Concurrent Weighted Logic[☆]

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Abstract

We introduce Concurrent Weighted Logic (CWL), a multimodal logic for concurrent labeled weighted transition systems (LWSs). The synchronization of LWSs is described using dedicated functions that, in various concurrency paradigms, allow us to encode the compositionality of LWSs. To reflect these, CWL contains modal operators indexed with rational numbers to predicate over the numerical labels of LWSs as well as a binary modal operator that encodes properties concerning the (de-) composition of LWSs. We develop a Hilbert-style axiomatic system for CWL and we prove weak- and strong-completeness results for this logic. To complete these proofs we involve advanced topological techniques from Model Theory.

Keywords: non-compact modal logic, model theory, Rasiowa-Sikorski Lemma, weighted transition systems, concurrency

1. Introduction

In industrial practice, particularly in the area of embedded systems, a key challenge is to handle the growing complexity of systems, while meeting requirements on correctness, predictability, performance and also time and cost constraints. In this context model-driven and component-based development (MDD) is seen as a valuable approach, as

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it allows early design-space exploration and verification and may be used as the basis for systematic and unambiguous testing of a final product. However, for embedded systems, verification should not only address functional properties but also properties related to timing and resource constraints. Moreover, such complex systems are often modular in nature, consisting of parts, which are systems in their own; their global behavior depends on the behavior of their parts and on the links connecting them.

Consider, e.g., an energy-powered system; it might be composed of various modules such as an energy-provider and an energy-consumer component. It is crucial that its energy-level is always above 0. Clearly, the behavior of the whole system depends on that of both the consumer and the provider, and the way they interact. Understanding such systems requires compositional reasoning, integration of local information in a formal way, in order to address questions such as: ”to what extent is it possible to derive global properties of the system from the local properties of its modules?”

Within the area of model checking, a number of state-machine based formalisms have emerged, which allow for quantitative aspects to be modeled and efficiently analyzed; e.g., *timed automata* [1], *weighted automata* [2, 3] and *weighted timed automata* [4, 5]. Regarding modularity, MDD requires semantically well-defined models of composition, which preserve suitable notions of behavioral equivalence or preorder - as found in process calculi[6]. But this approach is quite restrictive, as process algebras cannot express basic Boolean operations that are necessary for systems' specifications[7, 8]. Usually, to do this, people use logics such as temporal logics, modal μ-calculus or Hennessy-Milner logic. These, however, only encode global properties and no logical framework developed so far allows one to reason on weighted systems and subsystems at the same time.

Similarly, axiomatization and theorem proving techniques have been developed for some modular logics for concurrent nodeterministic systems, such as Spatial Logic [9, 10] or Separation Logic [11, 12, 13, 14], or probabilistic/stochastic extensions of these [15] for Markov processes. However, the case of weighted systems is different in many respects, since the mathematical mechanism required to encode resource consumption is very different from the one used to encode probabilistic behaviours. For Markov processes the transition function maps states to probabilistic distributions over the state space, which are additive with respect to immediate successors; for weighted systems, the transition function is additive with respect to traces of computation.

To address these problems altogether, we consider in this paper a logical framework called Concurrent Weighted Logic (CWL) designed for concurrent labeled weighted transition systems (LWS), which are transition systems having both their states and their transitions labeled with real numbers. The state label denotes the energy-level of a state, while the numeric transition label represents the energy consumption required for taking that transition. We impose no restriction on the values of the labels, and they can be either positive or negative reals; in addition, the transition may also be labeled with actions. Moreover, we have incorporated in these models a general concept of composition that subsumes most of the classic paradigms of synchronization such as *interleaving*, *CCS-style* [16], *CSP-style* [17] or *ACP-style* [18]. All these features guarantee a robust modeling framework for which CWL is used to specify qualitative, quantitative and modular properties. Observe however that LWSs are not just instances of weighted automata [3], since they can be infinite or infinitely branching; on the other hand, any computational loop is guaranteed to have a null overall cost, since the state is characterized by the value of the available resources.

CWL is a multimodal boolean logic endowed with modalities indexed with rational numbers that express/approximate the numeric labels of LWSs and with a composition modality "|" that allows composing properties of the modules of a system. While in an LWS we can have real-valued labels, the modalities only encode rational values and often we are in the situation of characterizing a state or a transition label using an infinite convergent sequence of rationals. By combining these operators, CWL allows one to specify and reason about the discrete behavior of an LWS as well as its consumption of resources, in hierarchical/modular ways, thereby proving global properties from the local properties of modules. Formally, denoting *"system S has the property ϕ"* by $S \models \phi$ and letting "$\otimes$" be the composition operator on models, CWL aims to establish proof rules of the following form, where C is a logical constraint, e.g., of type

$\vdash \phi_1|...|\phi_k \to \rho$,

$$\frac{S_1 \models \phi_1, ..., S_k \models \phi_k}{S_1 \otimes ... \otimes S_k \models \rho} \; C(\rho, \phi_1, .., \phi_k).$$

We propose a weakly complete axiomatization for CWL that can prove constraints as C, guaranteeing that a logical formula is satisfied in all models iff it is provable in our axiomatic system. This provides a useful tool to turn validity problems into provability problems. We demonstrate that CWL enjoys the *finite model property* and this can support future development for model-checking and synthesis algorithms.

Since LWSs can have real labels while CWL can only express rational values, the logic is not compact[2]. This feature presents CWL as an appropriate framework to approximate properties of LWSs. In this context, we also provide a strongly complete extension of the axiomatic system: given a theory (a set of formulas, possibly infinite) Φ and a formula ϕ, if the models satisfying Φ also satisfy ϕ, then ϕ can be proved from Φ. Our completeness proof is innovative and follows a proof strategy recently introduced by the first two authors together with Kozen and Panangaden in [19, 20]. Our proof avoids the infinitary rules with uncountable set of instances usually used to obtain strong completeness for similar logics in [21, 22, 23, 24]. Instead, we propose a lighter rule with a countable set of instances, which allows us to apply some advanced model theoretical techniques such as the Rasiowa-Sikorski Lemma [25, 21, 26] to achieve our goal. Moreover, we prove an extension of the Rasiowa-Sikorski Lemma in general settings that allows us to demonstrate the Lindenbaum's Lemma for every Boolean logic with operators that admits a countable axiomatization. This result is rather general and can be used in other contexts as well.

The paper is organized as follows. Section 2 introduces LWSs and their bisimulation and compositionality. Section 3 defines CWL and its LWS-semantics. Section 4 presents Rasiowa-Sikorski Lemma and an extension of it, which are important for our construction of models. Section 5 presents the axiomatization and develops its metatheory including finite model property, soundness and completeness results.

[2]A logic is compact if whenever all the finite subsets of a given set of formulas have models, the entire set has a model.

2. Labeled Weighted Transition Systems

A *labeled weighted transition system* (LWS) is a transition system that has the states labeled with real numbers and the transitions labeled with both real numbers and actions - see Figure 1 (on Page 7). The state label is interpreted as the amount of resources available (or required if the number is negative) in the state, e.g., energy or time. During a transition the resources are updated: they can increase in value if the transition has a positive label - as it happens during a synchronization with a provider of resources, - or the state resources can be consumed when the transition label is negative. However, the label of the final state of a transition is the sum of the labels of the initial state and of the transition. In addition, the transitions are also labeled with actions that allow us to encode synchronizations.

Definition 1 (Labeled Weighted Transition System)*: An LWS is a tuple*

$$\mathcal{W} = (M, \Sigma, \theta, l)$$

where M is a non-empty set of states*, Σ a non-empty set of* actions*, $\theta : M\times(\Sigma\times\mathbb{R}) \to 2^M$ is the* labeled transition function *and $l : M \to \mathbb{R}$ is a state-labeling function satisfying that whenever $m' \in \theta(m, a, x)$, $l(m') = l(m) + x$.*

Instead of $m' \in \theta(m, a, x)$, we write $m \xrightarrow{x}_a m'$. Notice that if $m \xrightarrow{x_1}_a m'$ and $m \xrightarrow{x_2}_b m'$, the condition on state labels guarantees that $x_1 = x_2$.

Given an initial state, its labeling and a transition sequence, we can infer the labeling of the subsequent states; hence, the labeling function of definition 1 contains redundant information. However, since each state is related to a specific weight by the labeling function, only cycles with overall weight zero can appear in our systems: consider, for instance, the sequence $m_1 \xrightarrow{x}_a m_2 \xrightarrow{y}_b m_3$; we have that $l(m_3) = l(m_1) + x + y$, and if $m_1 = m_3$, then $l(m_3) - l(m_1) = x + y = 0$ – the cycle has the weight zero.

The concept of *weighted bisimulation* is a relation between LWSs that equates systems with identical state labels and identical weighted behaviors. It is defined similar to [27, 28, 29, 30, 31] as follows.

Definition 2 (Weighted Bisimulation)*: Given an LWS $\mathcal{W} = (M, \Sigma, \theta, l)$, a* weighted bisimulation *is an equivalence relation* $R \subseteq M \times M$ *such that whenever* $(m, m') \in R$*,*

1. $l(m) = l(m')$*;*

2. if $m \xrightarrow{x}_a m_1$*, then there exists* $m'_1 \in M$ *s.t.* $m' \xrightarrow{x}_a m'_1$ *and* $(m_1, m'_1) \in R$*;*

3. if $m' \xrightarrow{x}_a m'_1$*, then there exists* $m_1 \in M$ *s.t.* $m \xrightarrow{x}_a m_1$ *and* $(m_1, m'_1) \in R$*.*

If there exists a weighted bisimulation relation R *such that* $(m, m') \in R$*, we say that* m *and* m' *are* bisimilar*, denoted by* $m \sim m'$*.*

As for the other types of bisimulation, the previous definition can be extended to define the weighted bisimulation between distinct LWSs by considering bisimulation relations on their disjoint union. Weighted bisimilarity is the largest weighted bisimulation relation; if $\mathcal{W}_i = (M_i, \Sigma_i, \theta_i, l_i)$, $m_i \in M_i$ for $i = 0, 1$ and m_0 and m_1 are bisimilar, we write $(m_0, \mathcal{W}_0) \sim (m_1, \mathcal{W}_1)$.

