Alternation-Free Weighted Mu-Calculus: Decidability and Completeness

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Keywords: weighted modal Mu-Calculus, non-compact modal logics, weighted transition systems, satisfiability, complete axiomatization.

1 Introduction

For more than two decades, specification and modelling formalisms have been sought that address essential non-functional properties of embedded and cyber-physical systems. In particular, timed automata [4] were used for expressing and analysing timing constraints of systems with respect to timed logics such as TCTL [3], T_μ [17], L_ν [23] and MTL [19]. However, equally important non-functional properties of embedded or cyber-physical systems are related to consumption of resources, in particular that of energy. This lead initially to weighted extensions of timed automata [5, 6] and most recently to energy automata [9]. However, whereas the problems of cost-optimal reachability and infinite runs have been shown to be efficiently computable, the general model checking problem with respect to a weighted extension of TCTL turns out to be undecidable [11].

In this paper, we consider the purely weighted setting, in which the quantitative information of systems is modelled as weighted transition systems (WTSs) with transitions being decorated with non-negative reals besides actions. We study the problems of satisfiability and axiomatization of weighted logics in the most general setting. We develop WMC, a weighted version of the alternation-free modal mu-calculus, that subsumes WCTL and resembles the previously studied timed extension of the modal mu-calculus T_μ and L_ν. WMC is a multi-modal logic with fixed-point operators, where modalities either constrain

[1] This research is partially supported by the Sino-Danish Basic Research Center IDEA4CPS.

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This paper is electronically published in
Electronic Notes in Theoretical Computer Science
URL: www.elsevier.nl/locate/entcs

discrete transitions or the amount of resources in a given state. For the latter, WMC uses resource-variables, similar to the clock-variables used in timed logics, see e.g. [10].

Our first main contribution is to show decidability of satisfiability for WMC. In previous work [27], we proved decidability and finite model property for restriction of WMC with only one resource-variable for each resource and only maximal fixed points. This restriction bounds severely the expressiveness of the logic. In [25, 26], we studied two sub-logics of WMC with multiple resource-variables for each resource and only maximal fixed points. These logics are shown decidable by using the filtration construction, but are significantly weaker than WMC in that resource-variables are restricted to be event-recording. In contrast to these fragments and to modal mu-calculus, WMC does not posses finite model property, thus decidability does not follow from classical arguments. As an alternative, we propose here notions of symbolic model and semantics for which the finite model property does hold. Fortunately – as demonstrated in the paper – the notion of satisfiability coincides with that of symbolic satisfiability, from which our decidability result follows. This should be contrasted to the resembling timed logics for which satisfiability is undecidable.

The fact that the two semantics have the same validities is a remarkable property and a powerful tool that allows us to transport meta-results between the two semantics, in particular computability and complexity results for satisfiability checking and completeness results for proof systems.

Our second main contribution is a complete axiomatization of WMC, allowing all valid properties to be derived as theorems. At the best of our knowledge, this is the first complete axiomatization for a fixed point weighted modal logic in the literature. The axiomatization is remarkably simple, combining modal axioms of non-recursive weighted logic with classic axioms of fixed points [20, 28, 30]. The finite model property provides the arguments to demonstrate that the axiomatization is complete for the symbolic semantics and hence, the completeness result can be extrapolated to the WTS-semantics.

Our third main contribution is the completeness proof itself, which is non-standard and novel in many aspects. Since the logic is non-compact, it requires infinitary proof rules. To cope with this, we involve topological techniques for model theory, inspired by the work of Rasiowa and Sikorski [16, 29]. These techniques were previously developed by the first two authors in colaboration with Kozen and Panangaden for proving the strong completeness for Markovian logics [21, 22]. Our completeness proof avoids the tableaux method used in [32] for the general Mu-Calculus and it is sufficiently robust to be reused in similar contexts. On the other hand, our proof is designed for alternation-free calculi and it is not clear whether it can be used in a general unrestricted context.

2 Alternation-Free Weighted Mu-Calculus

Definition 2.1 *A weighted transition system is a tuple* $\mathcal{W} = (M, \mathcal{K}, \Sigma, \theta)$ *where* M *is a non-empty set of* states, $\mathcal{K} = \{e_1, \ldots, e_k\}$ *is a finite set of* resources, Σ *a non-empty set of* actions *and* $\theta : M \times \Sigma \times (\mathcal{K} \to \mathbb{R}_{\geq 0}) \to 2^M$ *is a* labelled transition function.

Instead of $m' \in \theta(m, a, f)$, we write $m \xrightarrow{f}_a m'$ and we call f the *weight function*. For simplicity, in what follows we assume that $\mathcal{K}$ is a singleton and we use the transition functions $\theta : M \times (\Sigma \times \mathbb{R}_{\geq 0}) \to 2^M$. However, the work can be straightforwardly extended to include multiple resources and all the following results hold in the extended case.

Alternation-Free Weighted Mu-Calculus (WMC) encodes properties of WTSs and involves modal operators and resource-variables similar to the ones used in timed logics [1, 3, 17]:
(i) *transition modalities* of type $[a]$ for $a \in \Sigma$;
(ii) *recursive-variables* that range over the set $\mathcal{X}$; they are used to define simultaneous recursive equations to express maximal and minimum fixed points, in the style of [12, 13, 24];
(iii) *resource-variables* ranging over the set $\mathcal{V}$;
(iv)) *state modalities* of type $x \trianglelefteq r$ for $\trianglelefteq \ \in \{\leq, \geq\}$ and $r \in \mathbb{Q}_{\geq 0}$, which approximates the resource-variable $x \in \mathcal{V}$;
(v) *reset* operators of type $x\ \underline{in}$ for the resource-variable.

Notation: we use both $\trianglelefteq$ and $\trianglerighteq$ to range over the set $\{\leq, \geq\}$ such that $\{\trianglelefteq, \trianglerighteq\} = \{\leq, \geq\}$. Similarly, we use $\triangleleft$ and $\triangleright$ to range over the set $\{<, >\}$ such that $\{\triangleleft, \triangleright\} = \{<, >\}$.

Definition 2.2 [Syntax] *The* formulas *of WMC are defined by the following grammar, for arbitrary $r \in \mathbb{Q}_{\geq 0}$, $a \in \Sigma$, $x \in \mathcal{V}$, $\trianglelefteq \in \{\leq, \geq\}$ and $X \in \mathcal{X}$.*

$$\mathcal{L}: \quad \phi := \ x \trianglelefteq r \mid \neg\phi \mid \phi \vee \phi \mid [a]\phi \mid x\ \underline{in}\ \phi \mid X\ .$$

We also consider the De Morgan duals of $x \trianglelefteq r$ and $[a]$, defined by

$$x \triangleleft r = \neg(x \trianglerighteq r) \quad \text{and} \quad \langle a\rangle\phi = \neg([a]\neg\phi) \text{ respectively.}$$

Given $\phi, \psi_1, ..., \psi_n \in \mathcal{L}$ and $X_1, ..., X_n \in \mathcal{X}$, let $\phi\{\psi_1/X_1, ..., \psi_n/X_n\}$ be the formula obtained by substituting each occurrence of the variable X_i in ϕ with ψ_i for each $i = 1..n$. If $\overline{\psi} = (\psi_1, ..., \psi_n)$ and $\overline{X} = (X_1, ..., X_n)$, let $\phi\{\overline{\psi}/\overline{X}\}$ denote $\phi\{\psi_1/X_1, ..., \psi_n/X_n\}$. Following [12, 13], we allow sets of the maximal or minimal *blocks* of mutually recursive equations in WMC.

Definition 2.3 [Equation Blocks] *An equation block B over the set $\mathcal{X}_B = \{X_1, \ldots, X_n\}$ of pairwise distinct variables has one of two forms –* $\min\{E\}$ *or* $\max\{E\}$*, where E is a system of (mutually recursive) equations such that for any $i, j \in \{1, ..., m\}$, ϕ_i is monotonic in X_j.*

$$E: \quad \langle\ X_1 = \phi_1,\ \ldots,\ X_n = \phi_n\ \rangle$$

If $B = \max\{E\}$ or $B = \min\{E\}$, the elements of $\mathcal{X}_B$ are called max-variables or min-variables respectively. Given the system E of equations in the previous definition, its *dual* is

$$\tilde{E}: \quad \langle\ X_1 = \neg\phi_1\{\neg X_1/X_1, \ldots, \neg X_n/X_n\},\ \ldots,\ X_n = \neg\phi_n\{\neg X_1/X_1, \ldots, \neg X_n/X_n\}\ \rangle$$

If $B = \max\{E\}$ or $B = \min\{E\}$, then its *dual* is $\tilde{B} = \min\{\tilde{E}\}$ or $\tilde{B} = \max\{\tilde{E}\}$ respectively.

Given a block B, a formula $\phi \in \mathcal{L}$ *depends on* B if it involves variables in $\mathcal{X}_B$. Given two blocks B and B' such that $\mathcal{X}_B \cap \mathcal{X}_{B'} = \emptyset$, we say that B is *dependent* on B' if the right hand side formulas of the equations of B depend on B'.

Definition 2.4 [Alternation-Free Block Sequence] *A sequence $\mathcal{B} = B_1, \ldots, B_m$ of $m \geq 1$ pairwise-distinct equation blocks is an* alternation-free block sequence *given that*

(i) $\mathcal{X}_{B_i} \cap \mathcal{X}_{B_j} = \emptyset$ *for $i \neq j$;* (ii) *if $i < j$, then B_i is not dependent on B_j.*

A formula $\phi \in \mathcal{L}$ is dependent on $\mathcal{B}$ if it is dependent of each block in the sequence.

Example 2.5 Anticipating the semantics, the sequence of blocks in WMC can be used to encode, for instance, the formula $A(\phi_1 U_{[r,r']}\phi_2)$ of WCTL: let $\phi = X$ be dependent on the alternation-free sequence $\mathcal{B} = B_1, B_2$ defined as follows

$$B_1 = \min\{Y = (\phi_2 \wedge r \leq x \leq r') \vee (\phi_1 \wedge \bigwedge_{a\in\Sigma}[a]Y)\}$$
$$B_2 = \max\{X = \bigwedge_{a\in\Sigma}[a]X \wedge (\phi_1 \to x\ \underline{in}\ Y\}$$

B_1 is a minimal equation block and B_2 is a maximal one. B_2 is dependent on B_1. ∎

3 Weighted Semantics for WMC

To provide a semantics for WMC in terms of WTSs, we define the notions of resource valuation, extended states and environments.

A *resource valuation* is a function $l : \mathcal{V} \to \mathbb{R}_{\geq 0}$ that assigns (non-negative) real numbers to the resource-variables in $\mathcal{V}$. We denote by L the class of resource valuations. For $l \in L$, $x \in \mathcal{V}$ and $s \in \mathbb{R}_{\geq 0}$, let $l[x \mapsto s] \in L$ be defined by $l[x \mapsto s](x) = s$ and $l[x \mapsto s](y) = l(y)$ for $y \neq x$; let $l + s \in L$ be defined by $(l + s)(x) = l(x) + s$.

Given a WTS $\mathcal{W} = (M, \Sigma, \theta)$, $m \in M$ and $l \in L$, the pair (m, l) is called an *extended state* of $\mathcal{W}$. Transitions between extended states are defined by:

$$(m, l) \to_a (m', l') \text{ iff } m \xrightarrow{u}_a m' \text{ and } l' = (l + u).$$

Given a WTS $\mathcal{W} = (M, \Sigma, \theta)$, an *environment* is a function $\rho : \mathcal{X} \to 2^{M\times L}$ that interpret the recursive-variables as sets of extended states. We use 0 as the empty environment that associates $\emptyset$ to all recursive-variables. Given an environment ρ and $S \subseteq M \times L$, let $\rho[X \mapsto S]$ be the environment that interprets X as S and all the other recursive-variables as ρ does. Similarly, for a pairwise-disjoint tuple $\overline{X} = (X_1, ..., X_n) \in \mathcal{X}^n$ and $\overline{S} = (S_1, ..., S_n) \subseteq (M \times L)^n$, let $\rho[\overline{X} \mapsto \overline{S}]$ be the environment that interprets X_i as S_i for all $i = 1..n$ and all the other variables as ρ does.

