SUM AND TENSOR OF QUANTITATIVE EFFECTS

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Abstract. Inspired by the seminal work of Hyland, Plotkin, and Power on the combination of algebraic computational effects via *sum* and *tensor*, we develop an analogous theory for the combination of quantitative algebraic effects.

Quantitative algebraic effects are monadic computational effects on categories of *metric spaces*, which, moreover, admit an algebraic presentation in the form of quantitative equational theories, a logical framework introduced by Mardare, Panangaden, and Plotkin that generalises equational logic to account for a concept of approximate equality. As our main result we show that the sum and tensor of two quantitative equational theories, correspond to the categorical sum (*i.e.*, coproduct) and tensor, respectively, of their effects qua monads. We further give a theory of *quantitative effect transformers* based on these two operations, essentially providing quantitive analogues to the following monad transformers due to Moggi: exception, resumption, reader, and writer transformers. Finally, as an application we provide the first quantitative algebraic axiomatizations to the following coalgebraic structures: Markov processes, labelled Markov processes, Mealy machines, and Markov decision processes, each endowed with their respective bisimilarity metrics. Apart from the intrinsic interest in these axiomatizations, it is pleasing they have been obtained as the composition, via sum and tensor, of simpler quantitative equational theories.

1. Introduction

The theory of computational effects began with the work of Moggi [34, 35] seeking a unified category-theoretic account of the semantics of higher-order programming languages. He modelled computational effects (which he called notions of computation) by means of strong monads on a base category with cartesian closed structure. With Cenciarelli [11], he later extended the theory by allowing a compositional treatment of various semantic phenomena

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[*] Extended and combined version of [3] (LICS'18) and [4] (CALCO'21).

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such as state, IO, exceptions, resumptions, etc, via the use of monad transformers. This work was followed up by the program of Plotkin and Power [36, 37] on an axiomatic understanding of computational effects as arising from operations and equations via the use of Lawvere theories (see also [22]). In a fundamental contribution [20] together with Hyland they developed a unified modular theory for algebraic effects that supports their combination by taking the *sum* and *tensor* of their Lawvere theories. This allowed them to recover in a more pleasing structural algebraic way many of the monad transformers considered by Moggi.

Quantitative equational logic, introduced by Mardare, Panangaden, and Plotkin [30], is a logical framework generalising the standard concept of equational logic to account for a concept of approximate equality. The key idea is to introduce equations indexed by rational numbers

$$t =_\varepsilon s$$

where t, s are terms over a signature of operations. One reads this as "s is within ε of t". The model theory of quantitative equational logic is developed into quantitative universal algebras, that is, universal algebras with operations interpreted as non-expansive maps on a metric space. Quantitative equational logic is a logical framework providing quantitative analogues of the core results of equational logic, such as completeness theorems, constructions of free algebras, Cauchy completions of models, and Birkoff-like (quasi-)variety theorems [30, 31, 14]. Moreover and relevantly for the present paper, they are used to provide an algebraic presentation of quantitative effects as freely-generated monads on categories of metric spaces. As we will show in Section 4, quantitative theories are expressive enough to recover many quantitative effects of interest in computer science, such as exceptions, interactive input/output, read, write, non-determinism, and probabilistic choice.

Following Hyland et al. [20], in this paper we develop the theory for the *sum* and *tensor of quantitative equational theories*.

The sum combines two theories by taking their disjoint union. In this sense, it is the simplest combination supporting both given effects. In contrast, the tensor additionally imposes mutual commutation of the operations from each theory. As such it refines the sum of theories, which is just their unrestricted combination. Sum and tensor of theories arise in several contexts. For example, in the semantics of programming languages, the monad transformer for exception and resumption are given by a sum; and the transformer for global state, reader and writer are given by a tensor [20].

The main contributions of the present paper are:

(1) we prove that the sum and tensor of quantitative equational theories correspond to the categorical sum (*i.e.*, coproduct) and tensor, respectively, of their induced quantitative effects as strong monads;
(2) we provide a quantitative presentation to the *quantitative exception* and *interactive input monads*, and obtain quantitative analogues to their corresponding Moggi transformers at the level of theories using sum;
(3) we give quantitative axiomatisations to the *quantitative reader* and *writer monads*, from which we obtain analogues of their monad transformers at the level of theories using tensor;
(4) we provide the first axiomatizations of *Markov processes*, *labelled Markov processes*, *Mealy machines*, and *Markov decision processes with rewards*, each endowed with their respective discounted bisimilarity metric.

For the results in (1) we require the quantitative theories to be axiomatised by a set of quantitative inferences involving only quantitative equations between variables in the premises. As in [31], we call this type of theories *basic*. The monad transformers that we recovered in (2) and (3) are compelling evidence for the usefulness of our compositional framework for quantitative effects. Ideally, these transformers could implemented in future quantitative extension of effectful programming languages, such as, Eff, Koka, or Haskell.

The axiomatisations listed in (4) are major examples for our compositional theory of quantitative effects. On the one hand, we obtain the discounted bisimilarity metrics for Markov processes by starting from the theory of interpolative barycentric algebras, (used to axiomatise probability distributions with the Kantorovich metric) and by applying to it, in turn, the exception and interactive-input theory transformers, which are two examples of sum of theories. On the other hand, labelled Markov processes and Markov decision processes with rewards are obtained by complementing the axiomatization for Markov processes, by adding the missing computational effects. We model the effect of reacting to an action label by tensoring with the theory of quantitative reading computations (corresponding to the reader transformer); while the effect of recording a reward are is modelled by tensoring with the theory of quantitative writing computations (corresponding to the writer transformer). We illustrate our approach by decomposing the proposed axiomatisations into their basic components and showing how to combine them step-by-step to get the desired result. The axiomatisation of Mealy machines is done similarly and is further evidence for the generality and simplicity of our compositional approach to quantitative effects.

This article is an extended and combined version of [3] and [4]. Beyond providing all proofs which could not be published in [3, 4] because of space limitations, we refactorized and simplied several technical results. The main examples of this refactorization are Sections 4, 5.1, and 5.2. In the latter, the statements of Corollaries 5.10–5.12 could be considerably simplified by using that fact that quantitative theories induce only monads with (at most) countable rank, a result due to Ford et al. [14] that we did not know when writing [3]. Moreover, the axiomatisation of Mealy machines (Section 6.5) is new material not present in the conference version of [4].

Further Related Work. In [20, 19] the sum and tensor of (enriched) Lawvere theories are characterized as the colimit of certain cocones, and the correspondence with the sum and tensor of monads is obtained via the equivalence between Lawvere theories and monads. Since it is not hard to show that (basic) quantitative equational theories can be characterised as metric-enriched Lawvere theories, one may think to recover the correspondence with the operations on their monads via the equivalence with Lawvere theories. Alas, quantitative equational theories and Lawvere theories are not equivalent, as the latter allows generic operations with metric spaces as arities, while the framework of Mardare et al. [30] admits only operations with discrete arities. An equivalence with *discrete* Lawvere theories [21] (where arities are just countable ordinals) does not hold either, because quantitative equations implicitly generates morphisms (hence, operations in a Lawvere sense) with non-discrete arities which cannot be expressed in the framework of discrete Lawvere theories.

The above arguments required us to follow a different path which required us to prove the two correspondences directly. For the correspondence with the sum of monads we could follow Kelly [24], which characterises the Eilenberg-Moore algebras of the coproduct of monads as bialgebras. In contrast, the correspondence with the tensor of monads was more involved and led us to the introduction of *pre-operations of a strong functor*, which we use to conveniently characterise the commutative bialgebras for the monads (which correspond to the Eilenberg-Moore algebras for their tensor). Pre-operations of a strong functor F are related to Plotkin and Power's *algebraic operations* [38, 39] in the sense that their assignment to F-algebras are the appropriate version of algebraic operations for functors. Moreover, when considered over a strong monad T they correspond to generic effects of type $I \to Tv$ (*i.e.*, Kleisli maps of type $I \to v$, where I is the identity for the monoidal product). The reason why we consider pre-operations over functors, and not just monads, is to relate the operations of an algebraic monad with those of its signature. Crucially, this allowed us to carry out the technical development directly at the level of quantitative equational theories without passing via a correspondence with metric-enriched Lawvere theories.

Finally, we remark that quantitative equational theories, although not as general as metric-enriched Lavwere theories, are a natural and simpler form of enriched equational theory, which is still expressive enough to recover many examples of interest in computer science (see [30, 3, 33]). In this respect, it is pleasing that also this simpler subclass of enriched theories are closed under sum and tensor.

Synopsis. We start by recalling some preliminary categorical definitions that will be used in the rest of the paper (Section 2). In Section 3, we introduce the core definitions and results of the theory of quantitative algebras. In Section 4, we present several examples of algebraic quantitative effects and present their axiomatic quantitative equational theories. In Sections 5 and 6, we develop the theory for the sum and tensor of quantitative equational theories and show that such combinators correspond to the categorical sum and tensor of quantitative effects as monads, respectively. In each of these sections we propose several nontrivial examples of composite of quantitative effects. Finally, we collect some conclusions and propose possible future work in Section 7.

In the Appendices A and B we collect some technical results regarding the categories of metric spaces that we use in our development.

2. Preliminaries and Notation

In this paper, we deal with Eilenberg-Moore algebras of strong monads on the category of extended metric spaces. We assume familiarity with the basic notions of category theory, such as, functors, natural transformations, and adjunctions (see [27] for reference).

In section, for the only sake of fixing notation, we recall some basic definitions regarding metric spaces, monads, and monoidal closed categories. As these definitions are standard, a reader that is familiar with these notions can safely skip this section.

2.1. **Categories of Extended Metric Spaces.**

An *extended metric space* is a pair (X, d) consisting of a set X equipped with a distance function $d\colon X \times X \to [0, \infty]$ satisfying: (i) $d(x, y) = 0$ iff $x = y$, (ii) $d(x, y) = d(y, x)$ and (iii) $d(x, z) \leq d(x, y) + d(y, z)$. Note that the distance function is allowed to have infinite values, and as such, sum between positive real numbers is extended to $[0, \infty]$ by canonically imposing that $\infty + r = r + \infty = \infty$, for all $r \in [0, \infty]$ (hence, ∞ is the top element w.r.t. the extension of the canonical order $\leq$).

A sequence (x_i) in (X, d) *converges* to $x \in X$ if $\forall \epsilon > 0, \exists N, \forall i \geq N, d(x_i, x) \leq \epsilon$. A sequence (x_i) is *Cauchy* if $\forall \epsilon > 0, \exists N, \forall i, j \geq N, d(x_i, x_j) \leq \epsilon$. If every Cauchy sequence converges, the extended metric space (X, d) is said to be *complete*. If a space is not complete it can be completed by a well-known construction called *Cauchy completion*. We write $\overline{(X,d)}$ for the completion of (X,d). When the distance function is clear from the context, we will refer to (X,d) simply as X and to its completion as $\overline{X}$.

Let (X,d_X), (Y,d_Y) be extended metric spaces. A map $f\colon X \to Y$ is *c-Lipschitz continuous*, with constant $c \geq 0$, if for all $x,x' \in X$, $c \cdot d_X(x,x') \geq d_Y(f(x),f(x'))$. If $c = 1$, the function is called *non-expansive*, and if $0 \leq c < 1$ and f maps to itself, it is called a *contraction*. Note that Lipschitz continuous functions preserve convergence since they are continuous in the usual sense.

The categories of metric spaces that we consider are **Met**, with extended metric spaces as objects and non-expansive maps as morphism, and its full subcategory **CMet** with objects the complete extended metric spaces. These categories are complete and cocomplete, *i.e.*, have all limits and colimits (see Appendix A for details). Moreover, **CMet** is a reflective subcategory of **Met**, with reflection given by the Cauchy completion functor $\mathbb{C}\colon \mathbf{Met} \to \mathbf{CMet}$, mapping a metric space to its completion, being the left adjoint to the embedding $\mathbf{CMet} \hookrightarrow \mathbf{Met}$.

2.2. **Monads and their Algebras.** A monad on a category $\mathbf{C}$ is a triple (T,η,μ) consisting of an endofunctor $T\colon \mathbf{C} \to \mathbf{C}$ and two natural transformations: a *unit* $\eta\colon Id \Rightarrow T$ and a *multiplication* $\mu\colon TT \Rightarrow T$ that satisfy the laws

$$
\begin{tikzcd}
TX \arrow[r, "\eta T"] \arrow[dr, "id"'] & TTX \arrow[d, "\mu"] & TX \arrow[l, "T\eta"'] \arrow[dl, "id"] \\
& TX &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTTX \arrow[r, "\mu T"] \arrow[d, "M\mu"'] & TTX \arrow[d, "\mu"] \\
TTX \arrow[r, "\mu"'] & TX
\end{tikzcd}
$$

respectively called the *left/right unit laws* and *multiplication law* for the monad (T,η,μ). When the monad structure is clear from the context we will denote (T,η,μ) simply as T.

Given an endofunctor $H\colon \mathbf{C} \to \mathbf{C}$, the *free monad on H* is a monad H^* on $\mathbf{C}$ equipped with a natural transformation $\gamma\colon H \Rightarrow H^*$ that is initial among all such pairs $(S, \lambda\colon H \Rightarrow S)$.

A *monad map* from a monad (T,η,μ) on to a monad (H,ρ,ν) on the same category is a natural transformation $\sigma\colon T \Rightarrow H$ that makes the following diagrams commute,

$$
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "\rho"'] & TX \arrow[d, "\sigma"] \\
& HX
\end{tikzcd}
\qquad
\begin{tikzcd}
TTX \arrow[r, "\sigma\sigma"] \arrow[d, "\mu"'] & HHX \arrow[d, "\nu"] \\
TX \arrow[r, "\sigma"'] & HX
\end{tikzcd}
$$

If $\sigma\colon T \Rightarrow H$ is an epimorphism, than H is a *quotient* of T. If it is a monomorphism, then T is a *submonad* of T. If it is an isomorphism, the two monad are isomorphic. In the following, we consider monads to be the same up to isomorphism.