To implement a general concept of concurrency for LWSs, following [32, 15], we consider a *synchronization function* $*$ on actions that is a partial function

$$* : \Sigma \times \Sigma \rightharpoonup \Sigma$$

which associates to two actions a, b an action $a * b$ - the synchronization of a and b. The operation $*$ is required to be commutative, i.e., for all $a, b \in \Sigma$, $a * b$ is defined iff $b * a$ is defined and $a * b = b * a$. Such a function can mimic various synchronization paradigms. E.g., the CCS-style synchronization requires that $a * \overline{a} = \tau$, where $\tau \in \Sigma$ is a special action; CSP-style requires that $a * a = a$; for interleaving semantics and ACP-style synchronization, we need to assume the existence of a special transition label $\delta \in \Sigma$ such that for all a, $a * \delta = a$. Similarly, most classical notions of parallel composition in process algebras may be expressed by a suitable synchronization function.

Definition 3 (Product of LWSs)*: Let* $\mathcal{W}_i = (M_i, \Sigma_i, \theta_i, l_i)$ *be LWSs for* $i = 0, 1$*.* $\mathcal{W} = (M, \Sigma, \theta, l)$ *is the product of* $\mathcal{W}_0$ *and* $\mathcal{W}_1$*, written* $\mathcal{W} = \mathcal{W}_0 \otimes \mathcal{W}_1$*, if* $M = M_0 \times M_1$*,* $\Sigma = \Sigma_0 * \Sigma_1 = \{a_0 * a_1 \mid a_0 \in \Sigma_0, a_1 \in \Sigma_1, a_0 * a_1 \text{ is defined}\}$*,* $l((m_0, m_1)) = l_0(m_0) + l_1(m_1)$ *for all* $(m_0, m_1) \in M$*, and* θ *is defined, for all* $m = (m_0, m_1) \in M$*,* $a \in \Sigma$ *and* $x \in \mathbb{R}$*, by:*

$$\theta(m, a, x) = \{(m'_0, m'_1) \in M \mid \; m_i \xrightarrow{x_i}_{a_i} m'_i, \; i = 0, 1, \; x = x_0 + x_1, \; a = a_0 * a_1\}.$$

It is trivial to verify that $\mathcal{W}_0 \otimes \mathcal{W}_1$ is an LWS.

Here synchronization labels are specified in terms of addition of reals. Intuitively, this correspond to global resource consumption being the sum of the synchronising components' resource consumption. One might argue that some quantitative properties are not necessarily the sums of the synchronising components weights. However, adapting our idea to such non-additive cases does not rise particular problems.

Example 1*: In Figure 1 we show the product of $\mathcal{W}_0$ and $\mathcal{W}_1$, assuming that $a * c$, $b * c$ and $a * d$ are the only permitted synchronizations ($b * d$ is not legal). Observe that the label of the state (m_0, n_0) is the sum of that of m_0 and n_0. The transition from (m_0, n_0) to (m_1, n_1) is labeled with the action $a * c$ since it is the synchronization of the actions a of the transition from m_0 to m_1 and c from n_0 to n_1; its cost is 5, which is the sum of the costs of the two transitions that synchronize.* ∎

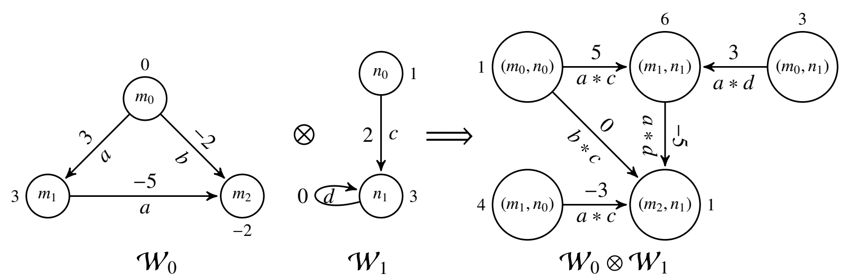

Figure 1: LWSs and their product

3. Concurrent Weighted Logic

In this section we introduce Concurrent Weighted Logic (CWL), which can express properties of LWSs, including properties regarding their compositionality. This logic extends Weighted Modal Logic [33] with a compositional operator similar to the one used in Spatial Logics [9] and in Separation Logics [11]. CWL is endowed, in addition to the classic boolean operators, with a class of modalities of arity 0 (predicates) called *state modalities* of type "$(\leq r)$" or "$(\geq r)$" for $r \in \mathbb{Q}$ that predicate about the value of the label of the current state; a class of modalities of arity 1 (unary modalities), named *transition modalities*, of type "$[\leq r]_a$" or "$[\geq r]_a$", for $r \in \mathbb{Q}$ and $a \in \Sigma$, which refer to the transitions; and a modality of arity 2 (binary modality), named *concurrent modality*, denoted by "|" that specifies compositional properties of LWSs.

Before proceeding with the formal definitions, we establish some useful notations. In the rest of this paper, we use $\trianglelefteq$ and $\trianglerighteq$ to range over $\{\leq, \geq\}$ such that $\{\trianglelefteq, \trianglerighteq\} = \{\leq, \geq\}$; these mean that $\trianglelefteq$ can either represent $\leq$ or $\geq$ and if $\trianglelefteq$ represents one, then $\trianglerighteq$ denotes the other. Similarly, we use $\triangleleft$ and $\triangleright$ to range over $\{<, >\}$ such that $\{\triangleleft, \triangleright\} = \{<, >\}$. Hereafter in the paper, we fix a set Σ of actions and we assume that we have a synchronization function $*$. Based on this, we introduce the syntax and semantics of CWL.

Definition 4 (Syntax of CWL)*: For arbitrary $r \in \mathbb{Q}$, $a \in \Sigma$ and $\trianglelefteq \in \{\leq, \geq\}$, let*

$$\mathcal{L}: \quad \phi := \bot \mid \phi \to \phi \mid (\trianglelefteq r) \mid [\trianglelefteq r]_a \phi \mid \phi|\phi.$$

We work with all boolean operators defined in the standard way from $\bot$ and $\to$. We also use the De Morgan duals of $[\trianglelefteq x]_a$ and $|$, defined by $\langle \trianglelefteq x \rangle_a \phi = \neg[\trianglelefteq x]_a \neg\phi$ and $\phi_0 || \phi_1 = \neg(\neg\phi_0 | \neg\phi_1)$.

The *LWS-semantics* of CWL is defined by *satisfiability relation*, for some LWS $\mathcal{W} = (M, \Sigma, \theta, l)$ with $m \in M$, inductively as follows. The boolean cases are omitted.

$\mathcal{W}, m \models (\trianglelefteq r)$ iff $l(m) \trianglelefteq r$,
$\mathcal{W}, m \models [\trianglelefteq r]_a \phi$ iff $\forall\, m' \in M$, $x \in \mathbb{R}$ s.t. $x \trianglelefteq r$ and $m \xrightarrow{x}_a m'$, $\mathcal{W}, m' \models \phi$,
$\mathcal{W}, m \models \phi_0|\phi_1$ iff there exist LWSs $\mathcal{W}_i = (M_i, \Sigma_i, \theta_i, l_i)$ and $m_i \in M_i$, $i = 0, 1$, such that $(m, \mathcal{W}) \sim ((m_0, m_1), \mathcal{W}_0 \otimes \mathcal{W}_1)$ and $\mathcal{W}_i, m_i \models \phi_i$, $i = 0, 1$.

As usual for modal logics, the universal quantification that appears in the semantics of $[\trianglelefteq r]_a \phi$ is trivially satisfied if the evaluated state admits no a-labelled transition. The semantics of the derived operators can be derived straightforwardly:

$\mathcal{W}, m \models \langle \trianglelefteq r \rangle_a \phi$ iff $\exists\, m' \in M$ and $x \in \mathbb{R}$ s.t. $x \trianglelefteq r$, $m \xrightarrow{x}_a m'$ and $\mathcal{W}, m' \models \phi$,
$\mathcal{W}, m \models \phi_0 || \phi_1$ iff for all LWSs $\mathcal{W}_i = (M_i, \Sigma_i, \theta_i, l_i)$, $i = 0, 1$ such that $(m, \mathcal{W}) \sim ((m_0, m_1), \mathcal{W}_0 \otimes \mathcal{W}_1)$, if $\mathcal{W}_i, m_i \not\models \phi_j$ then $\mathcal{W}_{1-i}, m_{1-i} \models \phi_{1-j}$, $i, j \in \{0, 1\}$.

Notice that "|" is a diamond-like modality and "||" is a box-like modality.

Whenever it is not the case that $\mathcal{W}, m \models \phi$, we write $\mathcal{W}, m \not\models \phi$. We say that a formula ϕ is *satisfiable* if there exists at least one LWS that satisfies it in one of its states; ϕ is a *validity* if it is satisfied in all states of all LWSs - in this case we write $\models \phi$.

In Example 1, observe that $\mathcal{W}_0, m_0 \models \langle\leq 3\rangle_a(\leq 3)$ and $\mathcal{W}_1, n_0 \models \langle\leq 2\rangle_c(\leq 3)$; this allows us to state that $\mathcal{W}_0 \otimes \mathcal{W}_1, (m_0, n_0) \models \langle\leq 3\rangle_a(\leq 3)|\langle\leq 2\rangle_c(\leq 3)$. On the other hand, we also have $\mathcal{W}_0 \otimes \mathcal{W}_1, (m_0, n_0) \models \langle\leq 5\rangle_{a*c}(\leq 6)$. The challenge we take further is to understand whether $\langle\leq 3\rangle_a(\leq 3)|\langle\leq 2\rangle_c(\leq 3)$ entails $\langle\leq 5\rangle_{a*c}(\leq 6)$.

