prove that this result can be obtained from the axioms in Table 2 (see Theorem 3). Hence, it is also true in the extended axiomatic system. ∎

Having these proved, we can proceed with the extended truth lemma.

Lemma 6 (Extended Truth Lemma)**.** *If* $\Phi \in \mathcal{L}$ *is a consistent set of formulas, then for arbitrary* $\Gamma \in S_{\mathcal{L}}$,

$$\Phi \subseteq \Gamma \quad \textit{iff} \quad \mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi.$$

Proof. We prove that for arbitrary $\psi \in \Phi$,

$$\psi \in \Gamma \quad \text{iff} \quad \mathcal{W}_{\mathcal{L}}, \Gamma \models \psi.$$

We do this by induction on ψ.

The Boolean cases and the cases $\psi = (\leq r)$ and $\psi = (\geq r)$ are proved as in the Truth Lemma in the previous section.

The case $\psi = [\leq r]\phi$**:** ($\Longrightarrow$) Suppose that $[\leq r]\phi \in \Gamma$. If there exists no $\Gamma' \in S_{\mathcal{L}}$ such that $\Gamma \xrightarrow{x} \Gamma'$ with $x \leq r$, then trivially $\mathcal{W}_{\mathcal{L}}, \Gamma \models [\leq r]\phi$. Otherwise, consider an arbitrary $\Gamma' \in S_{\mathcal{L}}$ such that $\Gamma \xrightarrow{x} \Gamma'$ with $x \leq r$. Since $x = \inf\{u \in \mathbb{Q} \mid [\leq u]\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} \leq r$, applying Lemma 1, we obtain that $[\leq r]\phi \in \Gamma$ implies $\phi \in \Gamma'$; and using the inductive hypothesis for ϕ, we obtain $\mathcal{W}_{\mathcal{L}}, \Gamma' \models \phi$. All these prove that $\mathcal{W}_{\mathcal{L}}, \Gamma \models [\leq r]\phi$.
($\Longleftarrow$) $\mathcal{W}_{\mathcal{L}}, \Gamma \models [\leq r]\phi$ iff for any $\Gamma' \in S_{\mathcal{L}}$ such that $\Gamma \xrightarrow{x} \Gamma'$ for some $x \leq r$, we have $\mathcal{W}_{\mathcal{L}}, \Gamma' \models \phi$. Using the inductive hypothesis, $\phi \in \Gamma'$, and since Γ' is an ultrafilter, this is equivalent to $\Gamma' \vdash \phi$. Applying (R5) we obtain further $[\leq r]\Gamma' \vdash [\leq r]\phi$.

Observe now that from the way we have defined $\theta_{\mathcal{L}}$, $\Gamma \xrightarrow{x} \Gamma'$ implies $\Gamma \vdash [\leq r]\Gamma'$ for any $r \geq x$.

Hence, $\Gamma \vdash [\leq r]\Gamma'$ and $[\leq r]\Gamma' \vdash [\leq r]\phi$ implying further, us using (R4), $\Gamma \vdash [\leq r]\phi$, which is equivalent to $[\leq r]\phi \in \Gamma$, since Γ is an ultrafilter.

The case $\psi = [\geq r]\phi$: It is proved similarly to the case $\psi = [\leq r]\phi$. ∎

The Extended Truth Lemma proves that $\mathcal{L}$ with the axiomatization presented in Tables 2 and 3 has a canonic model.

Theorem 8 (Canonic Model)**.** *For any $\mathcal{L}$-consistent set of formulas $\Phi \subseteq \mathcal{L}$, there exists $\Gamma \in S_{\mathcal{L}}$ such that*

$$\mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi.$$

Proof. Since Φ is consistent, using Lindenbaum's Lemma that we assumed as a meta-axiom, there exists a maximal consistent set $\Gamma \in S_{\mathcal{L}}$ such that $\Phi \subseteq \Gamma$. Applying Extended Truth Lemma, $\mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi$. ∎

The previous theorem proves the Strong-Completeness for our logic.

Theorem 9 (Strong Completeness)**.** *The logic $\mathcal{L}$ with the axiomatization presented in Tables 2 and 3 is strong-complete with respect to the WTS-semantics, i.e., for arbitrary $\Phi \subset \mathcal{L}$ and $\phi \in \mathcal{L}$,*

$$\Phi \models \phi \quad \textit{implies} \quad \Phi \vdash \phi.$$

Proof. The proof is a direct consequence of the Canonic Model Theorem and Extended Truth Lemma. ∎

5. Conclusions

In this paper we developed the metatheory of the Weighted Modal Logic. We have proved that it is sufficiently expressive to characterize WPA processes and we have developed a proof theory for it. We presented initially a weak-complete axiomatization and proved the finite model property; and eventually we have shown how one can extend the axiomatization to get strong-completeness. Interestingly enough, this technique is similar with the technique used with probabilistic and stochastic logics to prove strong completeness and it involves using the so-called countable additivity rule [12] and to assume Lindembaum's property as a meta-axiom.

In our recent paper, with Dexter Kozen and Prakash Panangaden [17], focused on proving a Stone duality for Markov Processes, we have shown that for probabilistic and stochastic logics one can get the strong completeness using a lighter version of the countable additivity axiom that also allowed us to prove Lindenbaum's property. We are confident that a similar approach can be taken with the weighted modal logic and this is a topic that we intend to study in the future.

An other promising direction where we plan to extend this paper is the compositionality method advertised by Glynn Winskel. The idea is to give a logical meaning of the concept of compositionality and quotienting and to use these principles to prove properties of composed systems relying on the properties of the components.

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