Let $F\colon \mathbf{C} \to \mathbf{C}$ an endofunctor. An algebra of F (or simply, F-algebra) is a pair (A,a) consisting of an object A, called *carrier*, and a morphism $a\colon FA \to A$ in $\mathbf{C}$, called *F-algebra structure*. A morphism of F-algebras (or simply, F-homomorphism) from (A,a) to (B,b) is an arrow $f\colon A \to B$ in $\mathbf{C}$ making the square below commute

$$
\begin{tikzcd}
FA \arrow[r, "a"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\
FB \arrow[r, "b"'] & B
\end{tikzcd}
$$

The algebras of a functor F and their homomorphisms form a category, denoted F-**Alg**. The category of F-algebras has an obvious forgetful functor $U^F\colon F\text{-}\mathbf{Alg} \to \mathbf{C}$ mapping an F-algebra (A, a) to its carrier A, hence forgetting the algebra structure. If the forgetful functor has a left adjoint $L^F\colon \mathbf{C} \to F\text{-}\mathbf{Alg}$, then the algebra $L^F(A)$ obtained from an object $A \in \mathbf{C}$, is called *free F-algebra for A*. The monad $U^F L^F$ arising from the adjunction is the free monad on F, denoted F^*. A monad arising from an adjunction involving free algebras is said *algebraic* [5].

An *Eilenberg-Moore* (EM) algebra for a monad (T, η, μ), is a T-algebra (A, a) making the two diagrams below commute

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "id"'] & TA \arrow[d, "a"] & TTA \arrow[l, "\mu"'] \arrow[d, "Ta"] \\ & A & TA \arrow[l, "a"] \end{tikzcd}$$

respectively called *unit law* (left diagram) and *multiplication law* (right diagram) for the T-algebra (A, a). The morphisms between EM algebras are the T-homomorphism of their T-algebras. The resulting category of EM algebras for the monad T is called *Eilenberg-Moore category* for the monad T, and it is denoted by $\mathbf{EM}(T)$.

The forgetful functor $U^T\colon \mathbf{EM}(T) \to \mathbf{C}$ has a left adjoint $F^T\colon \mathbf{C} \to \mathbf{EM}(T)$ associating the *free* EM algebra (TX, μ_X) with the object $X \in \mathbf{C}$. By construction, the monad $U^T F^T$ arising from the adjunction is isomorphic to T. Moreover, $\mathbf{EM}(T)$ has all limits which exists in $\mathbf{C}$, and they are created by the forgetful functor. The situation for colimits is more complicated, as colimits may not necessarily exist.

An important result that will be used later on in the paper, relates the holds if If the free monad H^* on a endofunctor H exists and is algebraic, the Eilenberg-Moore category $\mathbf{EM}(H^*)$ is isomorphic to the category H-**Alg** of algebras of H.

2.3. Monoidal Closed Categories and Strong Functors.

A category is *monoidal* when it comes equipped with a "product" structure. In detail, a *monoidal category* is a category $\mathbf{V}$ with a *monoidal product*[1] $\Box\colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, a *unit (or identity) object* $I \in \mathbf{V}$, and three natural isomorphisms: (*associator*) $\alpha_{V,W,Z}\colon V \Box (W \Box Z) \xrightarrow{\cong} (V \Box W) \Box Z$, (*left unitor*) $\lambda_V\colon I \Box V \xrightarrow{\cong} V$, (*right unitor*) $\rho_V\colon V \Box I \xrightarrow{\cong} V$, subject to the coherence conditions

$$\begin{tikzcd} V \Box (W \Box (Y \Box Z)) \arrow[r, "\alpha"] \arrow[d, "id \Box \alpha"'] & (V \Box W) \Box (Y \Box Z) \arrow[r, "\alpha"] & ((V \Box W) \Box Y) \Box Z \\ V \Box ((W \Box Y) \Box Z) \arrow[rr, "\alpha"'] & & (V \Box (W \Box Y)) \Box Z \arrow[u, "\alpha \Box id"'] \end{tikzcd} \tag{ASSOC}$$

$$\begin{tikzcd} V \Box (I \Box W) \arrow[rr, "\alpha"] \arrow[dr, "id \Box \lambda"'] & & (V \Box I) \Box W \arrow[dl, "\rho \Box id"] \\ & V \Box W & \end{tikzcd} \tag{UNIT}$$

expressing that the operation $\Box$ is associative, with left/right identity.

[1]The standard symbol for monoidal product is $\otimes$, however we prefer to denote it as $\Box$ to avoid confusion with other tensorial operations we will deal with in this paper, specifically, the tensor of monads.

A monoidal category is *symmetric* when in addition is equipped with a natural isomorphism (*braiding*) $s_{V,W} \colon V \Box W \xrightarrow{\cong} W \Box V$ such that the following diagrams commute:

$$\begin{tikzcd}
V \Box (W \Box Z) \arrow[r, "\alpha"] \arrow[d, "id \Box s"'] & (V \Box W) \Box Z \arrow[r, "s"] & Z \Box (V \Box W) \arrow[d, "\alpha"] \\
V \Box (Z \Box W) \arrow[r, "\alpha"'] & (V \Box Z) \Box W \arrow[r, "s \Box id"'] & (Z \Box V) \Box W
\end{tikzcd}$$

$$\begin{tikzcd}
I \Box V \arrow[rr, "s"] \arrow[dr, "\lambda"'] & & V \Box I \arrow[dl, "\rho"] \\
& V &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \Box W \arrow[rr, "s"] \arrow[dr, "id"'] & & W \Box V \arrow[dl, "s"] \\
& V \Box W &
\end{tikzcd}$$

A monoidal category is *closed*, if has an internal *hom-functor* $[-,-] \colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, such that for every object $V \in \mathbf{V}$, $[V,-] \colon \mathbf{V} \to \mathbf{V}$ is right adjoint to $(V \Box -) \colon \mathbf{V} \to \mathbf{V}$. We will denote the *counit* (or evaluation map) of the adjunction $(V \Box -) \dashv [V,-]$ by $ev^V \colon V \Box [V,-] \Rightarrow Id$ and the *unit* (or co-evaluatation map) by $\overline{ev}^V \colon Id \to [V, V \Box -]$.

Examples 2.1. The monoidal closed categories we will consider are **Set**, **Met**, and **CMet**.

(1) **Set** is a symmetric monoidal closed category with Cartesian product $X \times Y$ as monoidal product and internal hom $[X,Y]$ given by the set of functions from X to Y. Since the monoidal product coincides with the categorical product, **Set** is a *Cartesian closed*.
(2) **Met** is a symmetric monoidal closed category, with monoidal product $(X, d_X) \Box (Y, d_Y)$ being the extended metric space with underlying set $X \times Y$ and distance function $d_{X \Box Y}((x,y)(x',y')) = d_X(x,x') + d_Y(y,y')$; the internal hom $[(X,d_X),(Y,d_Y)]$ is given by the set of non-expansive maps from X to Y with point-wise supremum metric $d_{[X,Y]}(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$. Note that $\Box$ is not the categorical product in **Met**, for which the distance function would have max in place of $+$, as one can show that **Met** is not cartesian closed [28].
(3) **CMet** has the same symmetric monoidal closed structure of **Met**, as the monoidal product $\Box$ defined above preserves Cauchy completeness.

It is common to refer monoidal closed categories simply as *closed categories*. We will use this abbreviation hereafter.

A functor $F \colon \mathbf{V} \to \mathbf{V}$ is *strong* with *monoidal strength* $t_{V,W} \colon V \Box F(W) \to F(V \Box W)$, if t is a natural transformation satisfying the following coherence conditions w.r.t. the associator α and left unitor λ of $\mathbf{V}$:

$$\begin{tikzcd}
I \Box FV \arrow[r, "t"] \arrow[dr, "\lambda"'] & F(I \Box V) \arrow[d, "F\lambda"] \\
& FV
\end{tikzcd}
\qquad
\begin{tikzcd}
(U \Box V) \Box FW \arrow[r, "\alpha"] \arrow[dd, "t"'] & U \Box (V \Box FW) \arrow[d, "U \Box t"] \\
& U \Box F(V \Box W) \arrow[d, "t"] \\
F((U \Box V) \Box W) \arrow[r, "F\alpha"] & F(U \Box (V \Box W))
\end{tikzcd}$$

When $\mathbf{V}$ is symmetric, the dual strength $\hat{t}_{V,W} \colon F(W) \Box V \to F(W \Box V)$ is given by $\hat{t} = F(s) \circ t \circ s$, where $s_{V,W} \colon V \Box W \to W \Box V$ is the braiding of $\mathbf{V}$. A natural transformation $\theta \colon F \Rightarrow G$ is said *strong* if F, G are strong functors with strengths t, σ, respectively, and $\sigma \circ (id \Box \theta) = \theta \circ t$, meaning that θ interacts well with the strengths.

A monad (T, η, μ) with unit $\eta\colon Id \Rightarrow T$ and multiplication $\mu\colon TT \Rightarrow T$, is *strong* if T is a strong functor with strength t such that $t \circ (id \,\Box\, \eta) = \eta$ and $\mu \circ tt = t \circ (id \,\Box\, \mu)$.

Note that strong functors (resp. monads) on a symmetric monoidal closed category $\mathbf{V}$ are equivalent to $\mathbf{V}$-enriched functors (resp. monads) on the self-enriched category $\mathbf{V}$ [26].

3. Quantitative Equational Theories

Quantitative equations were introduced in [30]. In this framework equalities $t =_\varepsilon s$ are indexed by a positive rational number, to capture the idea that t is "within ε" of s. This informal notion is formalised in a manner analogous to traditional equational logic. In this section we review this formalism.

Let Σ be a signature of function symbols $f\colon n \in \Sigma$ of arity $n \in \mathbb{N}$. Let X be a countable set of variables, ranged over by $x, y, z, \ldots$. We write $\mathbb{T}(\Sigma, X)$ for the set of Σ-terms freely generated over X, ranged over by $t, s, u, \ldots$.

A *substitution of type* Σ is a function $\sigma\colon X \to \mathbb{T}(\Sigma, X)$, canonically extended to terms as $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$; we write $\mathcal{S}(\Sigma)$ for the set of substitutions of type Σ.

A *quantitative equation of type* Σ over X is an expression of the form $t =_\varepsilon s$, for $t, s \in \mathbb{T}(\Sigma, X)$ and $\varepsilon \in \mathbb{Q}_{\geq 0}$. We use $\mathcal{V}(\Sigma, X)$ to denote the set of quantitative equations of type Σ over X, and its subsets will be ranged over by $\Gamma, \Theta, \ldots$. Let $\mathcal{E}(\Sigma, X)$ be the set of *conditional quantitative equations* on $\mathbb{T}(\Sigma, X)$, which are expressions of the form

$$\{t_1 =_{\varepsilon_1} s_1, \ldots, t_n =_{\varepsilon_n} s_n\} \vdash t =_\varepsilon s\,,$$

for arbitrary $s_i, t_i, s, t \in \mathbb{T}(\Sigma, X)$ and $\varepsilon_i, \varepsilon \in \mathbb{Q}_{\geq 0}$. As in standard equational logic, we abbreviate $\emptyset \vdash t =_\varepsilon s$ to $\vdash t =_\varepsilon s$.

Definition 3.1 (Quantitative Equational Theory)**.** A *quantitative equational theory of type* Σ *over* X is a set $\mathcal{U} \subseteq \mathcal{E}(\Sigma, X)$ of conditional quantitative equations satisfying the following conditions, for arbitrary $x, y, z, x_i, y_i \in X$, terms $s, t \in \mathbb{T}(\Sigma, X)$, rationals $\varepsilon, \varepsilon' \in \mathbb{Q}_{\geq 0}$, and $\Gamma, \Theta \subseteq \mathcal{V}(\Sigma, X)$,

(Refl) $\vdash x =_0 x \in \mathcal{U}$,

(Symm) $\{x =_\varepsilon y\} \vdash y =_\varepsilon x \in \mathcal{U}$,

(Triang) $\{x =_\varepsilon z, z =_{\varepsilon'} y\} \vdash x =_{\varepsilon+\varepsilon'} y \in \mathcal{U}$,

(Max) $\{x =_\varepsilon y\} \vdash x =_{\varepsilon+\varepsilon'} y \in \mathcal{U}$, for all $\varepsilon' > 0$,

(f-NE) $\{x_i =_\varepsilon y_i \mid i = 1 \ldots n\} \vdash f(x_1, \ldots, x_n) =_\varepsilon f(y_1, \ldots, y_n) \in \mathcal{U}$, for $f\colon n \in \Sigma$,

(Cont) If $\{x =_{\varepsilon'} y \mid \varepsilon' > \varepsilon\} \subseteq \mathcal{U}$, then $\vdash x =_\varepsilon y \in \mathcal{U}$,

(Subst) If $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$, then $\sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(s) \in \mathcal{U}$, for $\sigma \in \mathcal{S}(\Sigma)$,

(Ass) If $t =_\varepsilon s \in \Gamma$, then $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$,

(Cut) If $\{\Gamma \vdash t =_\varepsilon s \mid t =_\varepsilon s \in \Theta\} \subseteq \mathcal{U}$ and $\Theta \vdash t =_\varepsilon s \in \mathcal{U}$, then $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$,

where $\sigma(\Gamma) = \{\sigma(t) =_\varepsilon \sigma(s) \mid t =_\varepsilon s \in \Gamma\}$.

The conditions (Subst), (Cut), (Ass) are the usual deductive rules of equational logic. The axioms (Refl), (Symm), (Triang) correspond, respectively, to reflexivity, symmetry, and the triangle inequality; (Max) represents inclusion of neighbourhoods of increasing diameter; (Cont) is the limiting property of a decreasing chain of neighbourhoods with converging diameters; and (f-NE) expresses non-expansiveness of $f \in \Sigma$.

A set $A \subseteq \mathcal{E}(\Sigma, X)$ of conditional quantitative equations *axiomatises* a quantitative equational theory $\mathcal{U}$, if $\mathcal{U}$ is the smallest quantitative equational theory containing A.

The models of these theories, called *quantitative* Σ*-algebras*, are Σ-algebras in **Met**.

Definition 3.2 (Quantitative Algebra)**.** A *quantitative* Σ*-algebra* is a tuple $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, where A is an extended metric space and $\Sigma^{\mathcal{A}} = \{f^{\mathcal{A}} \colon A^n \to A \mid f : n \in \Sigma\}$ is a set of non-expansive *interpretations* (*i.e.*, satisfying $\max_i d_A(a_i, b_i) \geq d_A(f^{\mathcal{A}}(a_1, \dots, a_n), f^{\mathcal{A}}(b_1, \dots, b_n))$).

The morphisms between quantitative Σ-algebras are non-expansive Σ-homomorphisms. Quantitative Σ-algebras and their morphism form a category, denoted by $\mathbf{QA}(\Sigma)$.