Example 2*: Consider an energy system composed by an energy-providing module and an energy-consuming module. We can express that the system is in the safe range (above* 0*) and it can remain in this range during the next N-step transitions by using the following CWL formulas, where* $Good^0 = (\geq 0)$*:*

$$Good^N = (\geq 0) \wedge \langle\geq 0\rangle_{a*a} Good^{N-1},$$

The provider and the consumer can be specified by P^N *and* C^N*, with* $P^0 = C^0 = \top$*:*

$$P^N = \langle\geq x\rangle_a P^{N-1}, \qquad C^N = \langle\geq -x\rangle_a C^{N-1}.$$

Observe now that the property $P^N|C^N$ *states that if the consumer always uses energy within the bound x, the producer can always produce enough to maintain the system safe. Consequently, systems which satisfy* $(\geq 0) \wedge P^N|C^N$ *must also satisfy* $Good^N$*:*

$$\models (\geq 0) \wedge P^N|C^N \rightarrow Good^N.$$

However, all these considerations are semantic observations. In the following sections, we will axiomatize CWL so that we can prove the previous results as theorems. This will provide us a different perspective on such a problem by involving and allow us, for instance, to use theorem proving techniques instead of model checking. ∎

4. An Extension of Rasiowa-Sikorski Lemma

Before continuing with the study of CWL, we state and prove a series of general results regarding the deducibility relations for non-compact logics; we follow the line of [34] on approaching a general concept of deducibility relation. The main contribution of this section is proving an extension of Rasiowa-Sikorski lemma [25, 35], which is a key technique for our completeness results for CWL.

Given a Boolean logic[3] $\mathcal{L}$, a relation $\vdash \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is a *conjunctive deducibility relation*

[3] By a Boolean logic in this context we mean a logic that has the Boolean operators and obey the axioms

on $\mathcal{L}$ if it satisfies the following conditions.

(D1) If $F \vdash \phi$ and $F \subseteq F'$, then $F' \vdash \phi$;

(D2) If $\phi \in F$, then $F \vdash \phi$;

(D3) If $F \vdash \phi$ and $F \cup \{\phi\} \vdash \bot$, then $F \vdash \bot$;

(D4) $F \cup \{\neg\phi\} \vdash \bot$ iff $F \vdash \phi$;

(D5) If Σ is finite and $F \cup \Sigma \vdash \phi$, then $F \cup \{\bigwedge \Sigma\} \vdash \phi$.

We call the elements of $\vdash$ *inferences*. The previous rules guarantee that a deducibility relation is, in particular, a Boolean deducibility, i.e.,

$$\phi \rightarrow \psi \text{ is a Boolean validity iff } \{\phi\} \vdash \psi.$$

We say that a deducibility relation on $\mathcal{L}$ is *countably-axiomatized* if there exists a countable set $(F_i \vdash \phi_i)_{i \in \mathbb{N}}$ of inferences such that from them using the rules (D1)-(D5) we can derive all the elements of $\vdash$. A set $F \subseteq \mathcal{L}$ is called $\vdash$-consistent if $F \neq\vdash \bot$ and finitely $\vdash$-consistent if each finite subset of F is $\vdash$-consistent. F is maximally $\vdash$-consistent if it is $\vdash$-consistent but has no *vdash*-consistent proper extension in $\mathcal{L}$ and maximally finitely $\vdash$-consistent if it is finitely $\vdash$-consistent but has no finitely $\vdash$-consistent proper extension in $\mathcal{L}$.

The Rasiowa-Sikorski lemma [25, 35] is a model-theoretic result that exploits the Baire category theorem and the Stone duality for boolean algebras with operators. Applied to logics, the Rasiowa-Sikorski lemma states that given a multimodal logic (possibly involving an infinite set of modalities) for which the deducibility relation admits an axiomatization such that the set of instances of the infinitary proof rules (if any) is countable, then for every consistent formula ϕ, there exists a maximally-consistent set of formulas containing ϕ.

Let $\mathcal{B}$ be a boolean algebra and let $T \subseteq \mathcal{B}$ be such that T has a greatest lower bound $\bigwedge T$ in $\mathcal{B}$. An ultrafilter (maximal filter) U is said to *respect* T if $T \subseteq U$ implies that $\bigwedge T \in U$. If $\mathcal{T}$ is a family of subsets of $\mathcal{B}$, we say that an ultrafilter U respects $\mathcal{T}$ if it

of propositional logic; in addition, these logics might have other operators too that could be used in the axiomatic system.

respects every member of $\mathcal{T}$.

Lemma 1 (Rasiowa–Sikorski lemma [25])*: For every boolean algebra $\mathcal{B}$ and every countable family $\mathcal{T}$ of subsets of $\mathcal{B}$, each member of which has a meet in $\mathcal{B}$, and for every nonzero $x \in \mathcal{B}$, there exists an ultrafilter in $\mathcal{B}$ that contains x and respects $\mathcal{T}$.*

This lemma was later proved by Tarski in a purely algebraic way. See [35] for a discussion of the role of the Baire category theorem in the proof of this lemma. The following theorem is a direct consequence of the Rasiowa-Sikorski Lemma.

Theorem 1*: Given a Boolean logic $\mathcal{L}$ and a countably-axiomatized deductibility relation $\vdash$, for all $\vdash$-consistent formulas $\phi \in \mathcal{L}$ there exists a $\vdash$-maximally consistent set of formulas that contains ϕ.*

Proof. Observe that the Lindenbaum algebra $\mathcal{L}^{\leftrightarrow}$ (the quotient of $\mathcal{L}$ w.r.t. logical equivalence) is a Boolean algebra. Each of the inferences in the class $(\Gamma_i \vdash \phi_i)_{i \in \mathbb{N}}$ states that the meet of the elements in Γ_i exists in $\mathcal{L}^{\leftrightarrow}$ and it is equal to ϕ_i, for each $i = 1, n$. Consequently, a $\vdash$-maximally consistent set of formulas is an ultrafilter that respects each element of the class $(\Gamma_i)_{i \in \mathbb{N}}$. Since this class is countable, applying Rasiowa-Sikorski lemma we get that there must exist an ultrafilter of $\mathcal{L}^{\leftrightarrow}$ that contains ϕ and respects each element in $(\Gamma_i)_{i \in \mathbb{N}}$ - this is a $\vdash$-maximally consistent set of formulas. ∎

The previous theorem states that each consistent formula is contained in a maximally-consistent set. This does not mean that each consistent set of formulas has a maximally-consistent extension (Lindenbaum's lemma). While in the case of compact logics this can be proved using Zorn's lemma, in the case of non-compact logics one cannot prove constructively the existence of a maximal-consistent extension. In what follows we prove that under the assumptions of the previous theorem Lindenbaum's lemma is satisfied also for the case of non-compact logics that admit a countable (possibly infinitary) axiomatization. In order to prove this, we firstly introduce a few concepts.

Definition 5*: A* completion *of a Boolean algebra $\mathcal{A}$ is a Boolean algebra $\mathcal{B}$ enjoying the following properties:*

1. $\mathcal{A}$ is a subalgebra of $\mathcal{B}$;

2. every subset of $\mathcal{A}$ has a supremum in $\mathcal{B}$;

3. every element in $\mathcal{B}$ is the supremum (in $\mathcal{B}$) of some subset of $\mathcal{A}$.

We state some basic facts about completion of a Boolean algebra (for more see [36]). First of all, every Boolean algebra $\mathcal{A}$ has a completion, namely (an isomorphic copy of) the Boolean algebra of complete ideals in $\mathcal{A}$. The completion is a complete Boolean algebra. Every element in the completion is the supremum (in the completion) of the set of elements in $\mathcal{A}$ that are below it. Condition 3 in Definition 5 is equivalent to the assertion that $\mathcal{A}$ is a *dense* subset of $\mathcal{B}$ in the sense that every non-zero element in $\mathcal{B}$ is above a non-zero element in $\mathcal{A}$. These results support us in proving Lemma 2.

Lemma 2 (Extended Rasiowa-Sikorski Lemma)*: Let $\mathcal{B}$ be a Boolean algebra, $\overline{\mathcal{B}}$ its completion and $\mathcal{T}$ a countable family of subsets of $\mathcal{B}$ each member of which has a meet in $\mathcal{B}$. If $S \subseteq \mathcal{B}$ is such that $\bigwedge S \neq 0$ in $\overline{\mathcal{B}}$, then there exists an ultrafilter $\mathcal{U}$ of $\mathcal{B}$ such that $S \subseteq \mathcal{U}$ and $\mathcal{U}$ respects $\mathcal{T}$.*

Proof. $\bigwedge S \neq 0$ in $\overline{\mathcal{B}}$, so there exists an ultrafilter $\overline{\mathcal{U}}$ of $\overline{\mathcal{B}}$ such that $\bigwedge S \in \overline{\mathcal{U}}$ and $\overline{\mathcal{U}}$ respects $\overline{\mathcal{T}}$. Let $\mathcal{U} = \overline{\mathcal{U}} \cap \mathcal{B}$. Because $\mathcal{B}$ is a subalgebra of $\overline{\mathcal{B}}$, $\mathcal{U}$ is an ultrafilter of $\mathcal{B}$. Since $\bigwedge S \in \overline{\mathcal{U}}$ and for every $s \in S$, $\bigwedge S \leq s$, we obtain that $S \subseteq \overline{\mathcal{U}}$, because $\overline{\mathcal{U}}$ is a filter. But we also know that $S \subseteq \mathcal{B}$. Hence, $S \subseteq \mathcal{U}$.
Consider an arbitrary set $T \in \mathcal{T}$ and assume that $T \subseteq \mathcal{U}$. Then, $T \subseteq \overline{\mathcal{U}}$. Because $\overline{\mathcal{U}}$ respects T, we get that $\bigwedge T \in \overline{\mathcal{U}}$. But $\bigwedge T \in \mathcal{B}$ from hypothesis. Hence, $\bigwedge T \in \mathcal{U}$. Consequently, $\mathcal{U}$ respects each element of $\mathcal{T}$ and contains S. ∎

A direct consequence of the previous lemma is the Lindenbaum's lemma for a countably–axiomatized deducibility relation.