Given a WTS $\mathcal{W} = (M, \Sigma, \theta)$ and an environment ρ, the WTS-semantics for $\mathcal{L}$ is defined, on top of the classic semantics for Boolean logic, as follows.

$\mathcal{W}, (m, l), \rho \models x \trianglelefteq r$ iff $l(x) \trianglelefteq r$;
$\mathcal{W}, (m, l), \rho \models [a]\phi$ iff for any $(m', l') \in M \times L$ s.t. $(m, l) \to_a (m', l')$, $\mathcal{W}, (m', l'), \rho \models \phi$;
$\mathcal{W}, (m, l), \rho \models x\ \underline{in}\ \phi$ iff $\mathcal{W}, (m, l[x \mapsto 0]), \rho \models \phi$;
$\mathcal{W}, (m, l), \rho \models X$ iff $(m, l) \in \rho(X)$.

Let $[\![\phi]\!]_\rho = \{(m, l) \in M \times L \mid \mathcal{W}, (m, l), \rho \models \phi\}$.

Following [12, 13, 24], we extend now the semantics to include the restrictions imposed by an alternation-free sequence of blocks and obtain the so-called block-semantics.

Given a set of equations E with variables $\overline{X} = (X_1, ..., X_n)$, an environment ρ and $\overline{\Upsilon} = (\Upsilon_1, ..., \Upsilon_n) \subseteq (M \times L)^n$, let the function $f_E^\rho : (2^{M\times L})^n \longrightarrow (2^{M\times L})^n$ be defined as follows:

$$f_E^\rho(\overline{\Upsilon}) = \langle [\![\phi_1]\!]_{\rho[\overline{X}\mapsto\overline{\Upsilon}]}, \ldots, [\![\phi_n]\!]_{\rho[\overline{X}\mapsto\overline{\Upsilon}]} \rangle.$$

Observe that $(2^{M\times L})^n$ forms a complete lattice with the ordering, join and meet operations defined as the point-wise extensions of the set-theoretic inclusion, union and intersection, respectively. Moreover, for any E and ρ, f_E^ρ is monotonic with respect to the order of the lattice and therefore, it has a greatest fixed point denoted by $\nu\overline{X}.f_E^\rho$ and a least fixed point denoted by $\mu\overline{X}.f_E^\rho$ [12]. These can be characterized as follows:

$$\nu\overline{X}.f_E^\rho = \bigcup\{\overline{\Upsilon} \mid \overline{\Upsilon} \subseteq f_E^\rho(\overline{\Upsilon})\}, \quad \mu\overline{X}.f_E^\rho = \bigcap\{\overline{\Upsilon} \mid f_E^\rho(\overline{\Upsilon}) \subseteq \overline{\Upsilon}\}.$$

The blocks $\max\{E\}$ and $\min\{E\}$ define environments that satisfy all the equations in E; $\max\{E\}$ is the greatest fixed point and $\min\{E\}$ is the least fixed point. The environment defined by the block B is denoted by $[\![B]\!]_\rho$.

Given an alternation-free block sequence $\mathcal{B} = B_1, \ldots, B_m$ and an environment ρ_0, let $\rho_1, \ldots, \rho_m$ be defined by $\rho_i = [\![B_i]\!]_{\rho_{i-1}}$ for $i = 1, \ldots, m$. The semantics of $\mathcal{B}$ is then given by

$$[\![\mathcal{B}]\!]_{\rho_0} = \rho_m.$$

Definition 3.1 [Block-Semantics] Given an alternation-free sequence $\mathcal{B}$ of blocks, the $\mathcal{B}$-semantics of a formula $\phi \in \mathcal{L}$ that depends on $\mathcal{B}$ is given for a WTS $\mathcal{W} = (M, \Sigma, \theta)$ with $m \in M$, a resource valuation $l \in L$ and an environment ρ, as follows

$$\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi \ \text{ iff } \ \mathcal{W}, (m, l), [\![\mathcal{B}]\!]_\rho \models \phi.$$

We say that a formula ϕ is *$\mathcal{B}$-satisfiable* if there exists at least one WTS that satisfies it for the alternation-free block sequence $\mathcal{B}$ in one of its states under some resource valuation and some environment; ϕ is a *$\mathcal{B}$-validity*, written $\models_{\mathcal{B}} \phi$, if it is satisfied in all states of any WTS under any resource valuation and any environment.

4 Symbolic Semantics for WMC

Consider an weighted system that can perform three actions a, b and c, and suppose that we are interested in the following specifications of the system:

1. it can do an a-action followed by an infinite sequence of alternations of the actions b and c with non-zero cost;
2. after an a-transition, the overall behaviour costs less than one unit of resource.

These requirements can be encoded in WMC, by using three resource-variables x_a, x_b and x_c, as follows:

$\phi = \langle a \rangle (x_a \ \underline{in}\ X)$,

$B = \max\{X = x_a < 1 \wedge \langle b \rangle (x_b \ \underline{in}\ (Y \wedge x_c > 0)),\ Y = x_a < 1 \wedge \langle c \rangle (x_c \ \underline{in}\ (X \wedge x_b > 0))\}$

We can see that there exists a WTS satisfying ϕ under the assumptions of B. But it cannot be satisfied by a finite WTS, since it must have at least one infinite trace of non-zero cost transitions with a bounded overall cost. However, all the WTSs that satisfy the requirements encoded by ϕ have something in common: the way the resource-variables behave under certain resource valuations and as a result of resetting.

This observation motivates the development of *symbolic weighted transition systems* (SWSs), which are similar to the ones used with timed automata in [2, 4, 23]. These are abstractions of WTSs: a symbolic model is a labelled transition system representing an infinite set of WTSs by involving the concept of *regions* that abstracts the quantitative information. One can provide an SWS-semantics for WMC (symbolic semantics) and can prove that there exists a relation between WTSs and SWSs such that any systems in this relation satisfy the same WMC properties. Moreover, the relation is complete, in the sense that to each WTS corresponds an SWS and reverse. An important consequence of this fact is that the validities for WTS-semantics coincide with the validities for SWS-semantics.

For any $s \in \mathbb{R}_{\geq 0}$, let $\lfloor s \rfloor = \max\{z \in \mathbb{N} \mid z \leq s\}$, $\{s\} = s - \lfloor s \rfloor$ and $\lceil s \rceil = \min\{z \in \mathbb{N} \mid z \geq s\}$.

Definition 4.1 *Given $N \in \mathbb{N}$, $l, l' \in L$ are equivalent w.r.t. N, denoted by $l \overset{N}{=} l'$ iff:*
1. $\forall x \in \mathcal{V}$, $l(x) > N$ iff $l'(x) > N$;
2. $\forall x \in \mathcal{V}$ s.t. $0 \leq l(x) \leq N$, $\lfloor l(x) \rfloor = \lfloor l'(x) \rfloor$ and $\{l(x)\} = 0 \Leftrightarrow \{l'(x)\} = 0$;
3. $\forall x, y \in \mathcal{V}$ s.t. $0 \leq l(x), l(y) \leq N$, $\{(l(x)\} \leq \{l(y)\} \Leftrightarrow \{(l'(x)\} \leq \{l'(y)\}$.

The equivalence classes under $\overset{N}{=}$ are called *N-regions*. Let $[l]$ be the region containing l and $\mathcal{R}_N^{\mathcal{V}}$ be the set of all N-regions for the set $\mathcal{V}$ of resource-variables and the constant N. For a given $N \in \mathbb{N}$, $\mathcal{R}_N^{\mathcal{V}}$ is finite whenever $\mathcal{V}$ is finite.

For $\delta \in \mathcal{R}_N^{\mathcal{V}}$, a *successor region* is the region δ' s.t. for any $l \in \delta$, there exists $d \in \mathbb{R}_{\geq 0}$ s.t. $l + d \in \delta'$, denoted by $\delta \rightsquigarrow \delta'$. For $\delta \in \mathcal{R}_N^{\mathcal{V}}$, $x \in \mathcal{V}$ and $n \in \mathbb{N}$, $\delta[x \mapsto n]$ denotes the region consisting of all the resource valuations l for which there exists $l' \in \delta$ s.t. $l = l'[x \mapsto n]$.

Example 4.2 In Figure 1 are represented some regions for $N = 1$ and $\mathcal{V} = \{x_a, x_b, x_c\}$.

$\delta_0 = [x_a = x_b = x_c = 0]$ $\quad$ $\delta_1 = [0 < x_a = x_b = x_c < 1]$
$\delta_2 = [x_b = 0, 0 < x_a = x_c < 1]$ $\quad$ $\delta_3 = [0 < x_b < x_a = x_c < 1]$
$\delta_4 = [x_c = 0, 0 < x_b < x_a < 1]$ $\quad$ $\delta_5 = [0 < x_c < x_b < x_a < 1]$
$\delta_6 = [x_b = 0, 0 < x_c < x_a < 1]$ $\quad$ $\delta_7 = [0 < x_b < x_c < x_a < 1]$

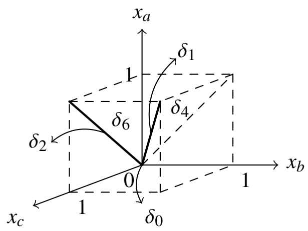

Fig. 1. Regions

δ_1 is a successor of δ_0, $\delta_2 = \delta_1[x_b \mapsto 0]$ and δ_3 is a successor of δ_2. Similarly, δ_5 is a successor of δ_4 and δ_7 is a successor of δ_6. Moreover, $\delta_2 = \delta_3[x_b \mapsto 0]$, $\delta_4 = \delta_3[x_c \mapsto 0] = \delta_5[x_c \mapsto 0] = \delta_7[x_c \mapsto 0]$ and $\delta_6 = \delta_5[x_b \mapsto 0] = \delta_7[x_b \mapsto 0]$. ∎

In what follows, we consider an extension of the concept of region to also include the case when $N = {}^p/_q$ with $p, q \in \mathbb{N}$. We firstly construct the regions for p and then divide each of the resource-valuation in it by q – the resulting set will be a region for $N = {}^p/_q$. For instance, if we take $N = {}^1/_2$ in Example 4.2, then $\delta_1 = [0 < x_a = x_b = x_c < {}^1/_2]$ and $\delta_2 = [x_b = 0, 0 < x_a = x_c < {}^1/_2]$ are regions in $\mathcal{R}_{1/2}^{\mathcal{V}}$.

Definition 4.3 [Symbolic Model] *Given $\mathcal{R}_N^{\mathcal{V}}$ and a non-empty set S, a* symbolic weighted transition system (SWS) *is a tuple $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ where $\Pi^s \subseteq S \times \mathcal{R}_N^{\mathcal{V}}$ is a non-empty set of* symbolic states, *$\Sigma^s = \{\varepsilon_x \mid x \in \mathcal{V}\} \cup \Sigma$ a non-empty set of actions, and $\theta^s : \Pi^s \times \Sigma^s \to 2^{\Pi^s}$ is a labeled transition function such that:*
1) *if $(s, \delta) \to_a (s', \delta')$ for $a \in \Sigma_a$, then $\delta \rightsquigarrow \delta'$;* 2) *if $(s, \delta) \to_{\varepsilon_x} (s, \delta')$ then $\delta' = \delta[x \mapsto 0]$.*

Note that if $(s, \delta) \to_{\varepsilon_x} (s, \delta)$, then for any $l \in \delta$, $l(x) = 0$.

For a given SWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$, a *symbolic environment* is a function $\rho^s : \mathcal{X} \to 2^{\Pi^s}$ which interprets the recursive-variables as sets of symbolic states.