$\mathcal{A} = (A, \Sigma^{\mathcal{A}})$ *satisfies* the conditional quantitative equation $\Gamma \vdash t =_\varepsilon s$ in $\mathcal{E}(\Sigma, X)$, written $\Gamma \models_{\mathcal{A}} t =_\varepsilon s$, if for any assignment $\iota \colon X \to A$, the following implication holds

$$\left(\forall t' =_{\varepsilon'} s' \in \Gamma \, , d_A(\iota(t'), \iota(s')) \leq \varepsilon'\right) \Rightarrow d_A(\iota(t), \iota(s)) \leq \varepsilon \, ,$$

where $\iota(t)$ is the homomorphic interpretation of t in $\mathcal{A}$.

A quantitative algebra $\mathcal{A}$ is said to *satisfy* (or be a *model* for) the quantitative theory $\mathcal{U}$, if $\Gamma \models_{\mathcal{A}} t =_\varepsilon s$ whenever $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$. We write $\mathbb{K}(\Sigma, \mathcal{U})$ for the collection of models of a theory $\mathcal{U}$ of type Σ.

3.1. **Free Monads on Quantitative Equational Theories.** To every signature Σ, one can associate a *signature endofunctor* (also called Σ) on **Met** by:

$$\Sigma = \coprod_{f:n \in \Sigma} Id^n \, .$$

It is easy to see that, by couniversality of the coproduct, quantitative Σ-algebras correspond to Σ-algebras for the functor Σ in **Met**, and the morphisms between them to non-expansive homomorphisms of Σ-algebras. In the rest of the paper, we will pass between these two points of view as convenient.

In [30] it is shown that any quantitative theory $\mathcal{U}$ of type Σ induces a monad $T_{\mathcal{U}}$ on **Met**, called the *free monad on* $\mathcal{U}$. The result leading to its definition is summarized in the following theorem.

Theorem 3.3 (Free Algebra [30])**.** *The forgetful functor* $\mathbb{K}(\Sigma, \mathcal{U}) \to$ **Met** *has a left adjoint.*

The left adjoint assigns to any $X \in$ **Met** a *free quantitative* Σ*-algebra* $(T_X, \psi^{\mathcal{U}}_X)$ satisfying the quantitative theory $\mathcal{U}$, from which one canonically obtains the monad $(T_{\mathcal{U}}, \eta^{\mathcal{U}}, \mu^{\mathcal{U}})$, with functor $T_{\mathcal{U}} \colon$ **Met** $\to$ **Met** mapping $X \in$ **Met** to the carrier T_X of the free algebra.

Directly from the universal property of the adjunction, we get that for any quantitative Σ-algebra $(A, a) \in \mathbb{K}(\Sigma, \mathcal{U})$ and non-expansive map $\beta \colon X \to A$, there exists a unique homomorphism $h \colon T_{\mathcal{U}} X \to A$ of quantitative Σ-algebras making the diagram below commute

$$\begin{tikzcd}
X \arrow[r, "\eta^{\mathcal{U}}_X"] \arrow[dr, "\beta"'] & T_{\mathcal{U}} X \arrow[d, dashed, "h"] & \Sigma T_{\mathcal{U}} X \arrow[l, "\psi^{\mathcal{U}}_X"'] \arrow[d, "\Sigma h"] \\
& A & \Sigma A \arrow[l, "a"]
\end{tikzcd}$$

The map h is called the *homomorphic extension of* a *along* β.

Notice that, homomorphic extensions provide us with a way of defining maps from $T_{\mathcal{U}} X$, for generic $X \in$ **Met**. For example, the multiplication $\mu^{\mathcal{U}} : T_{\mathcal{U}} T_{\mathcal{U}} \Rightarrow T_{\mathcal{U}}$ is defined at component X as the homomorphic extension of $\psi^{\mathcal{U}}_X$ along $id_{T_{\mathcal{U}}X}$ (*i.e.*, the unique map such that $\mu^{\mathcal{U}}_X \circ \eta^{\mathcal{U}}_X = id_{T_{\mathcal{U}}X}$ and $\mu^{\mathcal{U}}_X \circ \psi^{\mathcal{U}}_{T_{\mathcal{U}}X} = \psi^{\mathcal{U}}_X \circ \Sigma\mu^{\mathcal{U}}_X$).

Fact 3.4 (The Quantitative Term Monad)**.** In [30], the monad $T_{\mathcal{U}}$ has been characterised in the form of a "quantitative term monad". Concretely, $T_{\mathcal{U}}X$ is defined as the set of ground terms constructed over the signature Σ extended with constants from X. This set of terms can be (pseudo)metrized by the following distance function:

$$d_{T_{\mathcal{U}}X}(t,s) = \inf\{\varepsilon \mid \vdash t =_\varepsilon s \text{ is provable from } \mathcal{U} \text{ and } \Gamma_X\}$$

where $\Gamma_X = \{\vdash x =_\delta y \mid d_X(x,y) \le \delta\}$. Intuitively, the distance between two terms t, s is the smallest ε such that $\vdash t =_\varepsilon s$ is deducible by using quantitative equations from the theory $\mathcal{U}$ and axioms $\vdash x =_\delta y \in \Gamma_X$ between constant terms $x, y \in X$; if $\not\vdash t =_\varepsilon s$ (not provable) for any $\varepsilon \in \mathbb{Q}_{\ge 0}$, then the distance is ∞. The terms t, s are considered equal in $T_{\mathcal{U}}X$ if $\vdash t =_0 s$ can be deduced (hence, terms modulo 0-provability).

The unit and multiplication act as in a standard term monad: the unit $\eta^{\mathcal{U}}_X \colon X \to T_{\mathcal{U}}X$ interprets the elements of X as terms; the multiplication $\mu^{\mathcal{U}}_X \colon T_{\mathcal{U}}T_{\mathcal{U}}X \to T_{\mathcal{U}}X$ takes a term over terms in $T_{\mathcal{U}}T_{\mathcal{U}}X$ and flattens it out into a single term $T_{\mathcal{U}}X$ by term composition. The key detail is that these maps are non-expansive w.r.t. the distance defined above.

In [3], it is shown that, whenever the quantitative theory $\mathcal{U}$ is *basic*, *i.e.*, it can be axiomatised by a set of conditional equations of the form

$$\{x_1 =_{\varepsilon_1} y_1, \dots, x_n =_{\varepsilon_n} y_n\} \vdash t =_\varepsilon s\,,$$

where $x_i, y_i \in X$ (*cf.* [31]), then the EM algebras for $T_{\mathcal{U}}$ are in 1-1 correspondence with the quantitative algebras satisfying $\mathcal{U}$:

Theorem 3.5. *For any basic quantitative theory $\mathcal{U}$ of type Σ, $\mathbf{EM}(T_{\mathcal{U}}) \cong \mathbb{K}(\Sigma,\mathcal{U})$.*

3.2. **Completion of Quantitative Algebras.** Sometimes it is convenient to consider the quantitative Σ-algebras whose carrier is a complete extended metric space. This class of algebras forms a full subcategory of $\mathbf{QA}(\Sigma)$: the category of complete quantitative algebras, denoted $\mathbf{CQA}(\Sigma)$.

Then, it is natural to ask whether the standard Cauchy completion of metric spaces lifts to a notion of Cauchy completion of quantitative algebras. This is done as follows:

Definition 3.6. (Algebra Completion) The *Cauchy completion of a quantitative Σ-algebra* $\mathcal{A} = (A, \Sigma^{\mathcal{A}})$, is the quantitative Σ-algebra $\overline{\mathcal{A}} = (\overline{A}, \Sigma^{\overline{\mathcal{A}}})$, where $\overline{A}$ is the Cauchy completion of A and $\Sigma^{\overline{\mathcal{A}}} = \{f^{\overline{\mathcal{A}}} \colon \overline{A}^n \to \overline{A} \mid f \colon n \in \Sigma\}$ is such that for Cauchy sequences $(b^i_j)_j$ converging to $b^i \in \overline{A}$, for $1 \le i \le n$,

$$f^{\overline{\mathcal{A}}}(b^1, \dots, b^n) = \lim_j f^{\mathcal{A}}(b^1_j, \dots, b^n_j)\,.$$

The above definition extends to a functor $\mathbb{C} \colon \mathbf{QA}(\Sigma) \to \mathbf{CQA}(\Sigma)$, also called Cauchy completion functor, mapping a quantitative algebra to its completion. As it is for metric spaces, also this functor is the left adjoint to the embedding $\mathbf{CQA}(\Sigma) \hookrightarrow \mathbf{QA}(\Sigma)$.

Let $\mathcal{U}$ be a quantitative equational theory. Similarly to before, we consider the full subcategory of complete quantitative Σ-algebras that are models of $\mathcal{U}$, denoted by $\mathbb{CK}(\Sigma,\mathcal{U})$. Then, given an algebra $\mathcal{A}$ satisfying $\mathcal{U}$, an interesting question is whether its completion

$\overline{\mathcal{A}}$ is still a model for all the equations in $\mathcal{U}$. In other words, does the Cauchy completion functor restricts to $\mathbb{C}\colon \mathbb{K}(\Sigma,\mathcal{U}) \to \mathbb{CK}(\Sigma,\mathcal{U})$?

The answer is positive, whenever $\mathcal{U}$ can be axiomatised by a collection of *continuous schemata* of quantitative equations, *i.e.*, sets of quantitative equations of the form

$$\{x_i =_{\varepsilon_i} y_i \mid i = 1..n\} \vdash t =_\varepsilon s\,, \qquad \text{for all } \varepsilon \geq f(\varepsilon_1, \dots, \varepsilon_n),$$

where $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous real-valued function, $\varepsilon, \varepsilon_i \in \mathbb{Q}_{\geq 0}$, and $x_i, y_i \in X$. We call such a theory *continuous.*

Remark 3.7. Asking a theory to be continuous is necessary. For a counterexample consider the theory $\mathcal{U}$ for a signature having a single unary function symbol $g\colon 1$, with axioms

$$\begin{aligned} &(A1)\ \{x =_\varepsilon y\} \vdash g(x) =_\varepsilon y\,, \qquad \text{where } \varepsilon < 1\,, \\ &(A2)\ \{x =_\varepsilon y\} \vdash g(x) =_0 y\,, \qquad \text{where } \varepsilon \geq 1\,. \end{aligned}$$

A quantitative algebra that is a model of $\mathcal{U}$ is given by open interval $[0, 1)$ (with usual metric) by interpreting g as the identity function. The Cauchy completion of this algebra has the closed interval $[0, 1]$ as carrier and interprets g again as the identity. However, (A2) is not satisfied in $[0, 1]$ as this would require $g(0) = 1$.

When $\mathcal{U}$ is a continuous theory, the Cauchy completion functor $\mathbb{C}\colon \mathbb{K}(\Sigma,\mathcal{U}) \to \mathbb{CK}(\Sigma,\mathcal{U})$ is left adjoint to the functor embedding $\mathbb{CK}(\Sigma,\mathcal{U})$ into $\mathbb{K}(\Sigma,\mathcal{U})$.

Moreover, for this class of theories a similar result to Theorem 3.3 also holds.

Theorem 3.8 (Free Complete Algebra [30])**.** *For any continuous quantitative equational theory $\mathcal{U}$ of type Σ, the forgetful functor $\mathbb{CK}(\Sigma,\mathcal{U}) \to$ **CMet** has a left adjoint.*

As a direct consequence of the above and Theorem 3.3, when $\mathcal{U}$ is continuous, for any $X \in$ **CMet**, quantitative Σ-algebra (A, a) in $\mathbb{CK}(\Sigma,\mathcal{U})$ and non-expansive map $\beta\colon X \to A$, there exists a unique homomorphism $h\colon \mathbb{C}T_\mathcal{U} \to A$ making the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "\mathbb{C}\eta^{\mathcal{U}}_X"] \arrow[dr, "\beta"'] & \mathbb{C}T_\mathcal{U}X \arrow[d, dashed, "h"] & \Sigma\mathbb{C}T_\mathcal{U}X \arrow[l, "\mathbb{C}\psi^{\mathcal{U}}_X"'] \arrow[d, "\Sigma h"] \\
& A & \Sigma A \arrow[l, "a"']
\end{tikzcd}$$

This, in particular, tells us that $(\mathbb{C}T_\mathcal{U}X, \mathbb{C}\psi^\mathcal{U}_X)$ is the *free complete quantitative algebra* for an arbitrary metric space X, implying that $\mathbb{C}T_\mathcal{U}$ is the free monad on $\mathcal{U}$ in **CMet**.

Note that, by definition, continuous theories are basic. Thus, by essentially the same arguments of Theorem 3.5, we have a 1-1 correspondence between the EM algebras for $\mathbb{C}T_\mathcal{U}$ and the quantitative algebras satisfying a continuous theory $\mathcal{U}$.

Theorem 3.9. *For any continuous quantitative theory $\mathcal{U}$ of type Σ,* $\mathbf{EM}(\mathbb{C}T_\mathcal{U}) \cong \mathbb{CK}(\Sigma,\mathcal{U})$.

4. Algebraic Presentation of Monads over Metric Spaces: Examples

A presentation of a **Set** monad T is an algebraic theory (Σ, E) (*i.e.*, a signature Σ and a set E of equations $s = t$ between Σ-terms) such that the full subcategory of the universal algebras that satisfy the all equations in E is isomorphic to the Eilenberg-Moore category $\mathbf{EM}(T)$. If T has a presentation (Σ, E), then it is algebraic, because it is isomorphic to the (term) monad T_E freely generated from the equations in E.

As in this paper we deal with monads on **Met**, their presentations will be given in terms of *quantitative algebraic theories* $(\Sigma, \mathcal{U})$ (*i.e.*, a signature Σ and a quantitative equational theory $\mathcal{U}$ of type Σ) and, in complete analogy with the above, $(\Sigma, \mathcal{U})$ is a presentation for T, if the category of quantitative algebras that are models of $\mathcal{U}$ is isomorphic to the Eilenberg-Moore category of T (in short, $\mathbb{K}(\Sigma, \mathcal{U}) \cong \mathbf{EM}(T)$).

In this section, we propose quantitative versions of several **Set** monads classically used as computational effects in programming languages, and for each of them provide a quantitative equational presentation in the sense explained above. The computational effects we consider are: termination and exceptions (Section 4.1), interactive input (Section 4.2), reading/writing (Section 4.3), nondeterminism (Section 4.4), and probabilistic choice (Section 4.5).