Theorem 2 (Lindenbaum's Lemma for Countably-Axiomatized Logics)*: Given a Boolean logic $\mathcal{L}$ and a countably-axiomatized deductibility relation $\vdash$, then for every $\vdash$-consistent set of formulas $\Phi \subseteq \mathcal{L}$ there exists a $\vdash$-maximally consistent set of formulas that extends Φ.*

Proof. As in the case of Theorem 1, we consider the Lindenbaum algebra $\mathcal{L}^{\leftrightarrow}$, which is a Boolean algebra. Each of the countably many inferences $(\Gamma_i \vdash \phi_i)_{i \in \mathbb{N}}$ that axiomatize $\vdash$ state that the meet of the elements in Γ_i exists in $\mathcal{L}^{\leftrightarrow}$ and it is equal to ϕ_i, for each $i = 1, n$. Consequently, a $\vdash$-maximally consistent set of formulas is an ultrafilter that respects each element of the class $(\Gamma_i)_{i\in\mathbb{N}}$. Since this class is countable and Φ is consistent, meaning that in the completion $\overline{\mathcal{L}^{\leftrightarrow}}$ of $\mathcal{L}^{\leftrightarrow}$ it has a meet which is different than $\bot$, we can apply the Extended Rasiowa-Sikorski lemma 2 to get that there must exist an ultrafilter of $\mathcal{L}^{\leftrightarrow}$ that contains ϕ and respects each element in $(\Gamma_i)_{i\in\mathbb{N}}$ - this is a $\vdash$-maximally consistent set of formulas. ∎

5. Metatheory for Concurrent Weighted Logic

In this section we present a weakly and strongly complete axiomatization for CWL, we prove the finite model property and we construct the canonical model.

5.1. A Sound Axiomatization

The axiomatization of CWL includes, in addition to the axioms and rules of propositional logic, the axioms and rules presented in Table 1. For simplicity, we introduce the following notations first:

- $\trianglelefteq r \star s = \begin{cases} \trianglelefteq r + s & \text{if } \trianglelefteq = \leq \\ \trianglelefteq r - s & \text{if } \trianglelefteq = \geq \end{cases}$ $\qquad \trianglerighteq r \bar{\star} s = \begin{cases} \trianglerighteq r + s & \text{if } \trianglerighteq = \geq \\ \trianglerighteq r - s & \text{if } \trianglerighteq = \leq \end{cases}$
- *Modal prefixes* are words $w \in Mod^*$ over the alphabet of modal operators
$$Mod = \{[\trianglelefteq r]_a \mid r \in \mathbb{Q}, a \in \Sigma, \trianglelefteq \in \{\leq, \geq\}\},$$
e.g., $[\trianglelefteq r]_a[\trianglelefteq s]_b[\trianglerighteq t]_c, [\trianglelefteq r]_a[\trianglelefteq r]_a, \varepsilon \in Mod^*$, where ε is the empty word.
- For arbitrary sequences $\phi_1, .., \phi_k \in \mathcal{L}$ and $w_1, .., w_{k+1} \in Mod^*$ we construct the following generic formula involving a metavariable X quantifying over $\mathcal{L}$
$$C[X] = w_1(\phi_1 \| w_2(\phi_2 \| \cdots w_k(\phi_k \| w_{k+1} X) \cdots));$$
we call $C[X]$ a *context*, e.g., $C[X] = (\geq s_1)\|[\leq r_1]_a[\geq r_2]_b((\leq s_2)\|[\leq r_3]_c X)$ is a context and it can be instantiated to
$$C[(\leq x)] = (\geq s_1)\|[\leq r_1]_a[\geq r_2]_b((\leq s_2)\|[\leq r_3]_c(\leq x)) \in \mathcal{L};$$
also $\varepsilon[X]$ is a context - the empty one - and for $\phi \in \mathcal{L}$, $\varepsilon[\phi] = \phi$.Notice that the

metavariable X only appears once in the syntax of the context, i.e., we only consider contexts with one hole.

The axioms and rules in Table 1 are stated for arbitrary $\phi \in \mathcal{L}$, $r, s \in \mathbb{Q}$, $a \in \Sigma$ and context $C[X]$, where $\{\trianglelefteq, \trianglerighteq\} = \{\leq, \geq\}$, $[a]\phi = [\geq 0]_a\phi \wedge [\leq 0]_a\phi$. They, together with the propositional axioms and rules, axiomatize a deducibility relation on $\mathcal{L}$.

State and Transition Axioms

(A1): $\vdash (\trianglelefteq r) \vee (\trianglerighteq r)$

(A2): $\vdash (\trianglelefteq r) \rightarrow (\trianglelefteq r \star s)$, $s > 0$

(A3): $\vdash (\trianglelefteq r) \rightarrow \neg(\trianglerighteq r \overline{\star} s)$, $s > 0$

(A4): $\vdash [\trianglelefteq r]_a(\phi \rightarrow \psi) \rightarrow ([\trianglelefteq r]_a\phi \rightarrow [\trianglelefteq r]_a\psi)$

(A5): $\vdash \langle \trianglelefteq r\rangle_a\phi \rightarrow \langle \trianglelefteq r \star s\rangle_a\phi$, $s > 0$

(A6): $\vdash (\trianglelefteq r) \rightarrow [\trianglelefteq s]_a(\trianglelefteq r + s)$

(R1): If $\vdash \phi$, then $\vdash [\trianglelefteq r]_a\phi$

Compositional Axioms	Mixed Axioms
(A7) : $\vdash \phi \vert \psi \rightarrow \psi \vert \phi$	(A11): $\vdash (\trianglelefteq r) \vert (\trianglelefteq s) \rightarrow \trianglelefteq(r + s)$
(A8) : $\vdash \phi \vert (\psi \vert \rho) \rightarrow (\phi \vert \psi) \vert \rho$	(A12): $\vdash \langle \trianglelefteq r\rangle_a\phi \vert \langle \trianglelefteq s\rangle_b\psi \rightarrow \langle \trianglelefteq r + s\rangle_{a*b}(\phi \vert \psi)$
(A9) : $\vdash \phi \vert (\psi \vee \rho) \rightarrow (\phi \vert \psi) \vee (\phi \vert \rho)$	(A13): $\vdash \bigwedge_{a*b=c}([a]\phi \vert [b]\psi) \rightarrow [c](\phi \vert \psi)$
(A10): $\vdash \phi \vert \bot \rightarrow \bot$	(R3) : $\{C[(\trianglelefteq r)] \mid r \triangleright s\} \vdash C[(\trianglelefteq s)]$
(R2) : If $\vdash \phi \rightarrow \psi$, then $\vdash \phi \vert \rho \rightarrow \psi \vert \rho$	(R4) : $\{C[(\trianglelefteq r)] \mid r \in \mathbb{Q}\} \vdash C[\bot]$

Table 1: Axiomatization of Compositional Weighted Logic

As usual, we say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the given axioms and rules. We define

$$\overline{\Phi} = \{\phi \in \mathcal{L} \mid \Phi \vdash \phi\}.$$

Observe that the axiom (A6) states the relation between state modalities and transition modalities. Its instance $\vdash (\leq r) \rightarrow [\leq s]_a(\leq r + s)$ states that if the initial state of a transition is labeled with "at most r" and the transition is labeled with "at most s", then the final state has the label "at most $r+s$"; a similar instance exists for "at least".

The axiom (A11) states the relation between state modalities and concurrent modalities. Its instance $\vdash (\leq r)|(\leq s) \to (\leq r + s)$ states that if some states of two modules are labeled with "at most r" and "at most s" respectively, their composition is labeled with "at most $r + s$". Similarly, (A12) and (A13) state the relation between transition modalities and concurrent modalities. In (A13), the premise guarantees that for every possible splitting of a system in parallel subsystems and for every decomposition of the action c into a and b, one of the subsystem satisfies ϕ after all possible a-transitions and the other satisfies ψ after all possible b-transitions; this guarantees that the entire system will satisfy $\phi|\psi$ after all c-transitions. As we have emphasized in Section 2, if the synchronization of the system requires more complicated operations than addition, one only needs to modify correspondingly the operators in (A11), (A12) and (A13).

The rules (R3) and (R4) are infinitary and encode the Archimedean properties of rational numbers. For instance, the formula below is an instance of (R3) stating that if the resources available in a state are at least r for each $r < s$, then they are at least s.

$$\{(\geq r) \mid r < s\} \vdash (\geq s).$$

Similarly, the formula below is an instance of (R4) guaranteeing that the resources in a state cannot be infinite (bigger that every rational).

$$\{(\geq r) \mid r \in \mathbb{Q}\} \vdash \bot.$$

The rules (R3) and (R4) are closed under arbitrary contexts. Due to them, CWL is non-compact: infinite sets of formulas such as $\{(\geq r) \mid r < s\} \cup \{\neg(\geq s)\}$ and $\{(\geq r) \mid r \in \mathbb{Q}\}$ are inconsistent while every finite subset of them is consistent.

A simple induction on the structure of possible proofs within the axiomatic system shows that all the theorems are sound in the LWS-semantics.

Theorem 3 (Soundness)*: The axiomatic system in Table 1 is sound with respect to the semantics based on LWSs, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\vdash \phi \textit{ implies } \models \phi.$$

Proof. As usual, the soundness is proved by verifying that each axiom is sound and that the rules preserve soundness. These can be trivially verified. ∎

Example 3*: The axiomatic system allows us to prove the statement mentioned in Ex-*

ample 2, $\models (\geq 0) \wedge P^N | C^N \rightarrow Good^N$. *According to the soundness of the axiomatization, we need to prove* $\vdash (\geq 0) \wedge P^N | C^N \rightarrow Good^N$. *The proof can be done inductively on N:*

• *For* $N = 0$, *obviously* $\vdash (\geq 0) \wedge P^0 | C^0 \rightarrow Good^0$.

• *For* $N \neq 0$*:* $\vdash (\geq 0) \rightarrow [\geq 0]_{a*a}(\geq 0)$ *by (A6).*

$\vdash \langle \geq x \rangle_a P^{N-1} | \langle \geq -x \rangle_a C^{N-1} \rightarrow \langle \geq 0 \rangle_{a*a} P^{N-1} | C^{N-1}$ *by (A12).*

$\vdash [\geq 0]_{a*a}(\geq 0) \wedge \langle \geq 0 \rangle_{a*a} P^{N-1} | C^{N-1} \rightarrow \langle \geq 0 \rangle_{a*a}((\geq 0) \wedge P^{N-1} | C^{N-1})$ *by (A4) and propositional axioms. Then using the inductive hypothesis* $\vdash (\geq 0) \wedge P^{N-1} | C^{N-1} \rightarrow Good^{N-1}$, *we have that* $\vdash (\geq 0) \wedge P^N | C^N \rightarrow (\geq 0) \wedge \langle \geq 0 \rangle_{a*a} Good^{N-1}$.