The *symbolic satisfiability relation* $\models^s$ is defined for the non-Boolean operators as follows.
$\mathcal{W}^s, \pi, \rho^s \models^s x \trianglelefteq r$ iff for any $l \in \delta$, $l(x) \trianglelefteq r$;
$\mathcal{W}^s, \pi, \rho^s \models^s [a]\phi$ iff for arbitrary $\pi' \in \Pi^s$ such that $\pi \to_a \pi'$, we have $\mathcal{W}^s, \pi', \rho^s \models^s \phi$;

$\mathcal{W}^s, \pi, \rho^s \models^s x \ \underline{in} \ \phi$ iff there exists $\pi' \in \Pi^s$ such that $\pi \rightarrow_\varepsilon \pi'$ and $\mathcal{W}^s, \pi', \rho^s \models^s \phi$;
$\mathcal{W}^s, \pi, \rho^s \models^s X$ iff $\pi \in \rho^s(X)$.

Similarly as in Section 3, for a given alternation-free sequence of blocks $\mathcal{B}$ we can define the *symbolic $\mathcal{B}$-semantics* based on the $\mathcal{B}$-satisfiability relation $\models^s_{\mathcal{B}}$, as follows:

$$\mathcal{W}^s, \pi, \rho \models^s_{\mathcal{B}} \phi \ \text{ iff } \ \mathcal{W}^s, \pi, [\![\mathcal{B}]\!]_\rho \models^s \phi.$$

5 The Equivalence of the Two Semantics

In this section we prove that the two semantics introduced for WMC are equivalent, in the sense that the set of the WTS-validities coincides with the set of the SWS-validities. This result has important consequences: (i) if the satisfiability problem is decidable for one semantics, then it is also decidable for the other; and (ii) an axiomatization that is sound and complete for one semantics is sound and complete also for the other semantics. To prove the equivalence, we show that for any formula $\phi \in \mathcal{L}$ dependent on $\mathcal{B}$, if ϕ has a WTS-model, then we can also construct an SWS-model for it; and reversely, if it has an SWS-model, then we can construct a WTS-model for it.

Construction A: *Given a WTS $\mathcal{W} = (M, \Sigma, \theta)$ and $\mathcal{R}^{\mathcal{V}}_N$, we construct the SWS $\mathcal{W}^S = (\Pi^s, \Sigma^s, \theta^s)$, where $\Pi^s = M \times \mathcal{R}^{\mathcal{V}}_N$, $\Sigma^s = \{\varepsilon_x \mid x \in \mathcal{V}\} \cup \Sigma$ and θ^s is defined as follows:*
1. *$(m, [l]) \rightarrow_a (m', [l'])$ iff $(m, l) \rightarrow_a (m', l')$;*
2. *$(m, [l]) \rightarrow_{\varepsilon_x} (m, [l'])$ iff $[l'] = [l][x \mapsto 0]$.*

We call $\mathcal{W}^s$ the symbolic model *of $\mathcal{W}$ w.r.t. $\mathcal{R}^{\mathcal{V}}_N$, denoted by $\mathbb{S}(\mathcal{W}, \mathcal{R}^{\mathcal{V}}_N)$.*

Construction B: *Given an SWS $\mathcal{W}^S = (\Pi^s, \Sigma^s, \theta^s)$ on $\mathcal{R}^{\mathcal{V}}_N$ with $\Sigma^s = \{\varepsilon_x \mid x \in \mathcal{V}\} \cup \Sigma$, let $\mathcal{W} = (M, \Sigma, \theta)$ be a WTS s.t.*
- *the states are sets of type $\{(s, \delta_1, l_1), \dots, (s, \delta_k, l_k)\}$ where*
  (1) *$(s, \delta_i) \in \Pi^s$ and $l_i \in \delta_i$;* (2) *for any $i \in \{1, ..., k\}$ there exist $j \in \{1, ..., k\}$ and $x \subseteq \mathcal{V}$ s.t. either $\delta_j = \delta_i[x \mapsto 0]$ and $l_j = l_i[x \mapsto 0]$, or $\delta_i = \delta_j[x \mapsto 0]$ and $l_i = l_j[x \mapsto 0]$.*
- *θ is defined for any $m_1, m_2 \in M$, $m_1 \xrightarrow{u}_a m_2$ iff there exist $(s_1, \delta_1, l_1) \in m_1$ and $(s_2, \delta_2, l_2) \in m_2$ s.t. $(s_1, \delta_1) \rightarrow_a (s_2, \delta_2)$ and $l_2 = (l_1 + u)$.*

We call $\mathcal{W}$ the concrete model *of $\mathcal{W}^s$ on $\mathcal{R}^{\mathcal{V}}_N$, denoted by $\mathbb{C}(\mathcal{W}^s, \mathcal{R}^{\mathcal{V}}_N)$.*

We prove that the constructions preserve the $\mathcal{B}$-satisfiability of WMC properties, i.e., a formula ϕ is $\mathcal{B}$-satisfiable in the WTS-semantics iff it is $\mathcal{B}$-satisfiable in the SWS-semantics.

Consider an arbitrary formula $\phi \in \mathcal{L}$ dependent on $\mathcal{B}$.
- Let $\mathcal{V}[\phi, \mathcal{B}]$ be the set of the resource-variables in ϕ and $\mathcal{B}$. For any $x \in \mathcal{V}[\phi, \mathcal{B}]$, let $Q[\phi, \mathcal{B}] \subseteq \mathbb{Q}_{\geq 0}$ be the set of all $r \in \mathbb{Q}_{\geq 0}$ that occur in a construct of type $x \trianglelefteq r$ in ϕ or $\mathcal{B}$.
- Let g be the least common denominator of the elements of $Q[\phi, \mathcal{B}]$.
- Let $\mathcal{R}[\phi, \mathcal{B}]$ denote the set $\mathcal{R}^{\mathcal{V}[\phi, \mathcal{B}]}_{p/g}$ of p/q-regions, where $p/g = \max Q[\phi, \mathcal{B}]$.

Theorem 5.1 *Let ϕ depending of the alternating-free sequence of blocks $\mathcal{B} = B_1, \dots, B_m$.*
1. *If $\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi$, then $\mathcal{W}^s, (m, [l]), \rho^s \models^s_{\mathcal{B}} \phi$, where $\mathcal{W}^s = \mathbb{S}(\mathcal{W}, \mathcal{R}[\phi, \mathcal{B}])$ and $\rho^s(X) = \{(m, [l]) \mid (m, l) \in \rho(X)\}$ for any $X \in \mathcal{X}$.*
2. *If $\mathcal{W}^s, (s, \delta), \rho^s \models^s_{\mathcal{B}} \phi$, then $\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi$, where $\mathcal{W} = \mathbb{C}(\mathcal{W}^s, \mathcal{R}[\phi, \mathcal{B}])$, $m \in M$, $(s, \delta, l) \in m$ and for any $X \in \mathcal{X}$, $\rho(X) = \{(m, l) \mid (s, \delta) \in \rho^s(X), (s, \delta, l) \in m\}$.*

Consequently, the $\mathcal{B}$-validities for WTC-semantics coincide with that of SWS-semantics.

6 Decidability and finite symbolic model property

In this section, we prove that WMC enjoys the finite model property against the SWS-semantics, by involving the region construction technique and adapting the classical *tableau* method. A consequence of this result is that the $\mathcal{B}$-satisfiability problem for the SWS-semantics is decidable. In the light of Theorem 5.1, this means that $\mathcal{B}$-satisfiability is decidable also for the WTS-semantics even if, as we have emphasized in Section 4, WMC does not enjoy the finite model property for the WTS-semantics.

Given $\phi \in \mathcal{L}$ that depends on an alternation-free sequence $\mathcal{B}$, let $\Sigma[\phi, \mathcal{B}]$ be the set of all actions $a \in \Sigma$ that appears in some transition modality of type $\langle a \rangle$ or $[a]$ in ϕ or $\mathcal{B}$; let $Q[\phi]$ and $\mathcal{R}[\phi]$ be defined as in Section 5. Observe that $\Sigma[\phi]$, $Q[\phi]$ and $\mathcal{R}[\phi]$ are finite or empty.

We fix $\phi_0 \in \mathcal{L}$ dependent on $\mathcal{B}_0$. Let $\mathcal{L}[\phi_0, \mathcal{B}_0]$ be the set of the sub-formulas of ϕ_0 or $\mathcal{B}_0$. Let $\Omega[\phi_0, \mathcal{B}_0] \subseteq 2^{\mathcal{L}[\phi_0,\mathcal{B}_0]} \times \mathcal{R}[\phi_0, \mathcal{B}_0]$. Since $\mathcal{L}[\phi_0, \mathcal{B}_0]$ and $\mathcal{R}[\phi_0, \mathcal{B}_0]$ are both finite, $\Omega[\phi_0, \mathcal{B}_0]$ is finite. We construct a tableau for ϕ_0, which is similar to the standard construction with extra focus on the quantities.

The nodes of a tableau are pairs $(\Delta, \delta) \in \Omega[\phi_0, \mathcal{B}_0]$ and the tableau rules are listed in Table 1, where $\{\phi, \Delta\}$ denotes $\{\phi\} \cup \Delta$.

$$(\wedge)\ \frac{(\{\phi_1, \phi_2, \Delta\}, \delta)}{(\{\phi_1 \wedge \phi_2, \Delta\}, \delta)} \qquad (\vee)\ \frac{(\{\phi_1, \Delta\}, \delta) \quad (\{\phi_2, \Delta\}, \delta)}{(\{\phi_1 \vee \phi_2, \Delta\}, \delta)}$$

$$(\text{Res})\ \frac{(\{\phi, \Delta\}, \delta')}{(\{x\ \underline{in}\ \phi, \Delta\}, \delta)} \qquad (\text{Reg})\ \frac{(\{\phi_X, \Delta\}, \delta)}{(\{X, \Delta\}, \delta)}\quad X = \phi_X \in \mathcal{B}$$

$$(\text{Mod})\ \frac{(\{\psi\} \cup \{\psi' \mid [a]\psi' \in \Delta\}, \delta') \quad \text{for any } \langle a \rangle \psi \in \Delta}{(\Delta, \delta)}$$

Table 1
Tableau System $\mathcal{T}^{\phi}$

Because of the quantitative requirements must also be satisfied, not any pair (Δ, δ) is a node in the tableau. A tableau $\mathcal{T}(\phi, \delta)$ derived from the previous rules must be *region consistent*, meaning that any node $(\Delta, \delta') \in \mathcal{T}(\phi, \delta)$ must satisfy the following conditions:

(i) for any $x \trianglelefteq r \in \Delta$ and $l \in \delta$, $l(x) \trianglelefteq r$;

(ii) if $(\{x\ \underline{in}\ \phi, \Delta\}, \delta)$ is the conclusion and $(\{\phi, \Delta\}, \delta')$ is the assumption of (Res), then $\delta' = \delta[x \mapsto 0]$;

(iii) if (Δ, δ) is the conclusion of (Mod), then $\delta \rightsquigarrow \delta'$ for any assumption (Δ', δ').

If (Mod) is applied for an action a at the node t, the node obtained is called an $\langle a \rangle$-*son* of t. The tableaux may be infinite. However, because $\Omega[\phi_0]$ is finite, the pairs from $\Omega[\phi_0]$ that appear in $\mathcal{T}(\phi, \delta)$ are finitely many.

As in the classic method for mu-calculus [20, 31, 32], we use max-*trace*, min-*trace* to capture the idea of a history of the regeneration of a formula and *markings*, *consistent markings* to characterize $\mathcal{B}$-satisfiability of a formula in a state of an SWS (these classic definitions can be found in the appendix).

Lemma 6.1 *ϕ_0 is satisfied at state $\pi_0 = (s_0, \delta_0)$ in an SWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ if and only if there is a consistent marking of $\mathcal{T}(\phi_0, \delta_0)$ respect to $\mathcal{W}^s$ and π_0.*

The proof of Lemma 6.1 relies on notion of *signature*, similar to that considered by Streett and Emerson [31]. These notions come from the characterization of fixed point formulas by means of transfinite chains of approximations, which have been extended to the setting with fixed points defined with blocks in [12, 13]. Involving these, the previous lemma is proven similarly to the case of classic μ-calculus [20, 31, 32]. The correctness of the cases with weight is guaranteed by the region consistency.

This lemma allows us to prove the finite model property for SWS-semantics, by following the classic proof strategy of [20]; the only difference consists in managing the reset actions.