4.1. **Termination and Exceptions.** The monadic effect for termination in **Set** is given by the *termination monad* (a.k.a. *maybe monad*), denoted by $(- + 1)$, that maps a set X to $X + 1$, where $+$ denotes the coproduct (hence, disjoint union) and $1 = \{*\}$ is the terminal object in **Set**. The unit and multiplication are canonically defined from the universal property of the coproduct:

$$in_{(-)} \colon Id \to (- + 1) \qquad \textsc{unit}$$
$$[in_{(-)} \circ in_{(-+1)}, in_1] \colon ((- + 1) + 1) \to (- + 1) \qquad \textsc{multiplication}$$

A slight generalisation is given by the *exception monad*, denoted by $(- + E)$, that maps a set X to $X + E$, where E is a fixed set of exceptions. Intuitively, an effectful computation of this type, rather than just terminating, allows one to raise an exception $e \in E$ providing extra information on the causes of the termination. This monad have a very simple algebraic presentation in terms of a signature

$$\Sigma_E = \{\mathsf{raise}_e \colon 0 \mid e \in E\}\,,$$

having only nullary operation symbols (*i.e.*, constants) raise_e, for each exception $e \in E$, with equational theory being the trivial one (it contains only identities $t = t$ between terms constructed over the signature).

In the quantitative case, the corresponding exception monad on **Met** is still given by $Id + E$, with the only difference being that now E is an extended metric space with metric measuring the distance between exceptions. Computationally, this means that one may measure the difference of the different type of terminations. This interpretation can be useful, for example, in scenarios where exceptions carry the time-stamps of the moment they have been thrown. In this way one can compare program implementations by measuring the frequency of which exception are thrown.

For E an extended metric space of exceptions, we define the *quantitative algebraic theory of exceptions* over E, by taking the same signature as above, namely Σ_E, and adding to the theory the quantitative equations

$$\vdash \mathsf{raise}_{e_1} =_\delta \mathsf{raise}_{e_2}\,, \qquad \text{for } \delta \geq d_E(e_1, e_2)\,,$$

for any pair of exceptions $e_1, e_2 \in E$ and positive rational δ. The rôle of this axiom is to lift to the set of terms the underlying metric of E. We denote this quantitative theory by $\mathcal{E}_E$.

It is not difficult to show that for any $X \in \mathbf{Met}$, the quantitative Σ_E-algebra $(X+E, \phi_X)$ interpreting $\mathsf{raise}_e \colon 0 \in \Sigma_E$ as $e \in X + E$ for each exception $e \in E$, formally defined by

$$\phi_X \colon \Sigma_E(X + E) \to X + E \qquad\qquad \phi_X(in_{\mathsf{raise}_e}(*)) = e\,,$$

is isomorphic to the free quantitative algebra in $\mathbb{K}(\Sigma_E, \mathcal{E}_E)$. From this we obtain:

Theorem 4.1. *The monads $T_{\mathcal{E}_E}$ and $(- + E)$ on* **Met** *are isomorphic.*

As the quantitative theory $\mathcal{E}_E$ is basic, by Theorems 3.5 and 4.1, we have that $(\Sigma_E, \mathcal{E}_E)$ is a presentation of the exception monad $Id + E$ on **Met** (*i.e.*, $\mathbf{EM}(Id + E) \cong \mathbb{K}(\Sigma_E, \mathcal{E}_E)$).

The exception monad $(- + E)$ is well defined also in **CMet**, the only difference being that one assumes E to a complete metric space of exceptions. As the theory $\mathcal{E}_E$ is continuous, by similar arguments to the above, $(\Sigma, \mathcal{E}_E)$ provides us with a presentation of this monad also on **CMet**, that is $\mathbf{EM}((- + E)) \cong \mathbb{CK}(\Sigma_E, \mathcal{E}_E)$.

Theorem 4.2. *The monads $\mathbb{C}T_{\mathcal{E}_E}$ and $(- + E)$ on* **CMet** *are isomorphic.*

4.2. **Interactive Input.** Interactive input on a (nonempty) finite set $I = \{i_1, \dots, i_n\}$ of symbols, can be expressed by a n-ary operation $\mathsf{input}(t_1, \dots, t_n)$ representing the computation that proceeds as t_j on input i_j. In **Set**, the corresponding monadic effect is given in terms of the free monad on $Id^{|I|}$, with algebraic presentation given by the trivial equational theory with no axioms on the input operations.

In the quantitative setting, one may wish the input operation to be contractive (*i.e.*, c-Lipschitz continuous for some $0 \leq c < 1$) so that repeated input operations eventually converge to a fixed point on complete metric spaces (*cf.* Banach fixed point theorem). This can be expressed by means of the following quantitative equations

$$\{x_1 =_\varepsilon y_1, \dots, x_n =_\varepsilon y_n\} \vdash \mathsf{input}(x_1, \dots, x_n) =_\delta \mathsf{input}(y_1, \dots, y_n)\,, \qquad \text{for } \delta \geq c\varepsilon$$

expressing that the input operation is contractive (with contractive factor c).

The corresponding quantitative monadic effect on **Met** (and **CMet** too) is given by the free monad on $c \cdot Id^{|I|}$, where $c \cdot Id$ is the *rescaling functor*, mapping a metric space (X, d_X) to $(X, c \cdot d_X)$.

The quantitative algebras for interactive inputs described above is a particular instance of the *algebras of contractive operators* from [3], which we recall below.

4.2.1. *Algebras of contractive operators.* A signature of contractive operators Σ is a (at most countable) collection of function symbols f with associated arity $n \in \mathbb{N}$ and *contractive factor* $0 < c < 1$. We write this as $f : \langle n, c\rangle \in \Sigma$. The quantitative theory for Σ, written $\mathcal{O}(\Sigma)$, is the smallest theory satisfying, for each $f : \langle n, c\rangle \in \Sigma$, the quantitative equations

$$(f\text{-}\mathsf{Lip})\ \{x_1 =_\varepsilon y_1, \dots, x_n =_\varepsilon y_n\} \vdash f(x_1, \dots, x_n) =_\delta f(y_1, \dots, y_n)\,, \qquad \text{for } \delta \geq c\varepsilon\,.$$

The axiom (f-Lip) is just asking the interpretation of f to be c-Lipschitz continuous.

The quantitative algebras that are models for $\mathcal{O}(\Sigma)$ are called *algebras of the contractive signature* Σ, and we denote their category as $\mathbb{K}(\Sigma, \mathcal{O}(\Sigma))$.

4.2.2. *Monads of contractive operators.* For a contractive signature Σ, we define a modification of the signature endofunctor on **Met** by:

$$\tilde{\Sigma} = \coprod_{f\colon \langle n,c\rangle \in \Sigma} c \cdot Id^n \,. \tag{4.1}$$

It is not difficult to show that the quantitative Σ-algebras satisfying $\mathcal{O}(\Sigma)$ are in one-to-one correspondence with the algebras of $\tilde{\Sigma}$, that is $\mathbb{K}(\Sigma, \mathcal{O}(\Sigma)) \cong \tilde{\Sigma}$-**Alg**. Moreover, this isomorphism maps carrier identically. In virtue of this we will pass between these two categories of view as convenient, and say that an algebra of $\tilde{\Sigma}$ satisfies $\mathcal{O}(\Sigma)$.

Next we show that the free monad $T_{\mathcal{O}(\Sigma)}$ is isomorphic to $\tilde{\Sigma}^*$, the free monad on $\tilde{\Sigma}$. For this result, we first need some discussion about sufficient conditions for the existence free monads on an endofunctor.

Remark 4.3. Given any endofunctor H on a category **C**, we write $(\mu y.Hy, \alpha_H)$ for the initial H-algebra, if it exists. If **C** has binary coproducts, the free H-algebra on $X \in \mathbf{C}$ can be identified with $(\mu y.(Hy + X), \alpha_{H+X})$, and the one exists if and only if the other does. These free algebras exist if, for example, **C** is locally countably presentable and H has countable rank. In this case the forgetful functor $U^H\colon H$-**Alg** $\to$ **C** has a left adjoint, mapping **C**-objects to their corresponding free H-algebra.

We see from Remark 4.3 that, if **C** has binary sums, then H^* can be identified with $\mu y.(Hy + -)$ and the former exists if and only if the other does. We further see that if **C** is locally countably presentable and H has countable rank, then H^* exists [24]. Moreover, as H^* is algebraic, the Eilenberg-Moore category $\mathbf{EM}(H^*)$ is isomorphic to the category H-**Alg** of algebras of H (see Section 2.2).

Therefore, since **Met** is locally countably presentable [2] (see also Appendix B) and $\tilde{\Sigma}$ has countable rank, the free algebra for $\tilde{\Sigma}$ exists and so does the free monad $\tilde{\Sigma}^*$.

As $\mathbf{EM}(\tilde{\Sigma}^*)$ and $\mathbb{K}(\Sigma, \mathcal{O}(\Sigma))$ are isomorphic and $\mathcal{O}(\Sigma)$ is basic, by freeness of $T_{\mathcal{O}(\Sigma)}$ (Theorem 3.3) the following holds:

Theorem 4.4. *The monads $T_{\mathcal{O}(\Sigma)}$ and $\tilde{\Sigma}^*$ on* **Met** *are isomorphic.*

The situation is similar in the category **CMet** of complete extended metrics. As **CMet** has coproducts and finite products, and rescaling a metric by a factor $0 < c < 1$ preserves completeness, for any contractive signature Σ, the endofunctor $\tilde{\Sigma}$ defined as in (4.1) is well defined in **CMet**. Moreover, **CMet** is locally countably presentable [2] and, since $\tilde{\Sigma}$ has countable rank, by Remark 4.3 the free monad $\tilde{\Sigma}^*$ on **CMet** exists and is algebraic.

Similarly to the previous case, also in this time the Eilenberg-Moore category $\mathbf{EM}(\tilde{\Sigma}^*)$ is isomorphic to $\mathbb{CK}(\Sigma, \mathcal{O}(\Sigma))$. As $\mathcal{O}(\Sigma)$ is a continuous quantitative theory, by Theorem 3.9 and repeating the same argument we used before, we obtain:

Theorem 4.5. *The monads $\mathbb{C}T_{\mathcal{O}(\Sigma)}$ and $\tilde{\Sigma}^*$ on* **CMet** *are isomorphic.*

4.3. **Reading/Writing.** The monadic effects for reading and writing in **Set** are respectively given by the, so called, reader and writer monads. These effects, respectively, allow a computation to read from a finite list of globally declared variables, and write on an output tape though to record annotations or just used as standard output. Their formal definitions are recalled below.

Given a set E of input values, the *reader monad* on **Set**, denoted by $(-)^E$, maps a set X to X^E, the set all of functions from E to X, and acts on morphism $f\colon X \to Y$ as $f^E(g) = f \circ g$, for all $g \in X^E$. The unit $\kappa\colon Id \Rightarrow (-)^E$ and multiplication $\zeta\colon ((-)^E)^E \Rightarrow (-)^E$ are respectively given as follows, for $x \in X$, $e \in E$, and $g\colon E \to X^E$

$$\begin{aligned} &\kappa_X(x)(e) = x && \text{UNIT} \\ &\zeta_X(f)(e) = f(e)(e) && \text{MULTIPLICATION} \end{aligned} \tag{4.2}$$

Given a set Λ of output values having monoid structure $(\Lambda, *, 0)$, the *writer monad* on **Set**, denoted by $(\Lambda \times -)$, acts on sets X as $\Lambda \times X$, where $\times$ denotes the product (hence, Cartesian product), and on morphisms $f\colon X \to Y$ as $(\Lambda \times f)(\alpha, x) = (\alpha, x)$, for $x \in X$ and $\alpha \in \Lambda$. The unit $\tau\colon Id \Rightarrow (\Lambda \times -)$ and multiplication $\varsigma\colon (\Lambda \times (\Lambda \times -)) \Rightarrow (\Lambda \times -)$ are respectively given as follows, for $x \in X$ and $\alpha, \alpha' \in \Lambda$

$$\begin{aligned} \tau_X(x) &= (0, x)\,, && \text{UNIT} \\ \varsigma_X((\alpha, (\alpha', x))) &= (\alpha * \alpha', x) && \text{MULTIPLICATION} \end{aligned} \tag{4.3}$$

In the quantitative case, one wish to define analogous monads on the category **Met** of extended metric spaces. However, extra care has to be taken as the well definition of the above monads crucially exploits the Cartesian closed structure of **Set**, and we already have seen that **Met** is not Cartesian closed (Example 2.1).

Remark 4.6. The reader monad is always well defined in a cartesian closed category **C**. Fix an object $E \in \mathbf{C}$. The reader monad $(-)^E$ has unit and multiplication respectively given by

$$X \cong X^1 \xrightarrow{X^!} X^E \qquad \text{and} \qquad (X^E)^E \cong X^{E \times E} \xrightarrow{X^\delta} X^E\,,$$

where $!\colon E \to 1$ is the unique map to the terminal object and $\delta\colon E \to E \times E$ the diagonal map $\delta = \langle id, id \rangle$. However, this definition does not generalise to arbitrary monoidal closed categories, and **Met** is such a counterexample. The specific problem with **Met** is that $\delta\colon E \to E \,\Box\, E$ is not well-defined for arbitrary $E \in \mathbf{Met}$, as non-expansivness requires that

$$d_E(e, e') \geq d_{E \Box E}(\delta(e), \delta(e')) = d_E(e, e') + d_E(e, e')\,,$$

which holds only when E has the discrete metric, *i.e.*, the one that assigns infinite distance to any pair of distinct elements. From this we see that a quantitative analog of the reader monad can be obtained if we restrict our attention only to spaces E with discrete metric.

For a set E denote by $\underline{E}$ the corresponding extended metric space equipped with discrete metric. The reader monad on **Met**, denoted by $(-)^{\underline{E}}$, assigns to each $X \in \mathbf{Met}$ the internal hom $[E, X]$ of (necessarily non-expansive) maps from $\underline{E}$ to X with point-wise supremum metric (*cf.* Example 2.1(2)) and acts on morphism $f\colon X \to Y$ as $f^E(g) = f \circ g$, for all $g \in [E, X]$. The unit and multiplication are defined as in (4.2), where non-expansiveness for the multiplication's components at $X \in \mathbf{Met}$ follows because $\underline{E}$ has discrete metric.

As for a quantitative analog of the writer monad, we will assume the set of output values Λ to be an extended metric space and further require its monoid structure $(\Lambda, *, 0)$ to have non-expansive multiplication $*\colon \Lambda \times \Lambda \to \Lambda$. Then, the writer monad on **Met**, denoted by $(\Lambda \,\Box\, -)$, acts on objects $X \in \mathbf{Met}$ as $(\Lambda \,\Box\, X)$, where $\Box$ denotes the monoidal product discussed in Example 2.1(2), and on morphisms $f\colon X \to Y$ as $(\Lambda \,\Box\, f)(\alpha, x) = (\alpha, f(x))$, and $\alpha \in \Lambda$. The unit and multiplication are defined as in (4.3), where non-expansiveness for the multiplication's components at $X \in \mathbf{Met}$ crucially depends on the assumption that the monoidal operation $*$ on Λ is non-expansive.