Hence $\vdash (\geq 0) \wedge P^N | C^N \rightarrow Good^N$. ∎

5.2. Weak-Completeness for Compositional Weighted Logic

We prove in this sub-section that the axiomatic system in Table 1 is weakly complete for the LWS-semantics, meaning that all the validities can be proved, as theorems, from the proposed axioms and rules.

To prove this, Model Theory tells us that it is sufficient to show that every consistent formula of $\mathcal{L}$ has a model. In what follows we show how such a model can be constructed for an arbitrary consistent formula. In fact, our result is stronger: we construct a finite model (CWL enjoys the finite model property). The construction follows the line of the classical filtration method for modal logic [**?**].

Consider an arbitrary formula $\phi \in \mathcal{L}$. Let $R_\phi \subseteq \mathbb{Q}$ be the set of all $r \in \mathbb{Q}$ such that r is in the label of some state or transition modality $(\trianglelefteq r)$, $\langle \trianglelefteq r \rangle_a$ or $[\trianglelefteq r]_a$ that appears in the syntax of ϕ. Let Σ_ϕ be the set of all actions $a \in \Sigma$ such that a appears in some transition modality $\langle \trianglelefteq r \rangle_a$ or $[\trianglelefteq r]_a$ in the syntax of ϕ. Obviously, R_ϕ and Σ_ϕ are finite.

- The *granularity* of $\phi \in \mathcal{L}$, denoted by $gr(\phi)$, is the least common denominator of the elements of R_ϕ. Let $I(\phi)$ be the set of all rationals of type $\frac{p}{gr(\phi)}$ in the interval $[min(R_\phi), max(R_\phi)]$, $p \in \mathbb{Z}$. $I(\phi) = \emptyset$ when $R_\phi = \emptyset$.

- The *modal depth* of ϕ, denoted by $md(\phi)$, is defined inductively by

$$md(\phi) = \begin{cases} 0, & \text{if } \phi = \bot \text{ or } \phi = (\trianglelefteq r) \\ max\{md(\psi), md(\psi')\}, & \text{if } \phi = \psi \to \psi' \text{ or } \phi = \psi|\psi' \\ md(\psi) + 1, & \text{if } \phi = [\trianglelefteq r]_a\psi \end{cases}$$

- The *compositional depth* of $\phi \in \mathcal{L}$, denoted by $cd(\phi)$, is defined inductively by

$$cd(\phi) = \begin{cases} 0, & \text{if } \phi = \bot \text{ or } \phi = (\trianglelefteq r) \\ cd(\psi), & \text{if } \phi = [\trianglelefteq r]_a\psi \\ max\{cd(\psi), cd(\psi')\}, & \text{if } \phi = \psi \to \psi' \\ cd(\psi) + cd(\psi') + 1, & \text{if } \phi = \psi|\psi' \end{cases}$$

In what follows we fix a consistent formula $\rho \in \mathcal{L}$ and we construct a model for it. Let

$$\mathcal{L}[\rho] = \{\phi \in \mathcal{L} \mid I(\phi) \subseteq I(\rho), md(\phi) \leq md(\rho), cd(\phi) \leq cd(\rho), \Sigma_\phi \subseteq \Sigma_\rho\}.$$

Let Ω be the set of the $\mathcal{L}$-maximal consistent sets of formulas and $\Omega[\rho]$ the set of the $\mathcal{L}[\rho]$-maximal consistent sets of formulas. Since $\mathcal{L}[\rho]$ is finite (modulo logical equivalence), $\Omega[\rho]$ is finite.

Since the set of instances of the infinitary rules in Table 1 is countable[4], we can use the *Extended Rasiowa-Sikorski Lemma* (Lemma 2) to prove Lindenbaum's Lemma (Theorem 2) for our logic, as demonstrated in Section 4.

Suppose that for each $\Lambda \in \Omega[\rho]$ we chose one $\Gamma \in \Omega$ such that $\Lambda \subseteq \Gamma$; to identify it, we denote this Γ by Λ^+. Let $\Omega^+[\rho] = \{\Lambda^+ \in \Omega \mid \Lambda \in \Omega[\rho]\}$. In what follows we will construct an LWS $\mathcal{W}_\rho = (\Omega^+[\rho], \theta_\rho, l_\rho)$ that satisfies ρ in one of its states. To do this, we have to properly define the transition function and the labeling function. Besides, we also need to define the composition of arbitrary $\Gamma, \Gamma' \in \Omega^+[\rho]$.

In order to define the transition function, we state the following lemma that will be repeatedly used in the sequent proofs.

Lemma 3*: For arbitrary* $\Gamma, \Gamma' \in \Omega$ *and arbitrary* $r, s \in \mathbb{Q}$ *with* $s > 0$*,*

1. If $[\phi \in \Gamma'$ *implies* $\langle \leq r\rangle_a\phi \in \Gamma]$*, then* $[\phi \in \Gamma'$ *implies* $\langle \leq r + s\rangle_a\phi \in \Gamma]$*;*

2. If $[\phi \in \Gamma'$ *implies* $\langle \geq r\rangle_a\phi \in \Gamma]$*, then* $[\phi \in \Gamma'$ *implies* $\langle \geq r - s\rangle_a\phi \in \Gamma]$*;*

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[4]Observe that since Σ, $\mathbb{Q}$ and Mod^* are countable, the sets of instances of the infinitary rules (R3) and (R4) are countable as well.

3. *If* $x = \inf\{u \in \mathbb{Q} \mid \phi \in \Gamma'$ *implies* $\langle \leq u \rangle_a \phi \in \Gamma\}$ *and* $r \geq x$*, then*

$$[\phi \in \Gamma' \textit{ implies } \langle \leq r \rangle_a \phi \in \Gamma];$$

4. *If* $x = \sup\{u \in \mathbb{Q} \mid \phi \in \Gamma'$ *implies* $\langle \geq u \rangle_a \phi \in \Gamma\}$ *and* $r \leq x$*, then*

$$[\phi \in \Gamma' \textit{ implies } \langle \geq r \rangle_a \phi \in \Gamma].$$

Proof. 1. From (A5), we have that $\langle \leq r \rangle_a \phi \in \Gamma$ implies $\langle \leq r + s \rangle_a \phi \in \Gamma$. Similar for 2. 3. It is a direct consequence of 1 using the fact that x is an infimum. Similarly for 4. ∎

In Lemma 3 x might be irrational and consequently it cannot appear as an index of a modality. Nevertheless, it is the limit of some monotonic sequence of rationals.

Now we are ready to define the transition function $\theta_\rho : \Omega^+[\rho] \times (\Sigma \times \mathbb{R}) \to \Omega^+[\rho]$ for $\mathcal{W}_\rho$ as follows: for $a \in \Sigma_\rho$ and $\Gamma, \Gamma' \in \Omega^+[\rho]$, we define $\Gamma \xrightarrow{x}_a \Gamma'$ whenever

$$x = \sup\{r \in \mathbb{Q} \mid \phi \in \Gamma' \text{ implies } \langle \geq r \rangle_a \phi \in \Gamma\}$$
$$= \inf\{r \in \mathbb{Q} \mid \phi \in \Gamma' \text{ implies } \langle \leq r \rangle_a \phi \in \Gamma\} \in \mathbb{R}.$$

Lemma 4*: For arbitrary* $\Gamma \in \Omega$*,* $\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$.

Proof. Let $A = \{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}$ and $B = \{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$. We first prove that A and B are both non-empty: Axiom (A1) guarantees that for all $r \in \mathbb{Q}$, either $(\geq r) \in \Gamma$ or $(\leq r) \in \Gamma$. Suppose that there exists $r \in \mathbb{Q}$ such that $(\geq r) \in \Gamma$. Then, $A \neq \emptyset$. Suppose that $B = \emptyset$, then (A1) implies that for all $r \in \mathbb{Q}$, $(\geq r) \in \Gamma$. Using (R4) for $C[X] = X$, we derive that $\perp \in \Gamma$ - contradiction. Consequently, $B \neq \emptyset$. Similarly it can be proven that $B \neq \emptyset$ implies $A \neq \emptyset$.

Since the two sets are non-empty, the sup and inf exist. Suppose that $\sup A = \infty$. Then by applying (A2) we obtain that $\mathbb{Q} \subseteq A$ and (R4) for $C[X] = X$ proves the inconsistency of Γ - contradiction. Hence $\sup A \in \mathbb{R}$. Similarly one can prove that $\inf B \in \mathbb{R}$.

Let $\sup A = x$ and $\inf B = y$. We need to prove that $x = y$. If $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$. Since $x < r$, $(\geq r) \notin \Gamma$ and (A1) guarantees that $(\leq r) \in \Gamma$. But this contradicts the fact that $r \leq y$. If $y < x$, there exists $r_1, r_2 \in \mathbb{Q}$ such that $y < r_1 < r_2 < x$. Since $r_1 > y$, $(\leq r_i) \in \Gamma$ for $i = 1, 2$ (applying (A2)), and similarly, $r_i < x$ implies $(\geq r_i) \in \Gamma$ for $i = 1, 2$. Since $r_2 - r_1 > 0$, we apply (A3) and obtain $\vdash (\geq r_2) \to \neg(\leq r_1)$. This shows that Γ is inconsistent - contradiction. Consequently, $x = y$. ∎

Lemma 4 allows us to define, for all $\Gamma \in \Omega^+[\rho]$, $l_\rho(\Gamma) = \sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}$.

***Theorem* 4**: *The tuple $\mathcal{W}_\rho = (\Omega^+[\rho], \theta_\rho, l_\rho)$ defined above is an LWS.*

Proof. We need to prove that for all $\Gamma, \Gamma' \in \Omega^+[\rho]$, if $\Gamma \xrightarrow{x}_a \Gamma'$, then $l_\rho(\Gamma') = l_\rho(\Gamma) + x$. Let $l_\rho(\Gamma) = y, l_\rho(\Gamma') = y'$. Consider arbitrary $r_x, r_y \in \mathbb{Q}$ with $r_x \leq x$ and $r_y \leq y$. By (A2) we have $(\geq r_y) \in \Gamma$. Then using (A6), $[\geq r_x]_a(\geq r_x + r_y) \in \Gamma$.

Since $r_x \leq x = \sup\{r \in \mathbb{Q} \mid [\geq r]_a\phi \in \Gamma \text{ implies } \phi \in \Gamma'\}$, we have $(\geq r_x + r_y) \in \Gamma'$.