Theorem 6.2 (Finite Symbolic Model Property) *Let $\phi_0 \in \mathcal{L}$ be a formula that depends of $\mathcal{B}_0$. If ϕ_0 is $\mathcal{B}_0$-satisfiable, then there exists a finite SWS $\mathcal{W}^s_f = (\Pi^s_f, \Sigma^s_f, \theta^s_f)$ with $\pi_f \in \Pi^s_f$ and a symbolic environment ρ^s_f such that $\mathcal{W}^s_f, \pi_f, \rho^s_f \models_{\mathcal{B}_0} \phi_0$.*

According to Lemma 6.1 and Theorem 6.2, we can have an algorithm to decide the satisfiability of a given WMC formula. The following example shows how this works.

Example 6.3 Suppose that we want to verify the $\mathcal{B}$-satisfiability of the property discussed at the beginning of Section 4.

$\phi = \langle a \rangle (x_a \ \underline{in}\ X)$,

$B = \max\{X = x_a < 1 \wedge \langle b \rangle (x_b \ \underline{in}\ (Y \wedge x_c > 0)),\ Y = x_a < 1 \wedge \langle c \rangle (x_c \ \underline{in}\ (X \wedge x_b > 0))\}$.

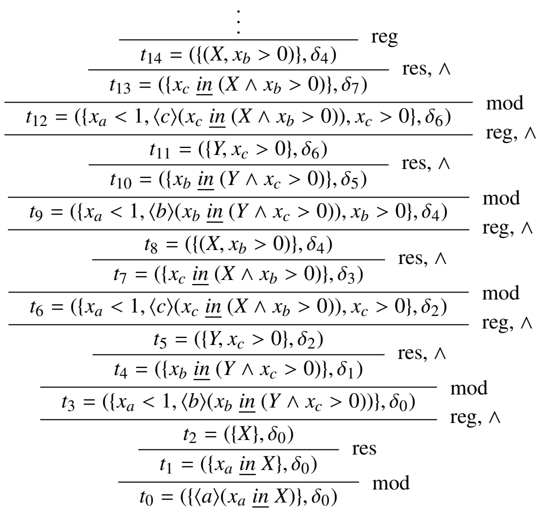

Fig. 2. Tableau $\mathcal{T}(\phi, B)$

$\delta_0 = [x_a = x_b = x_c = 0]$
$\delta_2 = [x_b = 0, 0 < x_a = x_c < 1]$
$\delta_4 = [x_c = 0, 0 < x_b < x_a < 1]$
$\delta_6 = [x_b = 0, 0 < x_c < x_a < 1]$
$\delta_1 = [0 < x_a = x_b = x_c < 1]$
$\delta_3 = [0 < x_b < x_a = x_c < 1]$
$\delta_5 = [0 < x_c < x_b < x_a < 1]$
$\delta_7 = [0 < x_b < x_c < x_a < 1]$

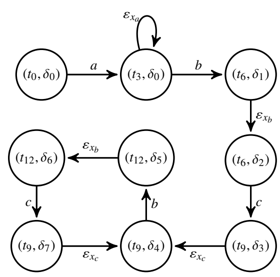

Fig. 3. SWS for ϕ dependent on B

In Figure 2 shows $\mathcal{T}(\phi, \delta_0)$. There is only one infinite trace – max-trace. We construct $\mathcal{W}^s$: $\Sigma^s = \{a, b, c, \varepsilon_{x_a}, \varepsilon_{x_b}, \varepsilon_{x_c}\}$, $\Pi^s = \{(t_0, \delta_0), (t_3, \delta_0), (t_6, \delta_1), (t_6, \delta_2), (t_9, \delta_3), (t_9, \delta_4), (t_{12}, \delta_5), (t_{12}, \delta_6), (t_9, \delta_7)\}$ and θ^s is defined as shown in Figure 3. From the symbolic model in Figure 3, one can generate a WTS, which in this case is infinite; ϕ is satisfied in some state of it. In Figure 4 it is shown part of this infinite model.

$l_0 = (0,0,0)$
$l_1 = (0.3,0.3,0.3)$ $\quad l_2 = (0.3,0,0.3)$
$l_3 = (\frac{\pi}{10},\frac{\pi}{10},\frac{\pi}{10})$ $\quad l_4 = (\frac{\pi}{10},0,\frac{\pi}{10})$
$l_5 = (0.1,0.1,0.1)$ $\quad l_6 = (0.1,0,0.1)$
$l_7 = (0.5,0.2,0.5)$ $\quad l_8 = (0.5,0.2,0)$
$l_9 = (0.3+\frac{\pi}{10},\frac{\pi}{10},0.3+\frac{\pi}{10})$ $\quad l_{10} = (0.3+\frac{\pi}{10},\frac{\pi}{10},0)$
$l_{11} = (0.3,0.2,0.3)$ $\quad l_{12} = (0.3,0.2,0)$
$l_{13} = (0.6,0.3,0.1)$ $\quad l_{14} = (0.6,0,0.1)$
$l_{15} = (0.5,0.4,0.2)$ $\quad l_{16} = (0.5,0,0.2)$
$l_{17} = (0.75,0.15,0.25)$ $\quad l_{18} = (0.75,0.15,0)$
$l_{19} = (0.6,0.1,0.3)$ $\quad l_{20} = (0.6,0.1,0)$
$\ldots$ $\quad \ldots$
$m_0 = \{(t_0,\delta_0,l_0)\}$ $\quad m_1 = \{(t_3,\delta_0,l_0)\}$
$m_2 = \{(t_6,\delta_1,l_1),(t_6,\delta_2,l_2)\}$
$m_3 = \{(t_6,\delta_1,l_3),(t_6,\delta_2,l_4)\}$
$m_4 = \{(t_6,\delta_1,l_5),(t_6,\delta_2,l_6)\}$
$m_5 = \{(t_9,\delta_3,l_7),(t_9,\delta_4,l_8)\}$
$m_6 = \{(t_9,\delta_3,l_9),(t_9,\delta_4,l_{10})\}$
$m_7 = \{(t_9,\delta_3,l_{11}),(t_9,\delta_4,l_{12})\}$
$m_8 = \{(t_{12},\delta_5,l_{13}),(t_{12},\delta_6,l_{14})\}$
$m_9 = \{(t_{12},\delta_5,l_{15}),(t_{12},\delta_6,l_{16})\}$
$m_{10} = \{(t_9,\delta_7,l_{17}),(t_9,\delta_4,l_{18})\}$
$m_{11} = \{(t_9,\delta_7,l_{19}),(t_9,\delta_4,l_{20})\}$
$\ldots$

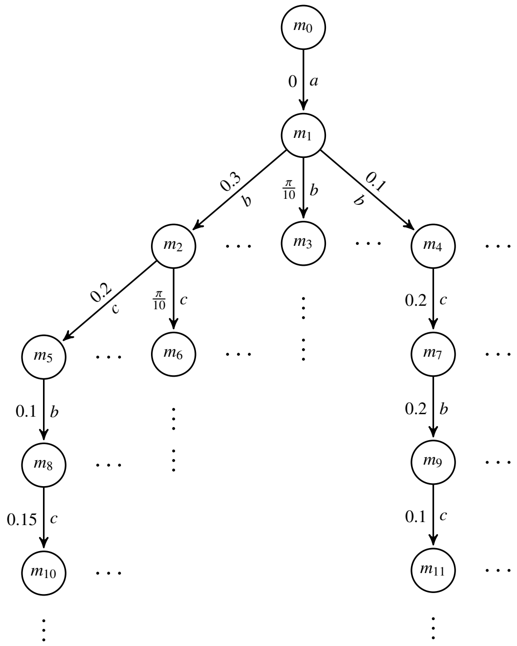

Fig. 4. Generalizing WTS from the symbolic model

It is not difficult to verify that it is a model for ϕ. ∎

Theorem 6.4 (Decidability of $\mathcal{B}$-Satisfiability) *For any alternation-free block sequence $\mathcal{B}$, the $\mathcal{B}$-satisfiability problem for WMC is decidable for both WTS- and SWS-semantics.*

7 Axiomatization

In this section, we focus on developing a sound and complete axiomatization for the validities of WMC with respect to the two semantics. Recall that the two sets of validities coincide. In the light of Theorem 5.1, it is sufficient to find such an axiomatization for the SWS-semantics and it is then sound and complete also for the WTS-semantics.

7.1 Sound axiomatization

In order to state the axioms for WMC we need to establish some notations.
• The *modal prefixes* are words $w \in Mod^*$ over the alphabet of modal operators of $\mathcal{L}$, $Mod = \{[a] \mid a \in \Sigma\} \cup \{x\ \underline{in} \mid x \in \mathcal{V}\}$. E.g., $[a], x\ \underline{in}\ [a][a], [a]x\ \underline{in}\ , \varepsilon \in Mod^*$.
• A *context* C is a word formed by a modal prefix $w \in Mod^*$ concatenated with the metavariable $\mathbb{X}$; e.g., $[a]\mathbb{X}, [a]x\ \underline{in}\ [b]\mathbb{X}, x\ \underline{in}\ [a][a]\mathbb{X}, [a]x\ \underline{in}\ \mathbb{X}$ are contexts. To emphasize the presence of the metavariable we will use the functional representation of type $C[\mathbb{X}]$ for contexts; this will allow us to instantiate the metavariable with elements from $\mathcal{L}$. E.g., if $C[\mathbb{X}] = [a]x\ \underline{in}\ [b]\mathbb{X}$ is a context, then $C[(x \geq r)] = [a]x\ \underline{in}\ [b](x \geq r) \in \mathcal{L}$. Also $\varepsilon[\mathbb{X}]$ is a context - the empty one - and for $\phi \in \mathcal{L}$, $\varepsilon[\phi] = \phi$.

The axiomatization of WMC is given in two phases. Firstly, we provide axioms for deriving the validities that do not depend on sequences of blocks; and secondly, we extend the axiomatization to recursive constructs.

The axioms and rules presented in Table 2 together with the axioms and the rules of propositional logic axiomatize a classic deducibility relation (see [16]) for the non-recursive validities of WMC denoted by $\vdash$. The axioms and the rules are stated for arbitrary $\phi, \psi \in \mathcal{L}$, $r, s \in \mathbb{Q}_{\geq 0}$, $a \in \Sigma$, $x, y \in \mathcal{V}$ and arbitrary context $C[\mathbb{X}]$, where $\{\trianglelefteq, \trianglerighteq\} = \{\leq, \geq\}$.

(A1): $\vdash x \geq 0$	(A8): $\vdash x\ \underline{in}\ x\ \underline{in}\ \phi \to x\ \underline{in}\ \phi$
(A2): $\vdash (x \geq r) \vee (x \leq r)$	(A9): $\vdash x\ \underline{in}\ y\ \underline{in}\ \phi \to y\ \underline{in}\ x\ \underline{in}\ \phi$
(A3): $\vdash x \leq r \to \neg(x \geq s),\ r < s$	(A10): $\vdash \neg(x\ \underline{in}\ \phi) \leftrightarrow x\ \underline{in}\ \neg\phi$
(A4): $\vdash x \geq r \to [a](x \geq r)$	(A11): $\vdash x\ \underline{in}\ \phi \to (x = 0 \to \phi)$
(A5): $\vdash x \trianglelefteq r \wedge y \trianglelefteq s \to [a](x \trianglelefteq r + t \to y \trianglelefteq s + t)$	(R1): If $\vdash \phi$, then $\vdash \Box\phi$
(A6): $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	(R2): $\{C[x \trianglelefteq r] \mid r \triangleright s\} \vdash C[x \trianglelefteq s]$
(A7): $\vdash x\ \underline{in}\ \bot \to \bot$	(R3): $\{C[x \geq r] \mid r \in \mathbb{Q}_{\geq 0}\} \vdash C[\bot]$

Table 2
Axiomatic System of WMC basic formulas

The axioms (A1)-(A3) state simple arithmetic facts. (A4) states that an action-transition has a positive cost. (A5) guarantees that all the resource-variables measure the same resource. The axiom (A6) and the rule (R1) state that all the box-like operators of WMC are normal in the sense of modal logic [8]. The nature of the reset operation is depicted by (A7)-(A11).