Below we provide quantitative equational representations for these two **Met** monads.

4.3.1. *Reader Algebras.* Let $E = \{e_1, \dots, e_n\}$ be a finite set of input values of which we assume a fixed enumeration. The *quantitative reader algebras* of type E are the algebras for the signature

$$\Sigma_{\mathcal{R}_E} = \{\mathsf{r} \colon |E|\}$$

having only one operator r of arity equal to the number of the input values in E, and satisfying the following quantitative equations

$$(\mathsf{Idem}) \vdash x =_0 \mathsf{r}(x, \dots, x)\,,$$
$$(\mathsf{Diag}) \vdash \mathsf{r}(x_{1,1}, \dots, x_{n,n}) =_0 \mathsf{r}(\mathsf{r}(x_{1,1}, \dots, x_{1,n}), \dots, \mathsf{r}(x_{n,1}, \dots, x_{n,n}))\,.$$

We call the quantitative theory induced by the equations above, written $\mathcal{R}_E$ (or simply $\mathcal{R}$ when E is clear), *quantitative theory of reading computations.*

Intuitively, the term $\mathsf{r}(t_1, \dots, t_n)$ can be interpreted as the computation that proceeds as t_i after reading the value e_i from its input. So r describes the operation of reading from an input with values in E. The equation (Idem) says that if we ignore the value of the input the reading of it is not observable; (Diag) says that the resulting computation after reading the input is the same no matter how many times we read it.

Remark 4.7. For the binary case ($|E| = 2$) we can think of r as an *if-then-else* statement $b?(S, T)$ checking for the value of a fixed global Boolean variable b and proceeding as S when $b = \text{true}$, and as T otherwise. In this case, (Idem) and (Diag) express the standard program equivalences

$$S \equiv b?(S, S) \qquad \text{and} \qquad b?(S, T) \equiv b?\big(b?(S, T), b?(S, T)\big)\,.$$

We should also remark that (Idem) and (Diag) are purely equational judgements and are the equations presenting the reader monad $(-)^E$ on **Set**. Shortly, we will see that they also provide a presentation for the reader monad $(-)^{\underline{E}}$ on **Met**. This should not surprise, since having input symbols equipped with the discrete metric makes the two monad essentially "equivalent".

Next we show that the reader monad $(-)^{\underline{E}}$ is isomorphic to the free monad $T_{\mathcal{R}}$ on $\mathcal{R}$. Consequently, as the theory $\mathcal{R}$ is basic, by Theorem 3.5, $\mathbf{EM}((-)^{\underline{E}}) \cong \mathbb{K}(\Sigma_{\mathcal{R}}, \mathcal{R})$. In other words, $(\Sigma_{\mathcal{R}}, \mathcal{R})$ is a quantitative equational presentation of the monad $(-)^{\underline{E}}$ on **Met**.

Theorem 4.8. *The monads $T_{\mathcal{R}}$ and $(-)^{\underline{E}}$ in* **Met** *are isomorphic.*

Proof. We prove this statement by showing that, for each $X \in \mathbf{Met}$, $X^{\underline{E}}$ has a quantitative algebraic structure that is free in $\mathbb{K}(\Sigma_{\mathcal{R}}, \mathcal{R})$ with universal natural arrow κ_X; and show that the units and multiplications of the two monads coincide (up-to iso).

For any $X \in \mathbf{Met}$, we define the quantitative $\Sigma_{\mathcal{R}}$-algebra $(X^{\underline{E}}, \rho_X)$ as follows, for arbitrary maps $f_1, \dots, f_n \colon \underline{E} \to X$

$$\rho_X \colon \Sigma_{\mathcal{R}} X^{\underline{E}} \to X^{\underline{E}} \qquad\qquad \rho_X(in_{\mathsf{r}}(f_1, \dots, f_n))(e_i) = f_i(e_i)\,.$$

Next we show that it satisfies $\mathcal{R}$. For convenience, let $\mathsf{r}^\rho = \rho_X \circ in_{\mathsf{r}}$ denote the interpretation of the operator symbol $\mathsf{r} \colon n \in \Sigma_{\mathcal{R}}$ in the algebra $(X^{\underline{E}}, \rho_X)$. Soundness for

(r-NE) follows by the fact that ρ_X is a well defined map in **Met** as shown below

$$\begin{aligned}
d_{X^{\underline{E}}}(\mathsf{r}^\rho(f_1,\dots,f_n),\mathsf{r}^\rho(g_1,\dots,g_n)) &= \sup_{e_i} d_X(\mathsf{r}^\rho(f_1,\dots,f_n)(e_i),\mathsf{r}^\rho(g_1,\dots,g_n)(e_i)) \\
&= \sup_{e_i} d_X(f_i(e_i),g_i(e_i)) \\
&\le \max_j \Big(\sup_{e_i\in E} d_X(f_j(e_i),g_j(e_i))\Big) \\
&\le \max_j d_{X^{\underline{E}}}(f_j,g_j)\,.
\end{aligned}$$

Soundness for (Idem) follows by definition of ρ as, for all $e_i \in E$

$$\mathsf{r}^\rho(f,\dots,f)(e_i) = f(e_i)\,.$$

Soundness for (Diag) also follows by definition, as

$$\begin{aligned}
\mathsf{r}^\rho(\mathsf{r}^\rho(f_{1,1},\dots,f_{1,n})),\dots,\mathsf{r}^\rho(f_{n,1},\dots,f_{n,n}))(e_i) &= \mathsf{r}^\rho(f_{i,1},\dots,f_{i,n})(e_i) \\
&= f_{i,i}(e_i) \\
&= \mathsf{r}^\rho(f_{1,1},\dots,f_{n,n})(e_i)\,.
\end{aligned}$$

Now we prove freeness. Let $(A,a) \in \mathbb{K}(\Sigma_{\mathcal{R}},\mathcal{R})$ and $\beta\colon X \to A$ a non-expansive map. Define $h\colon X^{\underline{E}} \to A$ as follows, for arbitrary $f\colon \underline{E} \to X$

$$h(f) = a(in_{\mathsf{r}}(\beta(f(e_1)),\dots,\beta(f(e_n))))\,.$$

As it is the composition of non-expansive maps, then also h is non-expansive. Next we prove that h is the only map making the diagram below commute.

$$\begin{tikzcd}
X \arrow[r, "\kappa_X"] \arrow[dr, "\beta"'] & X^{\underline{E}} \arrow[d, dashed, "h"] & \Sigma_{\mathcal{R}} X^{\underline{E}} \arrow[l, "\rho_X"'] \arrow[d, "\Sigma_{\mathcal{R}} h"] \\
& A & \Sigma_{\mathcal{R}} A \arrow[l, "a"]
\end{tikzcd}$$

Let $\mathsf{r}^\rho = \rho_X \circ in_{\mathsf{r}}$ and $\mathsf{r}^a = a \circ in_{\mathsf{r}}$ denote the interpretations of $\mathsf{r}\colon n \in \Sigma_{\mathcal{R}}$, respectively, in the algebras $(X^{\underline{E}},\rho_X)$ and (A,a). Let $x \in X$. Then

$$\begin{aligned}
(h\circ\kappa_X)(x) &= \mathsf{r}^a(\beta(\kappa_X(x)(e_1)),\dots,\beta(\kappa_X(x)(e_n))) && \text{(def. } h) \\
&= \mathsf{r}^a(\beta(x),\dots,\beta(x)) && \text{(def. } \kappa) \\
&= \beta(x)\,. && (\mathsf{Idem})
\end{aligned}$$

Let $f_1,\dots,f_n\colon \underline{E} \to X$. Then

$$\begin{aligned}
(h\circ\rho_X)(in_{\mathsf{r}}(f_1,\dots,f_n)) &= \mathsf{r}^a(\beta(f_1(e_1)),\dots,\beta(f_n(e_n))) && \text{(def. } h \text{ and } \rho) \\
&= \mathsf{r}^a\Big(\mathsf{r}^a(\beta(f_1(e_1)),\dots,\beta(f_1(e_n))),\dots \\
&\qquad \dots,\mathsf{r}^a(\beta(f_n(e_1)),\dots,\beta(f_n(e_1)))\Big) && (\mathsf{Diag}) \\
&= \mathsf{r}^a(h(f_1),\dots,h(f_n)) && \text{(def. } h) \\
&= (a\circ\Sigma_{\mathcal{R}} h)(in_{\mathsf{r}}(f_1,\dots,f_n))\,. && \text{(def. } \mathsf{r}^a \text{ and } \Sigma_{\mathcal{R}})
\end{aligned}$$

Hence h is a $\Sigma_{\mathcal{R}}$-homomorphism, that is, $h\circ\rho_X = a\circ\Sigma_{\mathcal{R}} h$.

It remains to prove the uniqueness. Assume exists $g\colon X^{\underline{E}} \to A$ such that $g \circ \kappa_X = \beta$ and $g \circ \rho_X = a \circ \Sigma_{\mathcal{R}} g$. Next we show $h = g$. Notice first that for any $f\colon X^{\underline{E}} \to X$, $f = \mathsf{r}^{\rho}(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))$, as for all $e_i \in E$, the following holds:

$$
\begin{aligned}
f(e_i) &= \kappa_X(f(e_i))(e_i) && \text{(def. } \kappa\text{)} \\
&= \mathsf{r}^{\rho}(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))(e_i)\,. && \text{(def. } \rho\text{)}
\end{aligned}
$$

From the above we have that, for all $f\colon X^{\underline{E}} \to X$,

$$
\begin{aligned}
h(f) &= h(\mathsf{r}^{\rho}(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))) \\
&= \mathsf{r}^{a}((h \circ \kappa)(f(e_1)), \dots, (h \circ \kappa)(f(e_1))) && (h \text{ homo}) \\
&= \mathsf{r}^{a}(\beta(f(e_1)), \dots, \beta(f(e_1))) && (h \circ \kappa = \beta) \\
&= \mathsf{r}^{a}((g \circ \kappa)(f(e_1)), \dots, (g \circ \kappa)(f(e_1))) && (g \circ \kappa = \beta) \\
&= g(\mathsf{r}^{\rho}(\kappa_X(f(e_1)), \dots, \kappa_X(f(e_n)))) && (g \text{ homo}) \\
&= g(f)
\end{aligned}
$$

Therefore, $g = h$.

By the proof of freeness above, the functors $(-)^{\underline{E}}$ and $T_{\mathcal{R}}$ are isomorphic and the units of the two monads coincide (up-to iso). We are left to prove that also the multiplications coincide (up-to iso). By the universal property of free algebras, this follows by showing that the following diagram commutes

$$
\begin{tikzcd}
X^{\underline{E}} \arrow[r, "\kappa_{X^{\underline{E}}}"] \arrow[dr, "id"'] & (X^{\underline{E}})^{\underline{E}} \arrow[d, dashed, "\zeta_X"] & \Sigma_{\mathcal{R}}(X^{\underline{E}})^{\underline{E}} \arrow[l, "\rho_{X^{\underline{E}}}"'] \arrow[d, "\Sigma_{\mathcal{R}}\zeta_X"] \\
& X^{\underline{E}} & \Sigma_{\mathcal{R}} X^{\underline{E}} \arrow[l, "\rho_X"']
\end{tikzcd}
$$

$\zeta_X \circ \kappa_X = id$ holds since $(-)^{\underline{E}}$ is a monad. The right square commutes as shown below

$$
\begin{aligned}
(\zeta_X \circ \rho_{X^{\underline{E}}})(in_{\mathsf{r}}(F_1, \dots, F_n))(e_i) &= \rho_{X^{\underline{E}}}(in_{\mathsf{r}}(F_1, \dots, F_n))(e_i)(e_i) && \text{(def. } \zeta\text{)} \\
&= F_i(e_i)(e_i) && \text{(def. } \rho\text{)} \\
&= \zeta_X(F_i)(e_i) && \text{(def. } \zeta_X\text{)} \\
&= \rho_X(in_{\mathsf{r}}(\zeta_X(F_1), \dots, \zeta_X(F_n)))(e_i) && \text{(def. } \rho\text{)} \\
&= (\rho_X \circ \Sigma_{\mathcal{R}}\zeta_X)(in_{\mathsf{r}}(F_1, \dots, F_n))(e_i) && \text{(def. } \Sigma_{\mathcal{R}}\text{)}
\end{aligned}
$$

for arbitrary $F_1, \dots, F_n \colon \underline{E} \to X^{\underline{E}}$. $\square$

Note that, the monad $(-)^{\underline{E}}$ is well-defined also in **CMet**. Indeed, as the functor $(-)^{\underline{E}}$ is isomorphic to the finite product $(-)^n$, for $n = |E|$, it preserves Cauchy completeness and can be restricted to an endofunctor on **CMet**. We further observe that it is isomorphic to the composite

$$
\mathbf{CMet} \hookrightarrow \mathbf{Met} \xrightarrow{(-)^{\underline{E}}} \mathbf{Met} \xrightarrow{\mathbb{C}} \mathbf{CMet}\,.
$$

Since $\mathcal{R}$ is a continuous quantitative theory, by Theorems 3.8 and 4.8 we obtain also the following isomorphism of monads.

Theorem 4.9. *The monads $\mathbb{C}T_{\mathcal{R}}$ and $(-)^{\underline{E}}$ in* **CMet** *are isomorphic.*

Consequently, by Theorem 3.9, $\mathbf{EM}((-)^{\underline{E}}) \cong \mathbb{CK}(\Sigma_{\mathcal{R}}, \mathcal{R})$, meaning that $(\Sigma_{\mathcal{R}}, \mathcal{R})$ is a quantitative equational presentation also of the monad $(-)^{\underline{E}}$ on **CMet**.

4.3.2. *Quantitative Writer Algebras.* Fix an extended metric space $\Lambda \in \mathbf{Met}$ of *output values* having monoid structure $(\Lambda, *, 0)$ with non-expansive multiplication $*\colon \Lambda \times \Lambda \to \Lambda$.