Consequently, $y' \geq r_x + r_y$ for arbitrary $r_x \leq x$ and $r_y \leq y$. Hence $y' \geq x + y$.

A similar argument proves that $y' \leq x + y$ and consequently, $y' = x + y$. ∎

Before proceeding with the Truth Lemma, we prove a couple of lemmas that establish essential correlations between the structure of $\mathcal{W}_\rho$ and the elements of the maximally consistent sets that form its support.

For all $\Gamma, \Gamma_0, \Gamma_1 \in \Omega^+[\rho]$, we write $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$ if:

$$\text{for all } \varphi_0 \in \Gamma_0 \text{ and } \varphi_1 \in \Gamma_1, \varphi_0|\varphi_1 \in \Gamma.$$

***Lemma* 5**: *For all $\Gamma, \Gamma_0, \Gamma_1 \in \Omega^+[\rho]$, if $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$ then*

$$(\Gamma, \mathcal{W}_\rho) \sim ((\Gamma_0, \Gamma_1), \mathcal{W}_\rho \otimes \mathcal{W}_\rho).$$

Proof. We need to prove that if $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$ then there exists a weighted bisimulation $R \subseteq \Omega^+[\rho] \times (\Omega^+[\rho] \times \Omega^+[\rho])$ s.t. $(\Gamma, (\Gamma_0, \Gamma_1)) \in R$. Let R be the binary relation defined as follows and we prove that R is a weighted bisimulation.

$$R = \{(\Gamma, (\Gamma_0, \Gamma_1)) \mid \Gamma, \Gamma_0, \Gamma_1 \in \Omega^+[\rho], (\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}\}.$$

Let $(\Gamma, (\Gamma_0, \Gamma_1)) \in R$. In what follows we verify the three conditions in Definition 2.

1. We prove that $l_\rho(\Gamma) = l_\rho((\Gamma_0, \Gamma_1))$.

Let $l_\rho(\Gamma) = u, l_\rho(\Gamma_0) = u_0$ and $l_\rho(\Gamma_1) = u_1$. Then $l_\rho((\Gamma_0, \Gamma_1)) = l_\rho(\Gamma_0) + l_\rho(\Gamma_1) = u_0 + u_1$. Suppose $u_0 + u_1 > u = \sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}$. Then there exists $s \in \mathbb{Q}$ s.t. $u < s \leq u_0 + u_1$ and $(\geq s) \notin \Gamma$. Then there exist $s_0, s_1 \in \mathbb{Q}$ s.t. $s = s_0 + s_1$, $s_0 \leq u_0$ and $s_1 \leq u_1$. For $i = 0, 1$, $u_i = \sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma_i\}$ implies for all $r_i \in \mathbb{Q}$ s.t. $r_i \leq u_i$, we have $(\geq r_i) \in \Gamma_i$. Therefore, $(\geq s_0) \in \Gamma_0$ and $(\geq s_1) \in \Gamma_1$. So $(\geq s_0)|(\geq s_1) \in \Gamma$ because $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$. Using (A11), $(\geq s_0 + s_1) \in \Gamma$ - contradiction!

Suppose $u_0+u_1 < u = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$. Then there exists $s \in \mathbb{Q}$ s.t. $u_0+u_1 \leq s < u$

and $(\leq s) \notin \Gamma$. Then there exist $s_0, s_1 \in \mathbb{Q}$ s.t. $s = s_0 + s_1$, $s_0 \geq u_0$ and $s_1 \geq u_1$. For $i = 0, 1$, $u_i = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma_i\}$ implies for all $r_i \in \mathbb{Q}$ s.t. $r_i \geq u_i$, we have $(\leq r_i) \in \Gamma_i$. Therefore, $(\leq s_0) \in \Gamma_0$ and $(\leq s_1) \in \Gamma_1$. So $(\leq s_0)|(\leq s_1) \in \Gamma$ because $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$. By (A11), $(\leq s_0 + s_1) \in \Gamma$ - contradiction! Consequently, $u = u_0 + u_1$.

2. We need to prove that: if $(\Gamma_0, \Gamma_1) \xrightarrow{x}_a (\Gamma'_0, \Gamma'_1)$, there exist $\Gamma' \in \Omega^+[\rho]$ and $y \in \mathbb{R}$ such that $\Gamma \xrightarrow{y}_a \Gamma'$, $y = x$ and $(\Gamma', (\Gamma'_0, \Gamma'_1)) \in R$.

• Suppose there exists no $\Gamma' \in \Omega^+[\rho]$ and $y \in \mathbb{R}$ s.t. $\Gamma \xrightarrow{y}_a \Gamma'$, which implies $[a]\bot \in \Gamma$. We prove that $[a]\bot \in \Gamma$ implies $[a_0]\bot \in \Gamma_0$ and $[a_1]\bot \in \Gamma_1$ for all $a_0 * a_1 = a$: suppose $\langle a_0 \rangle\top \in \Gamma_0$ and $\langle a_1 \rangle\top \in \Gamma_1$ with $a_0 * a_1 = a$. Then $\langle a_0 \rangle\top | \langle a_1 \rangle\top \in \Gamma$ since $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$. Hence $\langle a \rangle\top \in \Gamma$ by A(12) – contradiction!

So $[a_0]\bot \in \Gamma_0$ and $[a_1]\bot \in \Gamma_1$ for all $a_0 * a_1 = a$. Therefore, there do not exist $\Gamma'_i \in \Omega^+[\rho]$ and $x_i \in \mathbb{R}$ s.t. $\Gamma_i \xrightarrow{x_i}_{a_i} \Gamma'_i$, $i = 0, 1$. Hence, there do not exist $\Gamma'_0, \Gamma'_1 \in \Omega^+[\rho]$ and $x \in \mathbb{R}$ s.t. $(\Gamma_0, \Gamma_1) \xrightarrow{x}_a (\Gamma'_0, \Gamma'_1)$ – contradiction!

Consequently, there exists $\Gamma' \in \Omega^+[\rho]$ and $y \in \mathbb{R}$ s.t. $\Gamma \xrightarrow{y}_a \Gamma'$.

• Suppose for all $\Gamma' \in \Omega^+[\rho]$ s.t. $\Gamma \xrightarrow{y}_a \Gamma'$, and $(\Gamma', \Gamma'_0, \Gamma'_1) \in \mathcal{P}$, we have $y \neq x$.

$(\Gamma_0, \Gamma_1) \xrightarrow{x}_a (\Gamma'_0, \Gamma'_1)$ then exist $x_0, x_1 \in \mathbb{Q}$ and $a_0, a_1 \in \Sigma$ such that $x = x_0 + x_1$, $a = a_0 * a_1$ and $\Gamma_i \xrightarrow{x_i}_{a_i} \Gamma'_i$ for $i = 0, 1$.

If $x_0 + x_1 > y = \sup\{r \in \mathbb{Q} \mid \phi \in \Gamma' \text{ implies } \langle \geq r \rangle_a \phi \in \Gamma\}$, then there exists $s \in \mathbb{Q}$ s.t. $y < s \leq x_0 + x_1$ and for all $\phi \in \Gamma'$, $\langle \geq s \rangle_a \phi \notin \Gamma$. So there exist $s_0, s_1 \in \mathbb{Q}$ s.t. $s_0 \leq x_0$, $s_1 \leq x_1$ and $s = s_0 + s_1$, . For all $\phi_0 \in \Gamma'_0$ and $\phi_1 \in \Gamma'_1$, we have $\phi_0 | \phi_1 \in \Gamma'$ because $(\Gamma', \Gamma'_0, \Gamma'_1) \in \mathcal{P}$. This implies that $\langle \geq s \rangle_a \phi_0 | \phi_1 \notin \Gamma$ for $a = a_0 * a_1$. On the other hand, for all $\phi_0 \in \Gamma'_0, \phi_1 \in \Gamma'_1$, the definition of θ_ρ guarantees $\langle \geq s_0 \rangle_{a_0} \phi_0 \in \Gamma_0, \langle \geq s_1 \rangle_{a_1} \phi_1 \in \Gamma_1$. Hence, $\langle \geq s_0 \rangle_{a_0} \phi_0 | \langle \geq s_1 \rangle_{a_1} \phi_1 \in \Gamma$ since $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$. Now applying axiom (A12) we get $\langle \geq s \rangle_{a_0 * a_1} \phi_0 | \phi_1 \in \Gamma$ - contradicts the previous conclusion! Consequently, $x_0 + x_1 \not> y$. Similarly can be proven that it is not the case that $x_0 + x_1 < y$. Hence, $x = y$. Consequently, there exists $\Gamma' \in \Omega^+[\rho]$ such that $\Gamma \xrightarrow{x}_a \Gamma'$ and $(\Gamma', (\Gamma'_0, \Gamma'_1)) \in R$.

3. We need to prove that: if $\Gamma \xrightarrow{x}_a \Gamma'$, there exist $\Gamma'_0, \Gamma'_1 \in \Omega^+[\rho]$ and $y \in \mathbb{R}$ such that $(\Gamma_0, \Gamma_1) \xrightarrow{y}_a (\Gamma'_0, \Gamma'_1)$, $y = x$ and $(\Gamma', (\Gamma'_0, \Gamma'_1)) \in R$.

• Suppose there exist no $\Gamma'_0, \Gamma'_1 \in \Omega^+[\rho]$ and $y \in \mathbb{R}$ s.t. $(\Gamma_0, \Gamma_1) \xrightarrow{y}_a (\Gamma'_0, \Gamma'_1)$. So for all $a_0 * a_1 = a$, there exist no $\Gamma'_0, \Gamma'_1 \in \Omega^+[\rho]$ and $y_0, y_1 \in \mathbb{R}$ such that $\Gamma_i \xrightarrow{y_i}_{a_i} \Gamma'_i$, $i = 0, 1$, which further imply that $[a_0]\bot \in \Gamma_0$ and $[a_1]\bot \in \Gamma_1$ for all $a_0 * a_1 = a$. Consequently,

$[a_0]\bot|[a_1]\bot \in \Gamma$ because $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$. So, $[a]\bot \in \Gamma$ by A(13) – contradiction! Hence there exists $\Gamma_0', \Gamma_1' \in \Omega^+[\rho]$ and $y \in \mathbb{R}$ such that $(\Gamma_0, \Gamma_1) \xrightarrow{y}_a (\Gamma_0', \Gamma_1')$.