The rules (R2) and (R3) are infinitary and encode the Archimedean properties of rational numbers. For instance, the formula $\{(\geq r) \mid r < s\} \vdash (\geq s)$ is an instance of (R2) stating that if the resources available in a state are at least r for each $r < s$, then they are at least s.
Similarly, the formula $\{(\geq r) \mid r \in \mathbb{Q}\} \vdash \bot$ is an instance of (R3) guaranteeing that the resources in a state cannot be infinite (bigger that any rational).
The rules (R2) and (R3) are closed under arbitrary contexts. Due to them, WMC is non-compact: infinite sets of formulas such as $\{(\geq r) \mid r < s\} \cup \{\neg(\geq s)\}$ and $\{(\geq r) \mid r \in \mathbb{Q}\}$ are inconsistent while any finite subset of them is consistent.

Theorem 7.1 (Soundness) *The axiomatic system of $\vdash$ is sound with respect to the WTS-semantics, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\vdash \phi \text{ implies } \models \phi.$$

Consequently, the axioms are also sound for SWS-semantics. Now we can proceed with the recursive constructs.

Given a maximal equation block $B = \max\{X_1 = \phi_1, \ldots, X_n = \phi_n\}$ and an arbitrary clasical deducibility relation $\vdash^*$, we define the deducibility relation $\vdash^*_B$ as the extension of $\vdash^*$ given by the axioms and rules in Table 3, which are the equation-version of the fixed points axioms of Mu-calculus [20, 28, 30]. These are stated for arbitrary $\phi \in \mathcal{L}$ and $\overline{\Psi} = (\psi_1, ..., \psi_n) \in \mathcal{L}^n$, where $\overline{X} = (X_1, ..., X_n)$. Similarly, we define a classical deducibility relation $\vdash^*_B$ for a minimal equation block $B = \min\{X_1 = \phi_1, \ldots, X_n = \phi_n\}$ based on $\vdash^*$ by using the axioms and rules in Table 4.

(max-R1): If $\vdash^* \phi$, then $\vdash^*_B \phi$	(min-R1): If $\vdash^* \phi$, then $\vdash^*_B \phi$
(max-A1): $\vdash^*_B \bigwedge_{i=1,\ldots,n}(X_i \to \phi_i)$	(min-A1): $\vdash^*_B \bigwedge_{i=1,\ldots,n}(\phi_i \to X_i)$
(max-R2): If $\vdash^*_B \bigwedge_{i=1,\ldots,n}(\psi_i \to \phi_i\{\overline{\Psi}/\overline{X}\})$, then $\vdash^*_B \bigwedge_{i=1,\ldots,n}(\psi_i \to X_i)$	(min-R2): If $\vdash^*_B \bigwedge_{i=1,\ldots,n}(\phi_i\{\overline{\Psi}/\overline{X}\} \to \psi_i)$, then $\vdash^*_B \bigwedge_{i=1,\ldots,n}(X_i \to \psi_i)$
Table 3 Axiomatic System of Maximal Equation Blocks	Table 4 Axiomatic System of Minumum Equation Blocks

Given an alternation-free block sequence $\mathcal{B} = B_1, ..., B_m$, we define the classical deducibility relations $\vdash_0, \vdash_1, ..., \vdash_m$ as follows: $\vdash_0 = \vdash$, $\vdash_i = \vdash_{B_i}^{i-1}$ for $i = 1, ..m$. Consequently, $\vdash_{\mathcal{B}} = \vdash_m$.

As usual, we say that a formula ϕ (or a set Φ of formulas) is *$\mathcal{B}$-provable*, denoted by $\vdash_{\mathcal{B}} \phi$ (respectively $\vdash \Phi$), if it can be proven from the given axioms and rules of $\vdash_{\mathcal{B}}$. We denote by

$$\overline{\Psi} = \{\phi \in \mathcal{L} \mid \Psi \vdash_{\mathcal{B}} \phi\}.$$

An induction on the structure of the alternation-free blocks shows that all the theorems of $\vdash_{\mathcal{B}}$ are sound in the WTS-semantics, hence also in the SWS-semantics.

Theorem 7.2 (Extended Soundness) *The axiomatic system of $\vdash_{\mathcal{B}}$ is sound with respect to the semantics based on WTSs, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\vdash_{\mathcal{B}} \phi \textit{ implies } \models_{\mathcal{B}} \phi.$$

7.2 Completeness

In the rest of this section we prove that the axiomatic system of $\vdash_{\mathcal{B}}$ is not only sound, but also complete for the two semantics, meaning that all the $\mathcal{B}$-validities can be proved, as theorems, from the proposed axioms and rules, i.e., for arbitrary $\phi \in \mathcal{L}$, $\models_{\mathcal{B}} \phi$ implies $\vdash_{\mathcal{B}} \phi$. To complete this proof it is sufficient to show that any $\mathcal{B}$-consistent formula has a model.

For some set $S \subseteq \mathcal{L}$, Φ is *$(S, \mathcal{B})$-maximally consistent* if it is $\mathcal{B}$-consistent and no formula of S can be added to Φ without making it inconsistent. Φ is *$\mathcal{B}$-maximally-consistent* if it is $(\mathcal{L}, \mathcal{B})$-maximally-consistent.

In the following we fix a consistent formula ϕ_0 depending on a fixed alternation-free sequence $\mathcal{B}_0$ and we construct a model. Let Θ be the set of $\mathcal{B}_0$-maximally consistent sets.

The model construction is not standard, in the sense that we will not use Θ as the set of states in the canonical WTS model. This is because any state in a given WTS corresponds to a function from the set of valuations L to Θ: each resource valuation identifies a $\mathcal{B}_0$-maximally-consistent set of formulas satisfied by that model under the given resource valuation. Consequently, to construct the canonical model we will need to take as states not $\mathcal{B}_0$-maximally-consistent sets of formulas (as usual in modal logics), but some particular functions from L to $\mathcal{B}_0$-maximally-consistent sets, called *coherent functions*. Then, the construction will go as follows:

1. we construct a *canonical model* which takes coherent functions as states, similar to the construction made in [18] for timed logic;
2. we construct an SWS from the above model and prove the truth lemma, where the symbolic finite model property is used;
3. according to Theorem 5.1, there exists a WTS for any $\mathcal{B}_0$-consistent formula.

Lemma 7.3 *For arbitrary $\Lambda \in \Theta$ and $x \in \mathcal{V}$,*

$$\sup\{r \in \mathbb{Q}^+ \mid x \geq r \in \Lambda\} = \inf\{r \in \mathbb{Q}^+ \mid x \leq r \in \Lambda\} \in \mathbb{R}_{\geq 0}.$$

The previous lemma demonstrates that each $\mathcal{B}_0$-maximally-consistent set corresponds to a unique resource valuation of resource-variables, that we will identify using the function $\mathscr{I} : \Theta \longrightarrow L$ defined for arbitrary $\Lambda \in \Theta$ and $x \in \mathcal{V}$ by:

$$\mathscr{I}(\Lambda)(x) = \sup\{r \in \mathbb{Q}^+ \mid x \geq r \in \Lambda\} \in \mathbb{R}_{\geq 0}.$$

Since $\mathscr{I}(\Lambda)$ synthesizes only the information regarding the resource-variables, there exist distinct sets $\Lambda_1, \Lambda_2 \in \Theta$ s.t. $\mathscr{I}(\Lambda_1) = \mathscr{I}(\Lambda_2)$; this defines an equivalence relation on Θ and the equivalence classes are in one to one correspondence with the resource valuation in L.

Observe that not just any function $\gamma : L \to \Omega$ is a good candidate for becaming a state in the canonical model. To better understand this, we emphasize the essential role of resource valuations in the semantics of WMC. We start from analyzing how the formulas satisfied by a given WTS under a certain resource valuation change with the change of the valuation.

Let $\mathcal{F}(\phi)$ be the set of the free resource-variables in $\phi \in \mathcal{L}$ (i.e., those that are not bounded by reset operator $x\ \underline{in}$) defined by: $\mathcal{F}(\bot) = \mathcal{F}(X) = \emptyset$, $\mathcal{F}(x \trianglelefteq r) = \{x\}$, $\mathcal{F}(\phi \vee \psi) = \mathcal{F}(\phi) \cup \mathcal{F}(\psi)$, $\mathcal{F}(\neg\phi) = \mathcal{F}([a]\phi) = \mathcal{F}(\phi)$, $\mathcal{F}(x\ \underline{in}\ \phi) = \mathcal{F}(\phi) \setminus \{x\}$. Similarly, we denote the set of the free resource-variables in ϕ_0 and $\mathcal{B}$ by $\mathcal{F}[\phi_0, \mathcal{B}_0]$.

For $y \in \mathcal{V}$ that does not appear in the syntax of ϕ and $x \in \mathcal{F}(\phi)$, we denote by $\phi\{y/x\}$ the formula obtained by uniformly substituting all the occurrences of x in ϕ by y.

Definition 7.4 *Let f_-, f_+: $\mathcal{V} \to \mathbb{Q}$ be two rational resource valuations. For any formula $\phi \in \mathcal{L}$, let $\phi + {}^{f_-}\!/_{f_+}$ be defined as follows, where $x \trianglelefteq t$ for $t < 0$ should be read as $x \geq 0$:*

$$\begin{array}{llll}
\bot + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} \bot & (\phi \vee \psi) + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} (\phi + {}^{f_-}\!/_{f_+}) \vee (\psi + {}^{f_-}\!/_{f_+}) \\
(x \leq r) + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} x \leq (r + f_+(x)) & (x \geq r) + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} x \geq (r + f_-(x)) \\
(\neg\phi) + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} \neg(\phi + {}^{f_+}\!/_{f_-}) & ([a]\phi) + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} [a](\phi + {}^{f_-}\!/_{f_+}) \\
(x\ \underline{in}\ \phi) + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} x\ \underline{in}\ (\phi + {}^{f_-[x\mapsto 0]}\!/_{f_+[x\mapsto 0]}) & X + {}^{f_-}\!/_{f_+} & \stackrel{df}{=} X
\end{array}$$

Given a list of equations $E = (X_1 = \phi_1, .., X_n = \phi_n)$, let $E + {}^{f_-}\!/_{f_+} = (X_1 = \phi_1 + {}^{f_-}\!/_{f_+}, .., X_n = \phi_n + {}^{f_-}\!/_{f_+})$. Given an equation block $B = \max\{E\}$ or $B = \min\{E\}$, we define $B + {}^{f_-}\!/_{f_+}$ to be $\max\{E + {}^{f_-}\!/_{f_+}\}$ or $\min\{E + {}^{f_-}\!/_{f_+}\}$ respectively. Given an alternation-free block sequence $\mathcal{B} = B_1, \ldots, B_m$, let $\mathcal{B} + {}^{f_-}\!/_{f_+} = B_1 + {}^{f_-}\!/_{f_+}, \ldots, B_m + {}^{f_-}\!/_{f_+}$.

Whenever $f_- = f_+ = f$, we write $+f$.

For $S \subseteq \mathcal{L}$ and $\delta : \mathcal{V} \to \mathbb{R}$, let

$$S \boxplus \delta = \{\phi + {}^{f_-}\!/_{f_+} \mid \phi \in S, f_-, f_+ : \mathcal{K} \to \mathbb{Q} \text{ s.t. } f_- < \delta < f_+\}.$$

Definition 7.5 [Coherent function] *A function $\gamma : L \to \Theta$ is* coherent, *if for any $l, l' \in L$,*

1. $(\mathscr{I} \circ \gamma)(l) = l$; 2. $\gamma(l) \boxplus (l' - l) \subseteq \gamma(l')$.

The first fundamental result is that any $\mathcal{B}_0$-maximally-consistent set Λ belongs to the image $\gamma(L)$ of a coherent function γ. Eventually, we will construct a symbolic model from the WTS on the set of coherent functions, and this result will guarantee that any $\mathcal{B}_0$-maximally-consistent set is satisfied.