The *quantitative writer algebras* of type Λ are the algebras for the signature

$$\Sigma_{\mathcal{W}_\Lambda} = \{\mathsf{w}_\alpha \colon 1 \mid \alpha \in \Lambda\}$$

having a unary operator w_α, for each output value $\alpha \in \Lambda$, and satisfying the following axioms

$$\begin{aligned}
(\mathsf{Zero})\ & \vdash x =_0 \mathsf{w}_0(x)\,, \\
(\mathsf{Mult})\ & \vdash \mathsf{w}_\alpha(\mathsf{w}_{\alpha'}(x)) =_0 \mathsf{w}_{\alpha*\alpha'}(x)\,, \\
(\mathsf{Diff})\ & \{x =_\varepsilon x'\} \vdash \mathsf{w}_\alpha(x) =_\delta \mathsf{w}_{\alpha'}(x')\,,\ \text{for } \delta \geq d_\Lambda(\alpha, \alpha') + \varepsilon\,.
\end{aligned}$$

The quantitative theory induced by the axioms above, written $\mathcal{W}_\Lambda$ (or simply $\mathcal{W}$, when Λ is clear), is called *quantitative theory of writing computations.*

The term $\mathsf{w}_\alpha(t)$ represents the computation that proceeds as t after writing α on the output tape. The axiom (Zero) says that writing the identity element 0 is not observable on the tape; (Mult) says that consecutive writing operations are stored in the tape in the order of execution; (Diff) compares two computations w.r.t. the distance of their output values.

Next we show that the writer monad $(\Lambda \,\square\, -)$ is isomorphic to the free monad $T_{\mathcal{W}}$ on $\mathcal{W}$. Consequently, as the theory $\mathcal{W}$ is basic, by Theorem 3.5, $\mathbf{EM}((\Lambda \,\square\, -)) \cong \mathbb{K}(\Sigma_{\mathcal{W}}, \mathcal{W})$. Thus, $(\Sigma_{\mathcal{W}}, \mathcal{W})$ is a quantitative equational presentation of the writer monad on **Met**.

Theorem 4.10. *The monads $T_{\mathcal{W}}$ and $(\Lambda \,\square\, -)$ in* **Met** *are isomorphic.*

Proof. We show that, for each $X \in \mathbf{Met}$, $(\Lambda \,\square\, X)$ carries a quantitative algebraic structure that is free in $\mathbb{K}(\Sigma_{\mathcal{W}}, \mathcal{W})$ with universal arrow τ_X; and show that the units and multiplications of the two monads coincide (up-to iso).

For any $X \in \mathbf{Met}$, we define the quantitative $\Sigma_{\mathcal{W}}$-algebra $(\Lambda \,\square\, X, \omega_X)$ as follows, for arbitrary $\alpha, \alpha' \in \Lambda$ and $x \in X$

$$\omega_X \colon \Sigma_{\mathcal{W}}(\Lambda \,\square\, X) \to \Lambda \,\square\, X\,, \qquad \omega_X(in_{\mathsf{w}_\alpha}(\alpha', x)) = (\alpha * \alpha', x)\,.$$

Next we show that it satisfies $\mathcal{W}$. Let $\mathsf{w}^\omega_\alpha = \omega_X \circ in_{\mathsf{w}_\alpha}$ denote the interpretation of the operation $\mathsf{w}_\alpha \colon 1 \in \Sigma_{\mathcal{W}}$ in the algebra $(\Lambda \,\square\, X, \omega_X)$. Proving the soundness for (w_α-NE), for each $\alpha \in \Lambda$, is equivalent to show that the map ω is well-defined in **Met**.

$$\begin{aligned}
d_{(\Lambda\square X)}(\mathsf{w}^\omega_\alpha(\beta, x), \mathsf{w}^\omega_\alpha(\beta', x')) &= d_{(\Lambda\square X)}((\alpha * \beta, x), (\alpha * \beta', x')) && \text{(def. } \omega) \\
&= d_\Lambda(\alpha * \beta, \alpha * \beta') + d_X(x, x') && \text{(def. } \square) \\
&\leq \max\Big\{d_\Lambda(\alpha, \alpha), d_\Lambda(\beta, \beta')\Big\} + d_X(x, x') && (*\ \text{non-exp}) \\
&= d_\Lambda(\beta, \beta') + d_X(x, x') && \text{(metric)} \\
&= d_{(\Lambda\square X)}((\beta, x), (\beta', x'))\,. && \text{(def. } \square)
\end{aligned}$$

We are missing to prove that the algebra $((\Lambda \,\square\, X), \omega_X)$ satisfies the axioms (Zero), (Mult), and (Diff).

Soundness for (Zero) holds trivially as $(\alpha, x) = (0 * \alpha, x)$ because 0 is the identity element of the monoid Λ. The soundness of (Mult) follows directly by definition of ω as

$$\mathsf{w}^\omega_\alpha(\mathsf{w}^\omega_{\alpha'}(\beta, x)) = \mathsf{w}^\omega_\alpha((\alpha' * \beta, x))\,.$$

Finally, soundness for (Diff) follows by

$$\begin{aligned}
d_{(\Lambda\Box X)}(\mathsf{w}^{\omega}_{\alpha}(\beta,x),\mathsf{w}^{\omega}_{\alpha'}(\beta',x')) &= d_\Lambda(\alpha*\beta,\alpha'*\beta') + d_X(x,x') && (\text{def. } \omega \ \& \ \Box)\\
&= d_\Lambda(\alpha*\beta,\alpha*\beta') + d_\Lambda(\alpha*\beta',\alpha'*\beta') + d_X(x,x') && (\text{triang.})\\
&\leq d_\Lambda(\beta,\beta') + d_\Lambda(\alpha,\alpha') + d_X(x,x') && (*\ \text{non-exp})\\
&\geq d_\Lambda(\alpha,\alpha') + d_{(\Lambda\Box X)}((\beta,x),(\beta',x'))\,, && (\text{def. } \Box)
\end{aligned}$$

Now we prove freeness. Let (A,a) be a $\Sigma_{\mathcal{W}}$-algebra satisfying $\mathcal{W}$ and $\beta\colon X\to A$ a non-expansive map. We define $h\colon \Lambda\Box X\to A$ as follows, for arbitrary $\alpha\in\Lambda$ and $x\in X$

$$h((\alpha,x)) = a(in_{\mathsf{w}_\alpha}(\beta(x)))\,.$$

Non-expansiveness of h follows by the fact that (A,a) satisfies the axiom (Diff) as shown below, where $\mathsf{w}^a_\alpha = a\circ in_{\mathsf{w}_\alpha}$ denotes the interpretation of $\mathsf{w}_\alpha\colon 1\in\Sigma_{\mathcal{W}}$ in (A,a),

$$\begin{aligned}
d_A(h((\alpha,x)),h((\alpha,x))) &= d_A(\mathsf{w}^a_\alpha(\beta(x)),\mathsf{w}^a_{\alpha'}(\beta(x'))) && (\text{def. } h)\\
&\leq d_\Lambda(\alpha,\alpha') + d_A(\beta(x),\beta(x')) && (\mathsf{Diff})\\
&\leq d_\Lambda(\alpha,\alpha') + d_X(x,x') && (\beta\ \text{non-exp})\\
&= d_{\Lambda\Box X}((\alpha,x),(\alpha',x'))\,. && (\text{def. } \Box)
\end{aligned}$$

Next we prove that h is the unique map such that the diagrams below commute.

$$\begin{tikzcd}
X \arrow[r, "\tau_X"] \arrow[dr, "\beta"'] & \Lambda\Box X \arrow[d, dashed, "h"] & \Sigma_{\mathcal{W}}(\Lambda\Box X) \arrow[l, "\omega_X"'] \arrow[d, "\Sigma_{\mathcal{W}}h"] \\
& A & \Sigma_{\mathcal{W}}A \arrow[l, "a"']
\end{tikzcd}$$

The triangle to the left commutes because, for all $x\in X$

$$\begin{aligned}
(h\circ\tau_X)(x) &= h((0,x)) && (\text{def. } \tau)\\
&= \mathsf{w}^a_0(\beta(x)) && (\text{def. } h)\\
&= \beta(x)\,. && (\mathsf{Zero})
\end{aligned}$$

Let $x\in X$ and $\alpha,\alpha'\in\Lambda$. Then,

$$\begin{aligned}
&(h\circ\omega_X)(in_{\mathsf{w}_\alpha}(\alpha',x))\\
&\quad= \mathsf{w}^a_{\alpha*\alpha'}(\beta(x)) && (\text{def. } h \text{ and } \omega)\\
&\quad= \mathsf{w}^a_\alpha(\mathsf{w}^a_{\alpha'}(\beta(x))) && (\mathsf{Mult})\\
&\quad= \mathsf{w}^a_\alpha(h(\alpha',x)) && (\text{def. } h)\\
&\quad= (a\circ\Sigma_{\mathcal{W}}h)(in_{\mathsf{w}_\alpha}(\alpha',x))\,. && (\text{def. } \mathsf{w}^a_\alpha \text{ and } \Sigma_{\mathcal{W}})
\end{aligned}$$

Thus, h is a $\Sigma_{\mathcal{W}}$-homomorphism, *i.e.*, $h\circ\omega_X = a\circ\Sigma_{\mathcal{W}}h$.

It remains to show uniqueness. Notice first that, $(\alpha,x) = \mathsf{w}^\omega_\alpha(\tau(x))$, where $\mathsf{w}^\omega_\alpha = \omega_X\circ in_{\mathsf{w}_\alpha}$ denotes the interpretation of $\mathsf{w}_\alpha\colon 1\in\Sigma_{\mathcal{W}}$ in $(\Lambda\Box X,\omega_X)$. Indeed, the following holds

$$\begin{aligned}
(\alpha,x) &= (\alpha*0,x) && (0\ \text{identity})\\
&= \mathsf{w}^\omega_\alpha(0,x) && (\text{def. } \omega)\\
&= \mathsf{w}^\omega_\alpha(\tau(x))\,. && (\text{def. } \tau)
\end{aligned}$$

Assume there exists $g\colon \Lambda \,\Box\, X \to A$ such that $g \circ \tau_X = \beta$ and $g \circ \omega_X = a \circ \Sigma_{\mathcal{W}} g$. Then,

$$\begin{aligned}
h((\alpha, x)) &= h(\mathsf{w}^{\omega}_{\alpha}(\tau(x))) \\
&= \mathsf{w}^{a}_{\alpha}(h(\tau(x))) && (h \text{ homo}) \\
&= \mathsf{w}^{a}_{\alpha}(\beta(x)) && (h \circ \tau = \beta) \\
&= \mathsf{w}^{a}_{\alpha}(g(\tau(x))) && (g \circ \tau = \beta) \\
&= g(\mathsf{w}^{\omega}_{\alpha}(\tau(x))) && (g \text{ homo}) \\
&= g((\alpha, x))
\end{aligned}$$

Therefore, $h = g$.

By proof of freeness above, the functors $(\Lambda \,\Box\, -)$ and $T_{\mathcal{W}}$ are isomorphic and the units of the two monads coincide (up-to iso). We are left to prove that also the multiplications coincide (up-to iso). By the universal property of free algebras, this follows by showing that the following diagram commutes

$$\begin{tikzcd}
(\Lambda \,\Box\, X) \arrow[r, "\tau_{\Lambda \Box X}"] \arrow[dr, "id"'] & (\Lambda \,\Box\, (\Lambda \,\Box\, X)) \arrow[d, dashed, "\varsigma_X"] & \Sigma_{\mathcal{W}}(\Lambda \,\Box\, (\Lambda \,\Box\, X)) \arrow[l, "\omega_{\Lambda \Box X}"'] \arrow[d, "\Sigma_{\mathcal{W}} \varsigma_X"] \\
& \Lambda \,\Box\, X & \Sigma_{\mathcal{W}}(\Lambda \,\Box\, X) \arrow[l, "\omega_X"']
\end{tikzcd}$$

The triangle to the left holds as $(\Lambda \,\Box\, -)$ is a monad. The right square commutes by

$$\begin{aligned}
(\varsigma_X \circ \omega_{\Lambda \Box X})(in_{\mathsf{w}_\alpha}(\alpha', (\alpha'', x))) &= \varsigma_X((\alpha * \alpha', (\alpha'', x))) && (\text{def. } \omega) \\
&= (\alpha * \alpha' * \alpha'', x) && (\text{def. } \varsigma) \\
&= \omega_X(in_{\mathsf{w}_\alpha}(\alpha' * \alpha'', x)) && (\text{def. } \omega_X) \\
&= \omega_X(in_{\mathsf{w}_\alpha}(\varsigma_X(\alpha', (\alpha'', x)))) && (\text{def. } \varsigma) \\
&= (\omega_X \circ \Sigma_{\mathcal{W}} \varsigma_X)(in_{\mathsf{w}_\alpha}(\alpha', (\alpha'', x))) && (\text{def. } \Sigma_{\mathcal{W}})
\end{aligned}$$

for arbitrary $x \in X$ and $\alpha, \alpha', \alpha'' \in \Lambda$. $\square$

If we assume the monoid $(\Lambda, *, 0)$ to be over a complete extended metric space Λ, the writer monad $(\Lambda \,\Box\, -)$ is well defined also in **CMet**. We further observe that, as $\Box$ preserves completeness, the underling functor $(\Lambda \,\Box\, -)$ is isomorphic to the composite

$$\mathbf{CMet} \hookrightarrow \mathbf{Met} \xrightarrow{(\Lambda \Box -)} \mathbf{Met} \xrightarrow{\mathbb{C}} \mathbf{CMet}\,.$$

Since $\mathcal{W}$ is a continuous quantitative theory, by Theorems 3.8 and 4.10 we obtain also the following isomorphism of monads.

Theorem 4.11. *The monads $\mathbb{C}T_{\mathcal{W}}$ and $(\Lambda \,\Box\, -)$ in* **CMet** *are isomorphic.*

Thus, by Theorem 3.9, $(\Sigma_{\mathcal{W}}, \mathcal{W})$ is a quantitative equational presentation also of the monad $(\Lambda \,\Box\, -)$ on **CMet**.

4.4. **Nondeterminism.** The monadic effect for nondeterminism in **Set** is given by the *powerset monad*, denoted by $\mathcal{P}$, mapping a set X to $\mathcal{P}_f X = \{U \mid U \subseteq X\}$ and a function $f\colon X \to Y$ to $\mathcal{P}f(U) = \{f(u) \mid u \in U\}$, for $U \in \mathcal{P}X$. The unit $\sigma\colon Id \Rightarrow \mathcal{P}$ and multiplication $\upsilon\colon \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}$ are given, for $x \in X$ and $S \in \mathcal{P}\mathcal{P}X$, by

$$\begin{aligned} \sigma_X(x) &= \{x\} && \text{UNIT} \\ \upsilon_X(S) &= \bigcup\{U \mid U \in S\} && \text{MULTIPLICATION} \end{aligned} \tag{4.4}$$

In the quantitative setting, a natural candidate for a distance on subsets is the Hausdorff metric. Formally, the Hausdorff metric is defined on nonempty compact subsets (equivalently, nonempty closed bounded subsets). Nonemptyness is assumed to ensure the distance is finite on metric spaces. However, as we are dealing with extended metric spaces the empty set can be considered.