• Suppose for all $\Gamma_0', \Gamma_1' \in \Omega^+[\rho]$ s.t. $(\Gamma_0, \Gamma_1) \xrightarrow{y}_a (\Gamma_0', \Gamma_1')$, and $(\Gamma', \Gamma_0', \Gamma_1') \in \mathcal{P}$, we have $y \neq x$. $(\Gamma_0, \Gamma_1) \xrightarrow{y}_a (\Gamma_0', \Gamma_1')$ then exist $y_0, y_1 \in \mathbb{Q}$ and $a_0, a_1 \in \Sigma$ s.t. $y = y_0 + y_1$, $a = a_0 * a_1$ and $\Gamma_i \xrightarrow{y_i}_{a_i} \Gamma_i'$ for $i = 0, 1$. If $y_0 + y_1 > x = \sup\{r \in \mathbb{Q} \mid \phi \in \Gamma' \text{ implies } \langle \geq r\rangle_a \phi \in \Gamma\}$, then there exists $s \in \mathbb{Q}$ s.t. $x < s \leq y_0 + y_1$ and for all $\phi \in \Gamma'$, $\langle \geq s\rangle_a \phi \notin \Gamma$. So there exist $s_0, s_1 \in \mathbb{Q}$ s.t. $s_0 \leq y_0$, $s_1 \leq y_1$ and $s = s_0 + s_1$. For all $\phi_0 \in \Gamma_0', \phi_1 \in \Gamma_1'$, we have $\phi_0|\phi_1 \in \Gamma'$ because $(\Gamma', \Gamma_0', \Gamma_1') \in \mathcal{P}$. This implies that $\langle \geq s\rangle_a \phi_0|\phi_1 \notin \Gamma$ for $a = a_0 * a_1$. On the other hand, for all $\phi_0 \in \Gamma_0', \phi_1 \in \Gamma_1'$, the definition of θ_ρ guarantees $\langle \geq s_0\rangle_{a_0}\phi_0 \in \Gamma_0, \langle \geq s_1\rangle_{a_1}\phi_1 \in \Gamma_1$. Hence, $\langle \geq s_0\rangle_{a_0}\phi_0|\langle \geq s_1\rangle_{a_1}\phi_1 \in \Gamma$ since $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$. Now applying axiom (A12) we get $\langle \geq s\rangle_{a_0*a_1}\phi_0|\phi_1 \in \Gamma$ - contradicts the previous conclusion! Consequently, $y_0 + y_1 \not> x$. Similarly can be proven that it is not the case that $y_0 + y_1 < x$. Hence, $y = x$. So there exist $\Gamma_0', \Gamma_1' \in \Omega^+[\rho]$ s.t. $(\Gamma_0, \Gamma_1) \xrightarrow{x}_a (\Gamma_0', \Gamma_1')$ and $(\Gamma', (\Gamma_0', \Gamma_1')) \in R$. ∎

Lemma 5 allows us to characterize $\Gamma \in \Omega^+[\rho]$ as the composition of $\Gamma_0, \Gamma_1 \in \Omega^+[\rho]$ iff $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$, i.e., iff for all $\phi_0 \in \Gamma_0$ and $\phi_1 \in \Gamma_1$, $\phi_0|\phi_1 \in \Gamma$. Now we look deeper into the structure of the formulas that compose the maximally consistent sets.

***Lemma** 6: Let $\Phi \subseteq \mathcal{L}$ be a consistent set of formulas.*

1. For arbitrary $\phi \in \mathcal{L}$, $r \in \mathbb{Q}$ and $a \in \Sigma$, if $[\trianglelefteq r]_a\phi \notin \overline{\Phi}$, then the set $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]_a\psi \in \overline{\Phi}\} \cup \{\neg\phi\}$ is consistent.

2. For arbitrary $f, g \in \mathcal{L}$, if $f \| g \notin \overline{\Phi}$, then the sets $\{\phi \in \mathcal{L} \mid \phi \| g \in \overline{\Phi}\} \cup \{\neg f\}$ and $\{\psi \in \mathcal{L} \mid f \| \psi \in \overline{\Phi}\} \cup \{\neg g\}$ are consistent.

Proof. 1. Let $A = \{\psi \in \mathcal{L} \mid [\trianglelefteq r]_a\psi \in \overline{\Phi}\}$. Suppose $A \cup \{\neg\phi\}$ is inconsistent, then there exists a set $F = \{f_i \in A \mid i \in I\}$ of formulas such that $F \vdash \phi$. If I is finite, (R1) guarantees that $[\trianglelefteq r]_a F \vdash [\trianglelefteq r]_a\phi$. Otherwise, $F \vdash \phi$ is (modulo Boolean reasoning possible involving infinite meets) an instance of (R3) or (R4); in both cases $[\trianglelefteq r]_a F \vdash [\trianglelefteq r]_a\phi$ is an instance of the same rule. Since $F \subseteq A$, $[\trianglelefteq r]_a F \subseteq \overline{\Phi}$, which further implies that $[\trianglelefteq r]_a\phi \in \overline{\Phi}$ - contradicts the assumption of consistency of Φ.

2. A similar proof strategy as for 1. ∎

***Corollary* 1**: *For all $\Gamma, \Gamma_0, \Gamma_1 \in \Omega^+[\rho]$, $\phi, f_0, f_1 \in \mathcal{L}[\rho]$, $a \in \Sigma$ and $r \in \mathbb{Q}$,*

1. $[\trianglelefteq r]_a\phi \in \Gamma$ iff for all $\Gamma' \in \Omega^+[\rho]$ s.t. $\Gamma \xrightarrow{x}_a \Gamma'$ with $x \trianglelefteq r, \phi \in \Gamma'$.

2. $[\trianglelefteq r]_a\bot \in \Gamma \cap \mathcal{L}[\rho]$ iff there exists no $\Gamma' \in \Omega^+[\rho]$ s.t. $\Gamma \xrightarrow{x}_a \Gamma'$ with $x \trianglelefteq r$.

3. $f_0 \| f_1 \in \Gamma$ iff for all $\Gamma_0, \Gamma_1 \in \Omega^+[\rho]$ s.t. $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$, $f_i \in \Gamma_j$ or $f_{1-i} \in \Gamma_{1-j}$, $i, j \in \{0, 1\}$.

Proof. 1. ($\Longrightarrow$) This implication derives directly from the labeled transitions on $\mathcal{W}_\rho$.
($\Longleftarrow$) Let $B = \{\Gamma' \in \Omega^+[\rho] \mid \Gamma \xrightarrow{x}_a \Gamma', x \trianglelefteq r\}$ and $\bigcap B = \bigcap_{\Gamma' \in B} \Gamma'$.
Since all the elements of $\Omega^+[\rho]$ are maximal consistent sets, $\psi \notin \bigcap B$ iff there exists $\Gamma' \in \Omega^+[\rho]$ s.t. $\Gamma \xrightarrow{x}_a \Gamma'$ and $\neg\psi \in \Gamma'$. Let $\phi \in \bigcap B$ and suppose that $[\trianglelefteq r]_a\phi \notin \Gamma$. Applying Lemma 6, we obtain that the set $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]_a\psi \in \Gamma\} \cup \{\neg\phi\}$ is consistent. Since $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]_a\psi \in \Gamma\} \subseteq \bigcap B$ and $\neg\phi \in \mathcal{L}[\rho]$, there must exist $\Gamma'' \in \Omega^+[\rho]$ such that $\{\psi \in \mathcal{L} \mid [\trianglelefteq r]_a\psi \in \Gamma\} \cup \{\neg\phi\} \subseteq \Gamma''$. Since $\Gamma'' \supseteq \{\psi \in \mathcal{L} \mid [\trianglelefteq r]_a\psi \in \Gamma\}$, we obtain that $\Gamma'' \in B$. On the other hand, $\Gamma'' \ni \neg\phi$ - contradiction!
2. This is an instance of 1.
3. ($\Longrightarrow$) This implication derives directly from the composition of $\mathcal{W}_\rho$.
($\Longleftarrow$) Let $B = \{(\Gamma_0, \Gamma_1) \in \Omega^+[\rho] \times \Omega^+[\rho] \mid (\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}\}$ and $\bigcap B = \{\phi_0 \| \phi_1 \mid \phi_0 \in \Gamma_0, \phi_1 \in \Gamma_1, (\Gamma_0, \Gamma_1) \in B\}$. Since all the elements of $\Omega^+[\rho]$ are maximal consistent sets, $\phi_0 \| \phi_1 \notin \bigcap B$ iff there exists $\Gamma_0, \Gamma_1 \in \Omega^+[\rho]$ s.t. $(\Gamma, \Gamma_0, \Gamma_1) \in \mathcal{P}$ and $\neg\phi_i \in \Gamma_i$ for $i = 1, 2$. Let $f \| g \in \bigcap B$ and suppose that $f \| g \notin \Gamma$. Let $A_0 = \{\phi \in \mathcal{L} \mid \phi \| g \in \Gamma\}$, $A_1 = \{\psi \in \mathcal{L} \mid f \| \psi \in \Gamma\}$. By Lemma 6, the sets $A_0 \cup \{\neg f\}$ and $A_1 \cup \{\neg g\}$ are consistent. Since $A_0 \subseteq \{\phi_0 \mid \phi_0 \| \phi_1 \in \Gamma\}$ and $A_1 \subseteq \{\phi_1 \mid \phi_0 \| \phi_1 \in \Gamma\}$, we have $A_0 \| A_1 \subseteq \bigcap B$. Besides $f \| g \notin \Gamma$, so $\exists \Gamma'_0, \Gamma'_1 \in \Omega^+[\rho]$ s.t. $(\Gamma'_0, \Gamma'_1) \in B$ and $\neg f \in \Gamma'_i, \neg g \in \Gamma'_{1-i}$ for both $i = 0$ and 1. Then $A_0 \cup \{\neg f\} \subseteq \Gamma'_0$, $A_1 \cup \{\neg g\} \subseteq \Gamma'_1$ and $(\Gamma'_0, \Gamma'_1) \in B$. So $(\Gamma, \Gamma'_0, \Gamma'_1) \in \mathcal{P}$, which implies that for all $\phi \| \psi \in \Gamma$, $\phi_i \in \Gamma_j, i, j \in \{0, 1\}$ - contradiction! ∎