Lemma 7.6 *For any $\Lambda \in \Theta$, there exists a coherent function γ such that $\gamma(\mathscr{I}(\Lambda)) = \Lambda$.*

Firstly, we define a WTS using the state space

$$\Gamma = \{\gamma : L \to \Theta \mid \gamma \text{ is a coherent function}\}$$

and the transitions defined by

$$\gamma \xrightarrow{u}_a \gamma' \text{ if } [\forall l \in L, [a]\phi \in \gamma(l) \Rightarrow \phi \in \gamma'(l + u)].$$

Secondly, we apply Construction A from Section 5 and construct a SWS $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ for the above WTS w.r.t ϕ_0 that depends of $\mathcal{B}_0$, for a set of regions $\mathcal{R}[\phi_0, \mathcal{B}_0]$. We get

$\Pi^s = \Gamma \times \mathcal{R}[\phi_0, \mathcal{B}_0]$, $\Sigma^s = \Sigma[\phi_0, \mathcal{B}_0] \cup \{\varepsilon_x \mid x \in \mathcal{V}[\phi_0, \mathcal{B}_0]\}$ and
1. $(\gamma, [l]) \to_a (\gamma', [l'])$ iff $\gamma \to_a \gamma'$ and $l' = l + u$; 2. $(\gamma, [l]) \to_{\varepsilon_x} (\gamma, [l'])$ iff $[l'] = [l][x \mapsto 0]$.

Let $\mathcal{L}[\phi_0, B]$ be defined as:

$$\mathcal{L}[\phi_0, B] = \{\phi \in \mathcal{L} \mid \Sigma[\phi, B] \subseteq \Sigma[\phi_0, B], Q_i[\phi, B] \subseteq Q_i[\phi_0, B]\}.$$

Let ρ_0^s be the symbolic environment defined for any $X \in \mathcal{X}$, by $\rho_0^s(X) = \{(\gamma, [l]) \mid X \in \gamma(l)\}$.

Firstly, we prove the restricted truth lemma that does not consider recursive constructs. Its proof is similar to the proof presented in [18] for timed modal logic.

Lemma 7.7 (Restricted Truth Lemma) *For $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$, $l \in L$ and $\pi = (\gamma, [l]) \in \Pi^s$,*

$$\mathcal{W}^s, \pi, \rho_0^s \models \phi \text{ iff } \phi \in \gamma(l).$$

On the restricted truth lemma we can base the following two results that indicate how we can extend the results to include the recursive cases.

Lemma 7.8 *Let $B = \max\{X_1 = \phi_1, \dots, X_n = \phi_n\}$ be an equation block in the sequence $\mathcal{B}_0$ and ρ^s a symbolic environment such that $\rho^s(X_i) = \{(\gamma, [l]) \mid X_i \in \gamma(l)\}$ for any $i = 1, .., n$. For any $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$, $l \in L$ and $\pi = (\gamma, [l]) \in \Pi^s$,*

$$\text{if } [\mathcal{W}^s, \pi, \rho^s \models \phi \text{ iff } \phi \in \gamma(l)], \quad \text{then } [\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models \phi \text{ iff } \phi \in \gamma(l)].$$

Proof. Induction on ϕ. We prove here the case of the recursive-variables X_i, $i = 1, ..k$.

($\Longrightarrow$) Because WMC enjoys the finite symbolic model property, there exists a finite ordinal k_0 s.t. for all $i = 1, \dots, n$, $\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models X_i$ iff $\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models \phi_i^{k_0}$, where for all $i = 1, \dots, n$, ϕ_i^k are defined simultaneously by $\phi_i^0 = \bot$ and $\phi_i^{k+1} = \phi_i\{\overline{\Phi}^k / \overline{X}\}$, where $\overline{\Phi}^k = (\phi_1^k, .., \phi_n^k)$ and $\overline{X} = (X_1, ..X_n)$.

It is clear that in ϕ_i^k there is no recursive-variable from $\{X_1, \dots, X_n\}$. For any recursive-variable X other than $X_1, \dots, X_m$, $[\![B]\!]_{\rho^s}(X) = \rho^s(X)$. Hence, $\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models X_i$ implies $\mathcal{W}^s, \pi, \rho^s \models \phi_i^{k_0}$. Then, $\phi_i^k \in \gamma(l)$.

The finite symbolic model property also guarantees that for any $\pi' \in \Gamma$ and any $i = 1, \dots, n$,

$$\mathcal{W}^s, \pi', \rho^s \models \phi_i^k \to \phi_i\{\overline{\Phi}^k / \overline{X}\}.$$

So, for any $i = 1, ..n$, $\phi_i^k \to \phi_i\{\overline{\Phi}^k / \overline{X}\} \in \gamma'(l')$ for any $(\gamma', [l']) \in \Gamma$. This further implies that $\vdash \bigwedge_i(\phi_i^k \to \phi_i\{\overline{\Phi}^k / \overline{X}\})$, since $\bigwedge_i(\phi_i^k \to \phi_i\{\overline{\Phi}^k / \overline{X}\})$ is present in all the maximal-consistent sets. Hence, using (max-R2), for any i, $\phi_i^k \to X_i \in \gamma'(l')$ for any $(\gamma', [l']) \in \Gamma$.

As already proven above, $\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models X_i$ implies $\phi_i^k \in \gamma(l)$. Together with $\phi_i^k \to X_i \in \gamma'(l')$ for any $(\gamma', [l']) \in \Gamma$, provided by (max-A1), we get that $X_i \in \gamma(l)$.

($\Longleftarrow$) We prove that ρ^s is a post-fixed point of B as follows:

For any X_i, $i = 1, \dots, n$, suppose $\mathcal{W}^s, \pi, \rho^s \models X_i$. Then $X_i \in \gamma(l)$, which implies that $\phi_i \in \gamma(l)$ by (max-A1). So $\mathcal{W}^s, \pi, \rho^s \models \phi_i$. Since $[\![B]\!]_{\rho^s}$ is the maximal fixed point of B, we have $\rho^s \subseteq [\![B]\!]_{\rho^s}$. Therefore, $\mathcal{W}^s, \pi, \rho^s \models \phi$ implies $\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models \phi$. ∎

Since the minimal blocks are dual of the maximal blocks, we have a similar lemma for minimal blocks.

Lemma 7.9 *Let $B = \min\{X_1 = \phi_1, \ldots, X_n = \phi_n\}$ be an equation block in the sequence $\mathcal{B}_0$ and ρ^s a symbolic environment such that $\rho^s(X_i) = \{(\gamma, [l]) \mid X_i \in \gamma(l)\}$ for any $i = 1, ..n$. For any $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$, $l \in L$ and $\pi = (\gamma, [l]) \in \Pi^s$,*

$$\text{if} \quad [\mathcal{W}^s, \pi, \rho^s \models \phi \text{ iff } \phi \in \gamma(l)], \quad \text{then} \quad [\mathcal{W}^s, \pi, [\![B]\!]_{\rho^s} \models \phi \text{ iff } \phi \in \gamma(l)].$$

These lemmas allow us to prove the stronger version of the truth lemma.

Theorem 7.10 (Extended Truth Lemma) *For $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$, $l \in L$ and $\pi = (\gamma, [l]) \in \Pi^s$,*

$$\mathcal{W}^s, \pi, \rho_0^s \models_{\mathcal{B}} \phi \text{ iff } \phi \in \gamma(l).$$

A direct consequence of Theorem 7.10 is the completeness [3] of the axiomatic system.

Theorem 7.11 (Completeness) *The axiomatic system of $\vdash_{\mathcal{B}}$ is complete with respect to the WTS-semantics, i.e., for arbitrary $\phi \in \mathcal{L}$,*

$$\models_{\mathcal{B}} \phi \text{ implies } \vdash_{\mathcal{B}} \phi.$$

8 Conclusions

In this paper we have investigated the alternation-free weighted mu-calculus (WMC) for which we presented two semantics: one based on weighted transition systems (WTSs) and one based on the symbolic models (SWSs). We have demonstrated that the two semantics are equivalent in the sense that the WTS-validities coincide with the SWS-validities. This is a remarkable result that allows us to transport metaresults between the two semantics.

We firstly proved that even if WMC does not enjoy the finite model property for the WTS-semantics, it enjoys it for the SWS-semantics and thus we prove that satisfiability is decidable in both cases. To prove this we involve the tableau method. We suspect that a similar result can be extended to the entire weighted Mu-Calculus without the alternation-free restriction, but for now we have no evidence in this sense.

The finite model property is also used to prove that the axiomatization that combines modal axioms of weighted logic with the axioms of fixed points is complete for the SWS-semantics. Since the SWS-validities coincide with the WTS-validities, the completeness result can be extrapolated for the TWS-semantics.

The development of symbolic semantics that induces the same validities as the classic semantics is a powerful tool with potential applications also in other contexts. We intend to further apprehend these results to understand if some general technique can be proposed.

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Appendix

Related definitions for the tableau method

Definition [Trace] *Given a path* $\mathcal{P} = t_0 t_1 \dots$ *of a tableau* $\mathcal{T}(\phi, \delta)$*, a* trace *on* $\mathcal{P}$ *is a function* $\mathbb{T}$ *assigning a formula to every node* $t = (\Delta, \delta)$ *in some initial segment of* $\mathcal{P}$ *(possibly to all of* $\mathcal{P}$*), satisfying the following conditions:*
(i) *if* $\mathbb{T}(t)$ *is defined,* $\mathbb{T}(t) \in \Delta$*;*
(ii) *if* $\mathbb{T}(t)$ *is defined and* $t' \in \mathcal{P}$ *is a son of* t*; if a rule applied at* t *does not reduce the formula* $\mathbb{T}(t)$ *then* $\mathbb{T}(t') = \mathbb{T}(t)$*;if* $\mathbb{T}(t)$ *is reduced in* t *then* $\mathbb{T}(t')$ *is one of the results of the reduction.*

We say that there is a *regeneration* of a recursive-variable X on a trace $\mathbb{T}$ on some path of a tableau, if for t and its son t' on the path, $\mathbb{T}(t) = X$ and $\mathbb{T}(t') = \phi$, where $X = \phi \in \mathcal{B}$.

Definition [max-Trace and min-Trace] *We call a trace a* max-trace *iff it is an infinite trace (defined for the whole path) on which the recursive-variable regenerated infinitely often is a* max*-variable.*
Similarly, a trace will be called a min-trace *iff it is an infinite trace where the recursive-variable regenerated infinitely often is a* min*-variable.*

Every infinite trace is either a max-trace or a min-trace; all the rules except (Reg) decrease the size of formulas; hence, every formula is eventually reduced.

Definition [Marking] *For a tableau* $\mathcal{T}(\phi, \delta)$*, we define its* marking *with respect to an SWS* $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ *and state* $\pi_0 \in \Pi^s$ *to be a relation* $\mathfrak{M} \subseteq \Pi^s \times \mathcal{T}(\phi, \delta)$ *satisfying the following conditions:*
(i) $(\pi_0, t_0) \in \mathfrak{M}$*, where* t_0 *is the root of* $\mathcal{T}(\phi, \delta)$*;*
(ii) *if some pair* $(\pi, t) \in \mathfrak{M}$ *and a rule other than (mod) was applied at* t*, then for some son* t' *of* t*,* $(\pi, t') \in \mathfrak{M}$*;*
(iii) *if* $(\pi, t) \in \mathfrak{M}$ *and rule (mod) was applied at* t*, then for every action* a *for which exists* $\langle a \rangle \psi \in \Delta(t)$*:*
 (a) *for every* $\langle a \rangle$*-son* t' *of* t*, there exists a state* π' *s.t.* $\pi \rightarrow_a \pi'$ *and* $(\pi', t') \in \mathfrak{M}$*, and*
 (b) *for every state* π *s.t.* $\pi \rightarrow_a \pi'$*, there exists a* $\langle a \rangle$*-son* t' *of* t *s.t.* $(\pi', t') \in \mathfrak{M}$*.*

Definition [Consistent Marking] *A marking* $\mathfrak{M}$ *of* $\mathcal{T}(\phi, \delta)$ *is* consistent *with respect to* $\mathcal{W}^s = (\Pi^s, \Sigma^s, \theta^s)$ *and* $\pi \in \Pi^s$ *if and only if* $\mathfrak{M}$ *satisfies the following conditions:*
• local consistency*: for any node* $t = (\Delta_t, \delta_t) \in \mathcal{T}(\phi, \delta)$ *and state* $\pi' = (s', \delta') \in \Pi^s$*, if* $(\pi', t) \in \mathfrak{M}$ *then* $\delta_t = \delta'$ *and for any* $\psi \in \Delta(t)$*,* $\mathcal{W}^s, \pi' \models^s_{\mathcal{B}} \psi$*;*
• global consistency*: for every path* $\mathcal{P} = t_0, t_1, \dots$ *of* $\mathcal{T}(\phi, \delta)$ *s.t. there exist* π_i *with* $(\pi_i, t_i) \in \mathfrak{M}$ *for* $i = 0, 1, \dots$*, there is no* min*-trace on* $\mathcal{P}$*.*

Detailed Proofs

Proof. [Proof of Theorem 5.1] 1. $\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi$ iff there exist $\rho_0, \rho_1, \dots, \rho_m$ s.t.
• $\rho_0 = \rho$ and for any $i = 1, \dots, m$, $\rho_i = [\![B_i]\!]_{\rho_{i-1}}$;
• $\mathcal{W}, (m, l), \rho_m \models \phi$.
Let ρ^s_i for any $i = 0, \dots, m$ be defined as: $\rho^s_i(X) = \{(m, [l]) \mid (m, l) \in \rho_i(X)\}$ for any $X \in \mathcal{V}$.
It is not difficult to verify that $\rho^s_0 = \rho^s$ and $\rho^s_i = [\![B_i]\!]_{\rho^s_{i-1}}$ for any $i = 1, \dots, m$.