The *Hausdorff extended metric* on the set of all compact subsets of an extended metric space X is defined, for arbitrary closed sets $U, V \subseteq X$ by

$$\mathcal{H}(d_X)(U,V) = \max\left\{\sup_{u\in U} d_X(u,V), \sup_{v\in V} d_X(v,U)\right\}\ ,$$

where, $d_X(x,U) = \inf_{u\in U} d_X(x,U)$ denotes the distance from an element $x \in X$ to a set $U \subseteq X$ (we assume $\inf \emptyset = \infty$).

In this paper we will consider monads for quantitative nondeterminism both in **Met** and **CMet**.

On **Met**, the *finite (quantitative) powerset monad*, denoted by $\mathcal{P}_f$, maps an extended metric space X to $\mathcal{P}_f X = \{U \mid U \subseteq X, U \text{ finite}\}$ with Hausdorff metric (note that finite sets are compact) and acts on morphisms $f\colon X \to Y$ as $\mathcal{P}_f f(U) = \{f(u) \mid u \in U\}$, for $U \in \mathcal{P}_f X$. The unit and multiplication are defined as in (4.4). Another monad of interest on **Met** is the submonad $\mathcal{P}_{ne}$, of non-empty finite subsets, with same unit and multiplication.

On **CMet** the *the compact subsets monad*, denoted by $\mathcal{C}$, maps an extended metric space X to $\mathcal{C}X = \{U \mid U \subseteq X, U \text{ compact}\}$ with Hausdorff metric [18] and acts on morphisms $f\colon X \to Y$ as $\mathcal{C}f(U) = \{f(u) \mid u \in U\}$, for $U \in \mathcal{C}X$. The unit and multiplication are defined as in (4.4) (note that $\bigcup\{U \mid U \in S\}$ is closed in X for any $S \in \mathcal{C}\mathcal{C}X$).

Lemma 4.12. *The monads $\mathbb{C}\mathcal{P}_f$ and $\mathcal{C}$ on* **CMet** *are isomorphic.*

Proof. Let $X \in$ **CMet**. Clearly, $\mathcal{P}_f X \subseteq \mathcal{C}X$, as finite subsets are closed. Next, we show that $\mathcal{P}_f X$ is dense in $\mathcal{C}X$. Let $U \in \mathcal{C}X$ and $\epsilon > 0$. Let $B_\epsilon(u) = \{x \in X \mid d_X(u,x) < \epsilon\}$ be the open ball of radius ϵ centered in $u \in U$. As $\{B_\epsilon(u) \mid u \in U\}$ is an open cover for U, by compactness of U, there exists a finite subcover $\{B_\epsilon(v) \mid v \in V\}$ for some $V = \{v_0, \dots, v_n\} \subseteq U$. In particular, we have that for any $u \in U$, exists $v \in V$ such that $d_X(u,v) < \epsilon$ (equivalently, $\sup_{u\in U} d_X(u,V) < \epsilon$). Thus

$$\begin{aligned} \mathcal{H}(d_X)(U,V) &= \max\left\{\sup_{u\in U} d_X(u,V), \sup_{v\in V} d_X(v,U)\right\} \\ &= \sup_{u\in U} d_X(u,V) && \text{(by } V \subseteq U\text{)} \\ &< \epsilon \end{aligned}$$

So $\mathcal{P}_f X$ is dense in $\mathcal{C}X$. As X is complete, convergence and Cauchy convergence coincide. Thus $\overline{\mathcal{P}_f X} \cong \mathcal{C}X$. The correspondence between the units is trivial. The correspondence between multiplications follows because if (V_i) is a sequence of finite subsets of $\mathcal{C}X$ converging to $S \subseteq \mathcal{C}\mathcal{C}X$, then $(\bigcup\{U \mid U \in V_i\})$ converges to $\bigcup\{U \mid U \in S\}$. $\square$

4.4.1. *Quantitative Semilattices with Bottom.* In [30] it was shown that the quantitative powerset monads considered above have an algebraic presentation in terms of a simple quantitative extension to the equational theory of semilattices with bottom.

A quantitative semilattice with bottom is a quantitative algebra for the signature

$$\Sigma_{\mathcal{S}} = \{+ \colon 2, \mathbf{0} \colon 0\}$$

with a binary operator $+$ and a constant $\mathbf{0}$ that satisfying the quantitative equations

$$\begin{aligned}
&(\mathsf{S0}) \vdash x + \mathbf{0} =_0 x\,,\\
&(\mathsf{S1}) \vdash x + x =_0 x\,,\\
&(\mathsf{S2}) \vdash x + y =_0 y + x\,,\\
&(\mathsf{S3}) \vdash (x + y) + z =_0 x + (y + z)\,,\\
&(\mathsf{S4})\ \{x =_\varepsilon y, x' =_{\varepsilon'} y'\} \vdash x + x' =_{\max\{\varepsilon,\varepsilon'\}} y + y'\,.
\end{aligned}$$

We denote by $\mathcal{S}$ the above quantitative theory *of semilattices with bottom.* The axioms ($\mathsf{S0}$), ($\mathsf{S1}$), ($\mathsf{S2}$), ($\mathsf{S3}$) are those of *(join-)semilattices with bottom* and they are essentially standard "equational" axioms. The truly quantitative equation is the last one, ($\mathsf{S4}$).

Remark 4.13. Note that ($\mathsf{S4}$) is derivable from ($+$-NE) and (Max) and, conversely, ($+$-NE) is just an instance of ($\mathsf{S4}$) (when $\varepsilon = \varepsilon'$). Thus, in the quantitative theory $\mathcal{S}$, the axiom ($\mathsf{S4}$) is not necessary. We added it in the presentation above to stress the fact that $\mathcal{S}$ is not just the equational theory of semilattices with bottom.

For any $X \in \mathbf{Met}$, one can define a quantitative $\Sigma_{\mathcal{S}}$-algebra $(\mathcal{P}_f X, \phi_X)$ as follows, for arbitrary $U, V \in \mathcal{P}_f X$

$$\phi_X \colon \Sigma_{\mathcal{S}}\mathcal{P}_f X \to \mathcal{P}_f X \qquad \begin{aligned} \phi_X(in_+(U,V)) &= U \cup V\,,\\ \phi_X(in_{\mathbf{0}}) &= \emptyset\,. \end{aligned}$$

This quantitative algebra satisfies the quantitative theory $\mathcal{S}$, (*cf.* [30, Theorem 9.2]) and it is isomorphic to the free quantitative $\Sigma_{\mathcal{S}}$-algebra on $\mathcal{S}$ (*cf.* [30, Theorem 9.3]).

Thus, as shown in [30], $\mathcal{P}_f$ is isomorphic to the free monad $T_{\mathcal{S}}$ on the theory of quantitative semilattices with a bottom.

Theorem 4.14. *The monads $T_{\mathcal{S}}$ and $\mathcal{P}_f$ on* **Met** *are isomorphic.*

As a direct consequence of Lemma 4.12 and Theorem 4.14 we obtain the following result, which, in combination with Theorem 3.9, tells us that $(\Sigma_{\mathcal{S}}, \mathcal{S})$ is an algebraic presentation of the compact subsets monad $\mathcal{C}$ on complete metric spaces.

Theorem 4.15. *The monads $\mathbb{C}T_{\mathcal{S}}$ and $\mathcal{C}$ on* **CMet** *are isomorphic.*

4.5. **Probabilistic choice.** The monadic effect describing probabilistic choice (a.k.a., probabilistic nondeterminism) in **Set** is given by the *finitely supported probability distribution monad*, denoted by $\mathcal{D}$. This monad acts on sets X as

$$\mathcal{D}X = \left\{ p \colon X \to [0,1] \mid \sum_{x \in X} p(x) = 1, supp(p) \text{ is finite} \right\}.$$

i.e., the set of probability distributions p with finite support $supp(p) = \{x \mid p(x) \neq 0\}$ over X, and acting on morphisms $f \colon X \to Y$ as $\mathcal{D}(f)(p) = \sum_{x \in f^{-1}(y)} p(x)$, for $p \in \mathcal{D}X$. The unit $\delta \colon Id \Rightarrow \mathcal{D}$ is given by the Dirac distribution function $\delta_X(x) = (x \mapsto 1)$, for $x \in X$, and the multiplication $m \colon \mathcal{D}\mathcal{D} \Rightarrow \mathcal{D}$ by $m_X(P)(x) = \sum_{p \in supp(P)} P(p) \cdot p(x)$, for $P \in \mathcal{D}\mathcal{D}X$.

In the quantitative setting, given a metric space, one can more generally consider the set of Borel probability measures over it (those defined on the Borel σ-algebra induced by the metric). There are several ways of measuring the "difference" between probability measures, *e.g.*, using the Total Variation distance, Hellinger distance, Kullback–Leibler divergence, Jensen–Shannon divergence, etc. Here, however, we focus on one specific notion of distance: the *Kantorovich metric* [23] (a.k.a., Wasserstein-1 metric, or Earth Mover Distance). This distance has applications in optimisation and measure theory, as it is related to the concept of transportation problem [47] and metrizes weak-convergence of probability measures [8].

Formally, the Kantorovich distance is defined on Radon measures of finite moment, but as we are dealing with distances that may take infinite values, we won't require the latter condition and consider instead integration on nonnegative extended real-valued measurable functions (*cf.* [6] for the formal definition of the Lebesgue integration of extended real-valued functions). In detail, a Borel probability measure μ on an extended metric space X is *Radon* if for any Borel set $E \subseteq X$, $\mu(E)$ is the supremum of $\mu(K)$ over all compact subsets K of E.

The *Kantorovich extended metric* between Radon probability measures μ, ν over an extended metric space X is then defined as

$$\mathcal{K}(d_X)(\mu, \nu) = \min_{\omega} \int d_X \, \mathrm{d}\omega \,,$$

where ω runs over the set of all joint probability measures on $X \times X$ whose left and right marginals (= pushforwards along the projections) are, respectively, μ and ν.

Examples of Radon probability measures are: (i) finitely supported probability measures on any (extended) metric space, and (ii) generic Borel probability measures over complete separable (extended) metric spaces.

In this paper, we consider two distinct quantitative monads for probabilistic nondeterminism: one on **Met** and one on **CMet**.

On **Met**, the *finitely supported probability monad*, denoted by Π, assigns to an extended metric space X the space ΠX of finitely supported Borel probability measures with Kantorovich metric; and acts on morphisms $f \colon X \to Y$ as $\Pi(f)(\mu) = \mu \circ f^{-1}$ (the pushforward of f), for any $\mu \in \Pi(X)$. The unit $\delta \colon Id \Rightarrow \Pi$ and multiplication $m \colon \Pi\Pi \Rightarrow \Pi$, are given as follows, for $x \in X$, $\Phi \in \Pi\Pi X$, and Borel subset $E \subseteq X$

$$\begin{aligned} \delta_X(x) &= \delta_x \,, && \text{UNIT} \\ m_X(\Phi)(E) &= \int v_E \, \mathrm{d}\Phi && \text{MULTIPLICATION} \,, \end{aligned} \tag{4.5}$$

where δ_x is the Dirac delta measure at x, and $v_E \colon \Pi X \to [0,1]$ is the evaluation function, taking $\mu \in \Pi X$ to $\mu(E) \in [0,1]$.

On **CMet**, the *Radon probability monad*, denoted by Δ, maps a complete extended metric space X to the (complete) extended metric space ΔX of Radon probability measures with Kantorovich metric; and acts on morphisms $f\colon X \to Y$ as $\Delta(f)(\mu) = \mu \circ f^{-1}$, for $\mu \in \Delta(X)$. The unit $\delta\colon Id \Rightarrow \Delta$ and multiplication $m\colon \Delta\Delta \Rightarrow \Delta$, are defined as in (4.5).

These two monads are related as follows

Lemma 4.16. *The monads* $\mathbb{C}\Pi$ *and* Δ *on* **CMet** *are isomorphic.*

4.5.1. *Interpolative Barycentric Algebras.* In [30] it was shown that the quantitative probability monads considered above have an algebraic presentation in terms of a quantitative extension of barycentric algebras, which they called *interpolative barycentric algebras*.

Interpolative barycentric algebras are the quantitative algebras for the signature

$$\Sigma_{\mathcal{B}} = \{+_e\colon 2 \mid e \in [0,1]\}$$

with a binary operator $+_e$, for each $e \in [0,1]$ (a.k.a. *barycentric signature*), and satisfying the quantitative equations

$$\begin{aligned}
&(\mathsf{B1}) \vdash x +_1 y =_0 x\,,\\
&(\mathsf{B2}) \vdash x +_e x =_0 x\,,\\
&(\mathsf{SC}) \vdash x +_e y =_0 y +_{\bar{e}} x\,,\\
&(\mathsf{SA}) \vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)\,,\ \text{for } e, e' \in [0,1)\,,\\
&(\mathsf{IB}) \{x =_\varepsilon y, x' =_{\varepsilon'} y'\} \vdash x +_e x' =_\delta y +_e y',\ \text{ for } \delta \geq e\varepsilon + (1-e)\varepsilon'\,.
\end{aligned}$$

The quantitative theory axiomatised by the quantitative equations above, written $\mathcal{B}$, is called *interpolative barycentric quantitative equational theory.* The axioms (B1), (B2), (SC), (SA) are those of *barycentric algebras* (a.k.a. *abstract convex sets*) due to M. H. Stone [43] where (SC) stands for *skew commutativity* and (SA) for *skew associativity*; (IB) is the *interpolative barycentric axiom* introduced in [30].

For any $X \in$ **Met**, one can define a quantitative $\Sigma_{\mathcal{B}}$-algebra $(\Pi X, \phi_X)$ as follows, for arbitrary $\mu, \nu \in \Pi X$

$$\phi_X\colon \Sigma_{\mathcal{B}}\Pi X \to \Pi X \qquad\qquad \phi_X(in_{+_e}(\mu,\nu)) = e\mu + (1-e)\nu\,.$$

This quantitative algebra satisfies the interpolative barycentric theory $\mathcal{B}$ (*cf.* [30, Theorem 10.4]) and is isomorphic to the free quantitative $\Sigma_{\mathcal{B}}$-algebra on $\mathcal{B}$ (*cf.* [30, Theorem 10.5]).

Thus, as shown in [30], Π is isomorphic to the free monad $T_{\mathcal{B}}$ on the theory $\mathcal{B}$ of interpolative barycentric algebras.

Theorem 4.17. *The monads* $T_{\mathcal{B}}$ *and* Π *on* **Met** *are isomorphic.*

As a direct consequence of Lemma 4.16 and Theorem 4.17 we obtain the following result, which, in combination with Theorem 3.9, tells us that $(\Sigma_{\mathcal{B}}, \mathcal{B})$ is an algebraic presentation of the Randon probability monad Δ on complete metric spaces.