***Lemma* 7** (Truth Lemma): *If $\rho \in \mathcal{L}$ is consistent, then for all $\psi \in \mathcal{L}[\rho]$ and $\Gamma \in \Omega^+[\rho]$,*

$$\psi \in \Gamma \text{ iff } \mathcal{W}_\rho, \Gamma \models \psi.$$

Proof. Induction on the structure of ψ. The boolean cases are trivial.
case $\psi = (\leq r)$**:** Since $l_\rho(\Gamma) = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$, $(\leq r) \in \Gamma$ is equivalent to $r \geq l_\rho(\Gamma)$, which is further equivalent to $\mathcal{W}_\rho, \Gamma \models (\leq r)$. Similarly for $\psi = (\geq r)$.

case $\psi = [\leq r]_a\phi$**:** $[\leq r]_a\phi \in \Gamma$ iff for all $\Gamma' \in \Omega^+[\rho]$ s.t. $\Gamma \xrightarrow{x}_a \Gamma'$ with $x \leq r, \phi \in \Gamma'$ by Corollary 1, which is equal to $\mathcal{W}_\rho, \Gamma \models [\leq r]_a\phi$. Similarly for $\psi = [\geq r]_a\phi$.

case $\psi = \psi_1 \| \psi_2$**:** $\psi_1 \| \psi_2 \in \Gamma$ iff for all $\Gamma_1, \Gamma_2 \in \Omega^+[\rho]$ s.t. $(\Gamma, \Gamma_1, \Gamma_2) \in \mathcal{P}$ with $\psi_1 \in \Gamma_1$ or $\psi_2 \in \Gamma_2$ by Corollary 1, which is equal to $\mathcal{W}_\rho, \Gamma \models \psi_1 \| \psi_2$ by Lemma 5. ∎

The Truth Lemma proves that $\mathcal{L}$ with the axiomatization presented in Table 1 enjoys the finite model property.

Theorem 5 (Finite model property)*: For every $\mathcal{L}$-consistent formula ϕ, there exists a finite LWS $\mathcal{W} = (S, \theta, l)$ of cardinality bounded by the size of ϕ and a state $s \in S$ such that $\mathcal{W}, s \models \phi$.*

Proof. The consistency of $\phi \in \mathcal{L}$ and Rasiowa-Sikorski lemma guarantee the existence of a $\mathcal{L}$-maximally consistent set $\Gamma \in \Omega[\phi]$ such that $\phi \in \Gamma$. Using the truth lemma, $\mathcal{W}_\phi, \Gamma \models \phi$, which is a finite model by the construction. ∎

The finite model property proves the Weak-Completeness, as usual.

Theorem 6 (Weak Completeness)*: The logic $\mathcal{L}$ with the axiomatization presented in Table 1 is weakly complete with respect to the LWS-semantics, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\models \phi \quad \textit{implies} \quad \vdash \phi.$$

Proof. The proof is standard for logics that enjoy the finite model property:
[$\models \phi$ implies $\vdash \phi$] is equivalent to [$\nvdash \phi$ implies $\not\models \phi$], that is equivalent to [the consistency of $\neg\phi$ implies the existence of an LWS $\mathcal{W} = (S, \theta, l)$ and a state $s \in S$ such that $\mathcal{W}, s \models \neg\phi$]. And this is guaranteed by the finite model property. ∎

5.3. Strong Completeness for Concurrent Weighted Logic

In this section we prove the strong completeness, which means to prove that given a set Φ of formulas and a formula ϕ, if every model of Φ is a model of ϕ, then $\Phi \vdash \phi$.

Observe that CWL is not compact: for the set of formulas

$$\{(\leq r) \mid \text{ for all } r < s\} \cup \{\neg(\leq s)\},$$

every finite subset is consistent, but the entire set is not consistent since using rule (R3) in the context $C[X] = X$, we can prove $\perp$. Similarly, the following set is inconsistent,

$$\{(\trianglelefteq r) \mid \text{ for all } r \in \mathbb{Q}\}$$

as (R4) proves $\perp$ in $C[X] = X$, but each finite subset is consistent. This observation proves that strong completeness of CWL is not implied by weak completeness.

As in Section 5.2, the proof relies on showing that every consistent set $\Phi \subseteq \mathcal{L}$ has a model. The construction follows the general line used in 5.2 with the only difference that now we do not build a finite model, but the canonical model $\mathcal{W}_{\mathcal{L}} = (S_{\mathcal{L}}, \theta_{\mathcal{L}}, l_{\mathcal{L}})$ which is an LWS s.t. $S_{\mathcal{L}}$ is the set of all $\mathcal{L}$-maximally consistent sets of formulas, where consistency is defined with respect to the axioms in Table 1.

As before, we define the transition $\Gamma \xrightarrow{x}_a \Gamma'$ for $\Gamma, \Gamma' \in S_{\mathcal{L}}$ whenever

$$x = \sup\{r \in \mathbb{Q} \mid [\geq r]_a\phi \in \Gamma \text{ implies } \phi \in \Gamma'\}$$
$$= \inf\{r \in \mathbb{Q} \mid [\leq r]_a\phi \in \Gamma \text{ implies } \phi \in \Gamma'\} \in \mathbb{R}.$$

Since $\sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\} = \inf\{r \in \mathbb{Q} \mid (\leq r) \in \Gamma\}$, we can define

$$l_{\mathcal{L}}(\Gamma) = \sup\{r \in \mathbb{Q} \mid (\geq r) \in \Gamma\}.$$

A similar proof as for Theorem 4 gives us the next result.

Theorem 7*: The tuple $\mathcal{W}_{\mathcal{L}} = (S_{\mathcal{L}}, \theta_{\mathcal{L}}, l_{\mathcal{L}})$ defined above is an LWS.*

Before proceeding with the Extended Truth Lemma, we also need to prove a couple of lemmas that establish essential correlations between the structure $\mathcal{W}_{\mathcal{L}}$ and the elements of the maximally consistent sets that form its support. It is easy to see that Lemma 5, 6 and Corollary 1 also hold for arbitrary $\Gamma, \Gamma_0, \Gamma_1 \in S_{\mathcal{L}}$, where we extend the definition of $\mathcal{P}$ to all the elements in $S_{\mathcal{L}}$. With these results, we can proceed with the extended truth lemma that can be proved following the same strategy as for Lemma 7.

Lemma 8 (Extended Truth Lemma)*: If $\Phi \in \mathcal{L}$ is a consistent set, then for all $\Gamma \in S_{\mathcal{L}}$,*

$$\Phi \subseteq \Gamma \quad \textit{iff} \quad \mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi.$$

Proof. We prove that for arbitrary $\psi \in \Phi$, $\psi \in \Gamma$ iff $\mathcal{W}_{\mathcal{L}}, \Gamma \models \psi$. This can be done by induction on ψ, similarly to the proof of the truth lemma. ∎

The Extended Truth Lemma proves that $\mathcal{L}$ has a canonical model.

Theorem 8 (Canonical Model)*: If $\Phi \subseteq \mathcal{L}$ is $\mathcal{L}$-consistent, there exists $\Gamma \in S_{\mathcal{L}}$ s.t.*

$$\mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi.$$

Proof. Since Φ is consistent, and the infinitary rules (R3) and (R4) have countably many instances, we apply the Rasiowa-Sikorski lemma that guarantees the existence of $\Gamma \in S_{\mathcal{L}}$ such that $\Phi \subseteq \Gamma$. Applying the extended truth lemma, $\mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi$. ∎

The previous theorem proves the Strong-Completeness for our logic.

Theorem 9 (Strong Completeness)*: The logic $\mathcal{L}$ with the axiomatization presented in Table 1 is strongly complete with respect to the LWS-semantics, i.e., for arbitrary $\Phi \subset \mathcal{L}$ and $\phi \in \mathcal{L}$,*

$$\Phi \models \phi \quad \textit{iff} \quad \Phi \vdash \phi.$$

Proof. ($\Longleftarrow$) This direction is guaranteed by the soundness of Table 1.
($\Longrightarrow$) If Φ is inconsistent, the statement is trivially true. Otherwise, consider an arbitrary $\Gamma \in S_{\mathcal{L}}$. If $\Phi \subseteq \Gamma$, then from Extended Truth Lemma we get $\mathcal{W}_{\mathcal{L}}, \Gamma \models \Phi$. Since $\Phi \models \phi$, $\mathcal{W}_{\mathcal{L}}, \Gamma \models \phi$ meaning that $\phi \in \Gamma$. Consequently, for every maximally-consistent set Γ, if $\Gamma \supseteq \Phi$, then $\Gamma \ni \phi$. Hence, $\Phi \vdash \phi$. ∎

6. Conclusions

We introduce Concurrent Weighted Logic (CWL), which is a multimodal logic that expresses qualitative, quantitative and modular properties of compositional LWSs. This logic is meant to address major modeling challenges from model-driven and component-based development within the area of embedded systems. The compositionality features of CWL support modular reasoning and specifications. These can be used to prove properties of upper level systems from properties of subsystems and reverse, to decompose global properties of systems into properties of its modules. We specifically describe this mechanism by proposing a set of sound and complete axioms.

On the theoretical level, we developed a (weakly and strongly) complete axiomatization for CWL, proved the finite model property and constructed the canonical model using the classical filtration method. With respect to papers proving similar results, innovative in this paper is the way we manage to avoid the use of the uncountable rules that are usually applied to get the strong completeness for non-compact modal logics [21]. By following the pioneering direction proposed in [19, 20], we manage to use Rasiowa-Sikorski Lemma [25, 21, 26] and an extension of it that we demonstrate, which on one hand allows us to prove the Lindenbaum's Lemma and, on the other hand, to complete the strong completeness proof.

For future work, we intend to deeper study CWL in order to understand if the finite model property can provide some complexity results that can be applied maybe for designing algorithms for satisfiability checking. We also intend to extend the logic to include specifications of dynamic resources such as clock-like variables and (reset, update) operations on them, similar to the ones used in timed logics. Another interesting extension of CWL could be achieved by including fixed point operators and increasing its expressiveness. Also considering systems with more complex labels, e.g., vectors of numbers to specify multi-types resources, can be interesting to investigate. All these extensions have important practical applications.

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