We can prove that for any $i = 0, \dots, m$, if $\mathcal{W}, (m, l), \rho_i \models \phi$, then $\mathcal{W}^s, (m, [l]), \rho_i^s \models^s \phi$ by induction on ϕ. Moreover, $\mathcal{W}^s, (m, [l]), \rho^s \models^s_{\mathcal{B}} \phi$ iff $\mathcal{W}^s, (m, [l]), [\![\mathcal{B}]\!]_{\rho^s} \models^s \phi$, where $[\![\mathcal{B}]\!]_{\rho^s} = \rho_s^m$. Hence, $\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi$ implies $\mathcal{W}^s, (m, [l]), \rho^s \models^s_{\mathcal{B}} \phi$.

2. $\mathcal{W}^s, (s, \delta), \rho^s \models^s_{\mathcal{B}} \phi$ iff there exist $\rho_0^s, \rho_1^s, \dots, \rho_m^s$ s.t.
• $\rho_0^s = \rho^s$ and for any $i = 1, \dots, m$, $\rho_i^s = [\![B_i]\!]_{\rho_{i-1}^s}$;
• $\mathcal{W}^s, (s, \delta), \rho_m^s \models^s \phi$.
Let ρ_i for any $i = 0, \dots, m$ be defined as: $\rho_i(X) = \{(m, l) \mid (s, \delta) \in \rho^s(X), (s, \delta, l) \in m\}$ for any $X \in \mathcal{V}$. It is not difficult to verify that $\rho_0 = \rho$ and $\rho_i = [\![B_i]\!]_{\rho_{i-1}}$ for any $i = 1, \dots, m$.
We can prove that for any $i = 0, \dots, m$, if $\mathcal{W}^s, (s, \delta), \rho_i^s \models^s \phi$, then $W, (m, l), \rho_i \models \phi$ by induction on ϕ. Moreover, $\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi$ iff $\mathcal{W}, (m, l), [\![\mathcal{B}]\!]_\rho \models \phi$, where $[\![\mathcal{B}]\!]_\rho = \rho^m$.
Hence, $\mathcal{W}^s, (s, \delta), \rho \models^s_{\mathcal{B}} \phi$ implies $\mathcal{W}, (m, l), \rho \models_{\mathcal{B}} \phi$. ∎

Proof. [**Proof of Theorem 6.2**] Suppose $\phi_0 = (\pi_0, \delta_0)$ is satisfied at state π_0 in $\mathcal{W}^S$ under environment ρ^s. According to the above lemma, there is a consistent marking $\mathfrak{M}$ of $\mathcal{T}(\phi_0, \delta_0)$ respect to $\mathcal{W}^s$ and π_0. We construct a finite SWS $\mathcal{W}_f^s = (\Pi_f^s, \Sigma_f^s, \theta_f^s)$, with $\Sigma_f^s = \Sigma[\phi_0] \cup \{\varepsilon_x \mid x \in \mathcal{V}\}$.

Let A, B and C be the set of $\mathcal{T}(\phi_0, \delta_0)$ nodes that are leaves, where the (mod) rule is applied and where the (res) rule is applied respectively. For $t \in A \cup B \cup C$, let $U(t)$ be the set of nodes of $\mathcal{T}(\phi_0, \delta_0)$ consisting of t and all ancestors on the path back up to, but not including, the most recent ancestor in $A \cup B \cup C$; or back up to and including the root if no ancestor of t is in $A \cup B \cup C$. Similarly for $t \in A \cup B$, let $U'(t)$ be the set of nodes of $\mathcal{T}(\phi_0, \delta_0)$ consisting of t and all ancestors on the path back up to, but not including, the most recent ancestor in $A \cup B$; or back up to and including the root if no ancestor of t is in $A \cup B$.

Let $\Pi_1 = \{(t, \delta) \mid t = (\Delta, \delta) \in A \cup B\}$ and $\Pi_2 = \{(t, \delta') \mid t = (\Delta, \delta) \in A \cup B, t' = (\Delta', \delta') \in C \cap U'(t)\}$. The state set $\Pi_f^s = \Pi_1 \cup \Pi_2$. Notice that $\Omega[\phi]$ is finite, so Π^s is finite.

Then the transition relation θ_f^s is defined as:
• for any $(t_1, \delta_1), (t_2, \delta_2) \in \Pi_1$, $(t_1, \delta_1) \rightarrow_a (t_2, \delta_2)$ iff there exists an $\langle a \rangle$-son t' of t_1 s.t. $t' \in U(t_2)$;
• for any $(t_1, \delta_1) \in \Pi_1$ and $(t_2, \delta_2') \in \Pi_2$, $(t_1, \delta_1) \rightarrow_a (t_2, \delta_2)$ iff there exist an $\langle a \rangle$-son t' of t and $t_2' = (\Delta_2', \delta_2') \in C$ s.t. $t_2' \in U'(t)$ and $t' \in U(t_2')$;
• for any $(t, \delta'), (t, \delta'') \in \Pi_2$, $(t, \delta') \rightarrow_\varepsilon (t, \delta'')$ iff there exist $t' = (\Delta', \delta'), t'' = (\Delta'', \delta'') \in C$ and $(t, \delta) \in \Pi_1$ s.t. $t', t'' \in U'(t)$.
• for any $(t, \delta') \in \Pi_2, (t, \delta) \in \Pi_1$, $(t, \delta') \rightarrow_\varepsilon (t, \delta)$ iff there exist $t' = (\Delta', \delta') \in C$ s.t. $t \in U'(t)$.

For any $X \in \mathcal{X}$, let $\rho_f^s(X) = \{t = (\Delta, \delta) \mid X \in \Delta\}$. We need to prove that for any $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$ and $t = (\Delta, \delta) \in \Pi^s$,

$$t' = (\Delta', \delta') \in U(t), \phi \in \Delta' \text{ implies } \mathcal{W}^s, (t, \delta'), \rho_f^s \models^s_{\mathcal{B}} \phi.$$

This can be done in a similar way to that in [20, 32].The correctness of the cases with weight is guaranteed by the region consistency. ∎

Proof. [Proof of Lemma 7.3] Let $A = \{r \in \mathbb{Q}^+ \mid x \geq r \in \Lambda\}$ and $B = \{r \in \mathbb{Q}^+ \mid x \leq r \in \Lambda\}$. (A1) guarantees that $A \neq \emptyset$ and if $B = \emptyset$, we can derive a contradiction from (R3) for $C[\mathbb{X}] = \mathbb{X}$.

Since the two sets are non-empty, the sup and inf exist. Moreover, (R3) can also be used to prove that $\sup A < \infty$. Let $\sup A = u$ and $\inf B = v$. If $u < v$, there exists $r \in \mathbb{Q}^+$ such that $u < r < v$. Hence, $x \leq r \in \Lambda$, which contradicts $r \leq v$. If $v < u$, there exists $r_1, r_2 \in \mathbb{Q}^+$ such that $v < r_1 < r_2 < u$. Hence, $x \leq r_i, x \geq r_i \in \Lambda$ for $i = 1, 2$. Since $r_2 - r_1 > 0$, (A3)$\vdash x \geq r_2 \to \neg(x \leq r_1)$, which proves the inconsistency of Λ - contradiction. ∎

Proof. [Proof of Lemma 7.6] We prove the following properties first:

- For any $S \subseteq \mathcal{L}$ and $\delta, \delta_1, \delta_2 : \mathcal{V} \to \mathbb{R}$ such that $\delta = \delta_1 + \delta_2$, $S \boxplus \delta = (S \boxplus \delta_1) \boxplus \delta_2$.

  Proof: ($\Rightarrow$) Suppose $\psi' \in S \boxplus \delta$. Then there must exist $\psi \in S$, $f_-, f_+ : \mathcal{V} \to \mathbb{Q}$ s.t. $f_- < \delta < f_+$ and $\psi' = \psi + {}^{f_-}/_{f_+}$. Since $\delta = \delta_1 + \delta_2$, there exist $g_-, g_+, h_-, h_+ : \mathcal{V} \to \mathbb{Q}$ s.t. $g_- < \delta_1 < g_+, h_- < \delta < h_+$ and $f_- = g_- + h_-, f_+ = g_+ + h_+$. So $\psi' = \psi + {}^{g_- + h_-}/_{g_+ + h_+} = \psi + {}^{g_-}/_{g_+} + {}^{h_-}/_{h_+}$. Since $\psi + {}^{g_-}/_{g_+} \in S \boxplus \delta_1$ by definition, we have $\psi + {}^{g_-}/_{g_+} + {}^{h_-}/_{h_+} \in (S \boxplus \delta_1) \boxplus \delta_2$. Hence, $\psi' \in (S \boxplus \delta_1) \boxplus \delta_2$.

  ($\Leftarrow$) Suppose $\psi' \in (S \boxplus \delta_1) \boxplus \delta_2$. Then there must exist $\psi \in S$, $g_-, g_+, h_-, h_+ : \mathcal{V} \to \mathbb{Q}$ s.t. $g_- < \delta_1 < g_+, h_- < \delta < h_+$ and $\psi' = \psi + {}^{g_-}/_{g_+} + {}^{h_-}/_{h_+}$. Since $\delta = \delta_1 + \delta_2$, there exist $f_-, f_+ : \mathcal{V} \to \mathbb{Q}$ s.t. $f_- < \delta < f_+$ and $f_- = g_- + h_-, f_+ = g_+ + h_+$. So $\psi' = \psi + {}^{g_- + h_-}/_{g_+ + h_+} = \psi + {}^{f_-}/_{f_+}$. Hence, $\psi' \in S \boxplus \delta$.

- 2. Let $\Lambda_1, \Lambda_2 \in \Theta$ such that $\{\Lambda_1, \Lambda_2\}$ is coherent. Then, for any $l \in L$, $\Lambda_1 \boxplus (l - \mathscr{I}(\Lambda_1)) = \Lambda_2 \boxplus (l - \mathscr{I}(\Lambda_2))$.

  Proof: Let $l_1 = \mathscr{I}(\Lambda_1), l_2 = \mathscr{I}(\Lambda_2)$.

  ($\Rightarrow$) $\Lambda_1 \boxplus (l - l_1) = \Lambda_1 \boxplus ((l_2 - l_1) + (l - l_2))$, which implies $\Lambda_1 \boxplus (l - l_1) = (\Lambda_1 \boxplus (l_2 - l_1)) \boxplus (l - l_2))$ by the above property. Since $\{\Lambda_1, \Lambda_2\}$ is coherent, $\Lambda_1 \boxplus (l_2 - l_1) \subseteq \Lambda_2$. So $(\Lambda_1 \boxplus (l_2 - l_1)) \boxplus (l - l_2)) \subseteq \Lambda_2 \boxplus (l - l_2)$.

  Similarly for the other direction.

With these properties, we can prove the lemma.

I. Firstly, observe that $C \subseteq \Theta$ is coherent iff for any $\Lambda_1, \Lambda_2 \in C$, with $l_1 = \mathscr{I}(\Lambda_1), l_2 = \mathscr{I}(\Lambda_2)$,

$$\Lambda_1 \boxplus (l_2 - l_1) \subseteq \Lambda_2 \text{ and } \Lambda_2 \boxplus (l_1 - l_2) \subseteq \Lambda_1,$$

Moreover, $\Lambda_1 \boxplus (l_2 - l_1) \subseteq \Lambda_2$ iff $\Lambda_2 \boxplus (l_1 - l_2) \subseteq \Lambda_1$.

II. Secondly, we observe that all the infinitary rules of our axiomatization have countable sets of instances. We consider the Boolean-completion of $\mathcal{L}$ with the same axiomatization (see [14]), namely (an isomorphic copy of) the Boolean algebra of complete ideals in $\mathcal{L}$. The completion is a complete Boolean algebra. Every element in the completion is the supremum (in the completion) of the set of elements in $\mathcal{L}$ that are below it. Moreover, $\mathcal{L}$ is a *dense* subset of its completion in the sense that every non-zero element in the completion is above a non-zero element in $\mathcal{L}$. Since the axiomatization is countable, the Rasiowa-Sikorski lemma [15, 29] applied to the completion guarantees that any non-zero element of the completion belongs to an ultrafilter (of the completion). Since any consistent set S of $\mathcal{L}$ corresponds to a non-zero element $\bigwedge S$ in the completion, by applying Rasiowa-Sikorski lemma to the completion of $\mathcal{L}$, we obtain that there exists an ultrafilter u of the completion containing $\bigwedge S$. This is equivalent to the fact that there exists an ultrafilter $u \cap \mathcal{L}$ of $\mathcal{L}$ that includes S.

III. We prove that if $l = \mathscr{I}(\Lambda)$ and $l' \in L$, then there exists $\Lambda' \in \Theta$ s.t. $\mathscr{I}(\Lambda') = l'$ and $\{\Lambda, \Lambda'\}$ is coherent. To prove this, we firstly need to prove that $\Lambda \boxplus (l' - l)$ is consistent. The following two properties guarantee the consistency, which can be proved by induction on the structure of the formulas:
(a) If $\phi \in \Lambda$ and $f_-, f_+ : \mathcal{V} \to \mathbb{Q}$ s.t. for any $x \in \mathcal{V}(\phi)$, either $f_-(x) = f_+(x) = 0$ or $f_-(x) < (l' - l)(x) < f_+(x)$, then,

$$\vdash (\phi + {}^{f_-}/_{f_+}) + {}^{-f_-}/_{-f_+} \to \phi.$$

(b) For any $x \trianglelefteq r \in \mathcal{L}$,

$$\{(x \trianglelefteq r) + {}^{f_-}/_{f_+} \mid f_-, f_+ : \mathcal{V} \to \mathbb{Q}, f_- < 0 < f_+\} \vdash x \trianglelefteq r.$$

Since $\Lambda \boxplus (l' - l)$ is consistent, applying **II**, it must have a $\mathcal{B}_0$-maximal-consistent extension Λ'. Since $\Lambda \boxplus (l' - l) \subseteq \Lambda'$, we also have $\Lambda' \boxplus (l - l') \subseteq \Lambda$. Hence, $\{\Lambda, \Lambda'\}$ is coherent.

IV. Suppose $C = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_k, \ldots\}$ is a coherent set (possibly infinite), $l_i = \mathscr{I}(\Lambda_s)$, $i = 1, \ldots, k, \ldots$ and $l \in L$. Similarly with **III**, we can prove that $\Lambda_s \boxplus (l - l_i)$ is consistent. By Property 2 proven above, we have that $\Lambda_1 \boxplus (l - l_1) = \Lambda_2 \boxplus (l - l_2) = \ldots = \Lambda_k \boxplus (l - l_k) = \ldots$ Hence, in order to get a coherent function γ, we only need to get $\Lambda \boxplus (l' - l)$ for any $l' \in L$, and extend it to $\mathcal{B}_0$-maximal-consistent set $\Lambda_{l'}$ by applying **II**. Let $\gamma(l') = \Lambda_{l'}$. Obviously, γ is a coherent function. ∎

Proof. [Proof of Lemma 7.7] Induction on ϕ. $\phi \vee \psi$, $\neg\phi$ and X cases are straightforward.

[The case $x \trianglelefteq r$]:
($\Longrightarrow$) $\mathcal{W}^s, \pi, \rho_0^s \models x \trianglelefteq r$ implies for any $l' \in [l]$, $l'(x) \trianglelefteq r$. So $l(x) \trianglelefteq r$, which implies that $x \trianglelefteq r \in \gamma(l)$.

($\Longleftarrow$) $x \trianglelefteq r \in \gamma(l)$ implies $l(x) \trianglelefteq r$. Because $x \in \mathcal{V}[\phi_0, \mathcal{B}]$, so $r \in Q[\phi_0, \mathcal{B}_0]$. And since either $[l] = {}^{n}/_{g}$ or $[l] = ({}^{n}/_{g}, {}^{n+1}/_{g})$, it is obvious that for any $l' \in [l]$, $l'(x) \trianglelefteq r$. Hence $\mathcal{W}^s, \pi, \rho_0^s \models x \trianglelefteq r$.

[The case $[a]\phi$]:
$\mathcal{W}^s, \pi, \rho_0^s \models [a]\phi$ iff for any $\pi' = (\gamma', [l']) \in \Pi^s$ s.t. $\pi \to_a \pi'$, $\mathcal{W}^s, \pi', \rho_0^s \models \phi$, iff $\phi \in \gamma'(l')$ by induction hypothesis.

($\Longrightarrow$) Supp. $\langle a \rangle \neg\phi \in \gamma(l)$.

If γ cannot do any a-transition, then there should be no formula like $\langle a \rangle \psi$ in $\gamma(l)$ for all $l \in L$ – contradiction!

Suppose $\gamma \xrightarrow{u}_a \gamma'$. Let $A_l = \{\neg\phi\} \cup \{\psi \mid [a]\psi \in \gamma(l)\} \cup \Upsilon_{l+u}$ and $A_{l'} = \{\psi \mid [a]\psi \in \gamma(l')\} \cup \Upsilon_{l'+u}$ for any $l' \neq l$, where $\Upsilon_{l'} = \bigcup_{x \in \mathcal{V}} \{x \leq r \mid r \geq l'(x)\} \cup \{x \geq r \mid r \leq l'(x)\}$.

It is easy to see that $\{\psi \mid [a]\psi \in \gamma(l)\} \cup \Upsilon_{l+u}$ and $A_{l'}$ for any $l' \neq l$ are consistent.

Suppose that A_l is inconsistent. Then there exists a set $F \subseteq A_l$ s.t. $F \vdash \phi$. If F is finite, (R1) guarantees that $[a]F \vdash [a]\phi$, where $[a]F = \{[a]\psi \mid \psi \in F\}$. Otherwise, $F \vdash \phi$ is (modulo Boolean reasoning possible involving infinite meets) an instance of one of the rules (R2)-(R3); in all these cases, $[a]F \vdash [a]\phi$ is an instance of the same rule for the context $C[\mathbb{X}] = [a]\mathbb{X}$. Since $F \subseteq A_l$, $[a]F \subseteq \gamma(l)$ implying $[a]\phi \in \gamma(l)$, which contradicts the consistency of $\gamma(l)$. Hence, A_l is consistent.

Now we prove that for any $l_1, l_2 \in L$, A_{l_1} and A_{l_2} are such that $A_{l_1} + (l_2 - l_1) \subseteq A_{l_2}$. If $l_1 \neq l$, then for arbitrary $\psi' \in A_{l_1}$ either $[a]\psi' \in \gamma(l_1)$, or $\psi' = x \trianglelefteq r$.
In the first case, $[a]\psi' + {}^{f_-}/_{f_+} \in \gamma(l_2)$, for all $f_- \leq l_2 - l_1 \leq f_+$. So, $\psi' + {}^{f_-}/_{f_+} \in A_{l_2}$.
In the second case, since $\psi' = x \trianglelefteq r$ is closed under any resource valuation transformation, for any $f_- \leq l_2 - l_1 \leq f_+$, $\psi' + {}^{f_-}/_{f_+} \in A_{l_2}$.
If $l_1 = i$, consider an arbitrary $\psi' \in A_{l_1}$. If $\psi' \neq \neg\phi$, we get a similar case as above. Otherwise, $\langle a\rangle\psi' \in \gamma(l)$, which implies $\langle a\rangle\psi' + {}^{f_-}/_{f_+} \in \gamma(l_2)$ for all $f_- \leq l_2 - l_1 \leq f_+$. So, $\psi' + {}^{f_-}/_{f_+} \in A_{l_2}$.

At this point we can use a similar strategy as in Theorem 7.6 to prove that there exists $\gamma'' \in \Gamma$ s.t. for any $l' \in L$, $A_{l'} \subseteq \gamma''(l')$. Hence, $\neg\phi \in \gamma''(l + u)$. According to the definition of the model, $\gamma \xrightarrow{u}_a \gamma''$, which implies $\phi \in \gamma''(l + u)$ - contradiction!

Hence, $[a]\phi \in \gamma(l)$.

($\Longleftarrow$) derives from the definition of θ^s.

[The case x *in* ϕ]:
($\Longrightarrow$) $\mathcal{W}^s, \pi, \rho_0^s \models x\ \underline{in}\ \phi$ implies that there exists $\pi' \in \Pi^s$ s.t. $\pi \rightarrow_{\varepsilon_x} \pi'$ and $\mathcal{W}^s, \pi', \rho_0^s \models \phi$, which implies that $\phi \in \gamma(l[x \mapsto 0])$ by inductive hypothesis. Since $l[x \mapsto 0](x) = 0$, we have $x\ \underline{in}\ \phi \in \gamma(l[x \mapsto 0])$. Because γ is coherent function, it is not difficult to prove that $x\ \underline{in}\ \phi \in \gamma(l)$.

($\Longleftarrow$) $x\ \underline{in}\ \phi \in \gamma(l)$ implies that $x\ \underline{in}\ \phi \in \gamma(l[x \mapsto 0])$ by Definition 7.5. Therefore, $\phi \in \gamma(l[x \mapsto 0])$ by (A11). By inductive hypothesis, $\mathcal{W}^s, (\gamma, [l][x \mapsto 0]), \rho_0^s \models \phi$, which implies $\mathcal{W}^s, (\gamma, [l]), \rho_0^s \models x\ \underline{in}\ \phi$. ∎

Proof. [**Proof of Theorem 7.10**] By the semantics of the alternation-free block sequence, given an environment ρ_0, $\mathcal{B}$ defines a series of environments: $\rho_1^s, \ldots, \rho_m^s$, where $\rho_i^s = [\![B_i]\!]_{\rho_{i-1}^s}$ for any $i = 1, \ldots, m$. And $[\![\mathcal{B}]\!]_{\rho_0^s} = \rho_m^s$.

We prove that for $\rho_i^s, i = 0, 1, \ldots, m$,
$$\mathcal{W}^s, \pi, \rho_i^s \models \phi \text{ iff } \phi \in \gamma(l)$$
by induction on i. The case $i = 0$ is given by Lemma 7.7. Suppose the statement holds for $k \geq 0$. Then it is still true according to Lemma 7.8 and Lemma 7.9.

And $\mathcal{W}^s, \pi, \rho_0^s \models_{\mathcal{B}} \phi$ iff $\mathcal{W}^s, \pi, \rho_m^s \models \phi$. Therefore, $\mathcal{W}^s, \pi, \rho_0^s \models_{\mathcal{B}} \phi$ iff $\phi \in \gamma(l)$. ∎