Theorem 4.18. *The monads* $\mathbb{C}T_{\mathcal{B}}$ *and* Δ *on* **CMet** *are isomorphic.*

5. Sum of Quantitative Theories

In this section we develop the theory of the *sum* (or disjoint union) of quantitative equational theories and show it to correspond to the sum quantitative algebraic effects whose presentation is given in terms of basic quantitative theories.

Our leading examples of the sum of quantitative effects are given by the combination of termination/exceptions with arbitrary quantitative effects; and the combination of interactive inputs (more generally, a collection of contractive operators) with arbitrary quantitative effects. We conclude this section by showing how we can recover theory of quantitative Markov processes (*i.e.*, the usual theory Markov processes but now enriched with metric reasoning principles for the underlying probability distributions) in terms of these two generic combinators of quantitative effects.

Let Σ, Σ' be two disjoint signatures. The sum of two quantitative theories $\mathcal{U}$, $\mathcal{U}'$ of respective types Σ and Σ', written $\mathcal{U}+\mathcal{U}'$, is the smallest quantitative theory containing $\mathcal{U}$ and $\mathcal{U}'$. Following Kelly [24], we show that any model for $\mathcal{U}+\mathcal{U}'$ is a $\langle\mathcal{U},\mathcal{U}'\rangle$-*bialgebra*: a metric space A with both a Σ-algebra structure $\alpha\colon \Sigma A \to A$ satisfying $\mathcal{U}$ and a Σ'-algebra structure $\beta\colon \Sigma' A \to A$ satisfying $\mathcal{U}'$. Formally, let $\mathbb{K}((\Sigma,\mathcal{U})\oplus(\Sigma',\mathcal{U}'))$ be the category of $\langle\mathcal{U},\mathcal{U}'\rangle$-bialgebras with non-expansive maps preserving the two algebraic structures. Then, the following isomorphism of categories holds.

Proposition 5.1. $\mathbb{K}(\Sigma+\Sigma',\mathcal{U}+\mathcal{U}')\cong\mathbb{K}((\Sigma,\mathcal{U})\oplus(\Sigma',\mathcal{U}'))$.

Proof. The isomorphism is given by the following pair of functors

$$\begin{tikzcd} \mathbb{K}(\Sigma+\Sigma',\mathcal{U}+\mathcal{U}') \arrow[r, bend left, "H"] & \mathbb{K}((\Sigma,\mathcal{U})\oplus(\Sigma',\mathcal{U}')) \arrow[l, bend left, "K"] \end{tikzcd}$$

defined, for an arbitrary quantitative $(\Sigma+\Sigma')$-algebra (A,γ) satisfying $\mathcal{U}+\mathcal{U}'$ and a $\langle\mathcal{U},\mathcal{U}'\rangle$-bialgebra (B,α,β), respectively as

$$H(A,\gamma)=(A,\gamma\circ in_l,\gamma\circ in_r)\,, \qquad\qquad K(B,\alpha,\beta)=(B,[\alpha,\beta])\,,$$

where $[\alpha,\beta]$ is the unique map induced by α and β by couniversality of the coproduct $\Sigma A+\Sigma' A$. On morphisms both functors map a morphism to itself; it is easy to see that a homomorphism in one sense is also a homomorphism in the other.

The fact that the functors are inverses is clear: $H\circ K=Id$ and $K\circ H=Id$ follow immediately from the couniversal property of coproducts. We are done, provided we show that the functors are indeed well defined. In order to show that the functors are well defined, we need to prove that the functors preserve the relevant quantitative equations.

To show that H is well defined we need to prove that whenever (A,γ) satisfies $\mathcal{U}+\mathcal{U}'$, then $(A,\gamma\circ in_l)$ and $(A,\gamma\circ in_r)$ satisfy $\mathcal{U}$ and $\mathcal{U}'$, respectively. We will prove only that $(A,\gamma\circ in_l)$ satisfies $\mathcal{U}$, since the other follows similarly. Let $\Gamma\vdash t=_\varepsilon s\in\mathcal{U}$ and $\iota\colon X\to A$ be an arbitrary assignment of the variables. Since $\mathcal{U}\subseteq\mathcal{U}+\mathcal{U}'$, we have:

$$\big(\text{for all } t'=_{\varepsilon'} s'\in\Gamma,\ d_A(\iota^\sharp(t'),\iota^\sharp(s'))\le\varepsilon'\big) \text{ implies } d_A(\iota^\sharp(t),\iota^\sharp(s))\le\varepsilon\,, \tag{5.1}$$

where $\iota^\sharp\colon \mathbb{T}(\Sigma+\Sigma',X)\to A$ is the homomorphic extension of ι on (A,γ).

Note that, by definition of coproduct of functors and homomorphic extension, we have that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\eta^{\Sigma}_X"] \arrow[dr, "\eta^{\Sigma+\Sigma'}_X"'] \arrow[ddr, bend right, "\iota"'] & \mathbb{T}(\Sigma, X) \arrow[d, hook, "i"] & & \Sigma\mathbb{T}(\Sigma, X) \arrow[ll, "\psi^{\Sigma}_X"'] \arrow[d, "\Sigma i"] \\
& \mathbb{T}(\Sigma{+}\Sigma', X) \arrow[d, "\iota^\sharp"] & (\Sigma{+}\Sigma')\mathbb{T}(\Sigma{+}\Sigma', X) \arrow[l, "\psi^{\Sigma+\Sigma'}"'] \arrow[d, "(\Sigma+\Sigma')\iota^\sharp"] & \Sigma\mathbb{T}(\Sigma{+}\Sigma', X) \arrow[l, "in_l"'] \arrow[d, "\Sigma\iota^\sharp"] \\
& A & (\Sigma{+}\Sigma')A \arrow[l, "\gamma"] & \Sigma A \arrow[l, "in_l"]
\end{tikzcd}$$

where i is the canonical inclusion of Σ-terms in $\mathbb{T}(\Sigma + \Sigma', X)$. The above implies also that $\iota^\sharp \circ i$ is the homomorphic extension of ι on $(A, \gamma \circ in_l)$. Recall that $\mathcal{U}$ is of type Σ. Thus in $\Gamma \vdash t =_\varepsilon s$ can occur only terms in $\mathbb{T}(\Sigma, X)$. Therefore, (5.1) implies that $(A, \gamma \circ in_l)$ satisfies $\Gamma \vdash t =_\varepsilon s$. This argument is general so it applies to the whole theory $\mathcal{U}$.

For K, we need to show that whenever (A, α) satisfies $\mathcal{U}$ and (A, β) satisfies $\mathcal{U}'$, then $(A, [\alpha, \beta])$ satisfies $\mathcal{U} + \mathcal{U}'$. By the definition of the disjoint union of quantitative theories, it suffices to prove that $(A, [\alpha, \beta])$ is a model for both $\mathcal{U}$ and $\mathcal{U}'$. We show the former case, since the other follows similarly. Let $\Gamma \vdash t =_\varepsilon s \in \mathcal{U}$ and $\iota\colon X \to A$ be an arbitrary assignment of the variables. Since (A, α) satisfies $\mathcal{U}$, we have that

$$\big(\text{for all } t' =_{\varepsilon'} s' \in \Gamma,\ d_A(\iota^\sharp(t'), \iota^\sharp(s')) \leq \varepsilon'\big) \text{ implies } d_A(\iota^\sharp(t), \iota^\sharp(s)) \leq \varepsilon\,, \tag{5.2}$$

where $\iota^\sharp\colon \mathbb{T}(\Sigma, X) \to A$ is the homomorphic extension of ι on (A, α). Note that, by definition of coproduct of functors and homomorphic extension, we have that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\eta^{\Sigma}_X"] \arrow[dr, "\eta^{\Sigma+\Sigma'}_X"'] \arrow[ddr, bend right, "\iota"'] & \mathbb{T}(\Sigma, X) \arrow[d, hook, "i"] & & \Sigma\mathbb{T}(\Sigma, X) \arrow[ll, "\psi^{\Sigma}_X"'] \arrow[d, "\Sigma i"] \\
& \mathbb{T}(\Sigma{+}\Sigma', X) \arrow[d, "\iota^\flat"] & (\Sigma{+}\Sigma')\mathbb{T}(\Sigma{+}\Sigma', X) \arrow[l, "\psi^{\Sigma+\Sigma'}"'] \arrow[d, "(\Sigma+\Sigma')\iota^\flat"] & \Sigma\mathbb{T}(\Sigma{+}\Sigma', X) \arrow[l, "in_l"'] \arrow[d, "\Sigma\iota^\flat"] \\
& A & (\Sigma{+}\Sigma')A \arrow[l, "{[\alpha,\beta]}"] & \Sigma A \arrow[l, "in_l"]
\end{tikzcd}$$

where i is the canonical inclusion of Σ-terms in $\mathbb{T}(\Sigma + \Sigma', X)$, and $\iota^\flat$ is the homomorphic extension of ι on $(A, [\alpha, \beta])$. Since $[\alpha, \beta] \circ in_l = \alpha$, the above implies also that $\iota^\flat \circ i = \iota^\sharp$. Since $\mathcal{U}$ is of type Σ, then $\Gamma \vdash t =_\varepsilon s$ contains only terms in $\mathbb{T}(\Sigma, X)$. Therefore, (5.2) implies that $(A, [\alpha, \beta])$ satisfies $\Gamma \vdash t =_\varepsilon s$; again this implies the result for all of $\mathcal{U}$. □

Let T, T' be two monads on a category $\mathbf{C}$. An Eilenberg-Moore bialgebra for $\langle T, T' \rangle$ (or simply, $\langle T, T' \rangle$-bialgebra) is an object $A \in \mathbf{C}$ with Eilenberg-Moore algebra structures $\alpha\colon TA \to A$ and $\beta\colon T'A \to A$. We write $\mathbf{EM}\langle T, T' \rangle$ for the category of Eilenberg-Moore bialgebras for $\langle T, T' \rangle$ with morphisms those in $\mathbf{C}$ preserving the two algebraic structures.

When the quantitative equational theories $\mathcal{U}$ and $\mathcal{U}'$ are basic, by Theorem 3.5, we get a refinement of Proposition 5.1 as follows.

Corollary 5.2. *For $\mathcal{U}, \mathcal{U}'$ basic quantitative theories,* $\mathbb{K}(\Sigma + \Sigma', \mathcal{U} + \mathcal{U}') \cong \mathbf{EM}\langle T_{\mathcal{U}}, T_{\mathcal{U}'} \rangle$.

Proof. Immediate from Theorem 3.5 and Proposition 5.1. □

The following result supports the construction of the sum of quantitative theories as a combinator of quantitative effects. It states that the free monad $T_{\mathcal{U}+\mathcal{U}'}$ on the sum $\mathcal{U}+\mathcal{U}'$ corresponds to the categorical sum (coproduct) $T_{\mathcal{U}} + T_{\mathcal{U}'}$ of the free monads on $\mathcal{U}$ and $\mathcal{U}'$, respectively. This isomorphism of monads stands under the assumption that the sum is taken over basic quantitative theories.

Theorem 5.3. *If $\mathcal{U}, \mathcal{U}'$ are basic quantitative theories, then $T_{\mathcal{U}+\mathcal{U}'}$ is isomorphic to $T_{\mathcal{U}} + T_{\mathcal{U}'}$.*

Proof. By Corollary 5.2 and Theorem 3.3 the obvious forgetful functor from $\mathbf{EM}\langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$ to **Met** has a left adjoint. The monad generated by this adjunction is isomorphic to $T_{\mathcal{U}+\mathcal{U}'}$. Thus, by [24] (cf. also [1, Proposition 2.8]), the monad $T_{\mathcal{U}+\mathcal{U}'}$ is isomorphic to $T_{\mathcal{U}} + T_{\mathcal{U}'}$. □

The above constructions do not use any specific property of the category **Met**, apart from requiring its morphisms to be non-expansive. Thus, we can reformulate an alternative version of Theorem 5.3 which is valid in **CMet**, under the assumption that the sum is taken over continuous quantitative theories.

Recall that, continuous theories are basic. Moreover, the disjoint union $\mathcal{U}+\mathcal{U}'$ of two continuous quantitative theories $\mathcal{U}, \mathcal{U}'$ is also continuous, so that, by Theorem 3.8, the free monad on it in **CMet** is $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$. Thus:

Theorem 5.4. *If $\mathcal{U}, \mathcal{U}'$ are continuous theories, then $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$ is isomorphic to $\mathbb{C}T_{\mathcal{U}} + \mathbb{C}T_{\mathcal{U}'}$.*

Proof. By Theorem 3.8, the monads $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$, $\mathbb{C}T_{\mathcal{U}}$, and $\mathbb{C}T_{\mathcal{U}'}$ are, respectively, the free monads on $\mathcal{U}+\mathcal{U}'$, $\mathcal{U}$, and $\mathcal{U}'$ in **CMet**.

Similarly to Corollary 5.2, one obtains that $\mathbb{CK}(\Sigma+\Sigma', \mathcal{U}+\mathcal{U}')$ and $\mathbf{EM}\langle \mathbb{C}T_{\mathcal{U}}, \mathbb{C}T_{\mathcal{U}'}\rangle$ are isomorphic. Thus, by Theorem 3.8 the forgetful functor from $\mathbf{EM}\langle \mathbb{C}T_{\mathcal{U}}, \mathbb{C}T_{\mathcal{U}'}\rangle$ to **Met** has a left adjoint, and the monad generated by this adjunction is isomorphic to $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$. Thus, by [24] (cf. also [1, Proposition 2.8]), $\mathbb{C}T_{\mathcal{U}+\mathcal{U}'}$ is the sum of $\mathbb{C}T_{\mathcal{U}}$ and $\mathbb{C}T_{\mathcal{U}'}$. □

5.1. **Sum with Exceptions.**

As remarked by Hyland, Plotkin, and Power (*cf.* [20, Corollary 3]), Moggi's exception monad transformer, sending a monad T to the composite $T(-+E)$ can be explained in terms of the sum of monads:

Proposition 5.5 (Sum with Exception Monad)**.** *Given a category $\mathbf{C}$ with finite coproducts, an object E of $\mathbf{C}$, and a monad T on $\mathbf{C}$, the sum of the monads $(-+E)$ and T exists and is given by a canonical monad structure on the composite $T(-+E)$.*

From the above result, in combination with Theorems 5.3, 3.5 and 4.1 we obtain an analogous transformer at the level of quantitative equational theories as follows.

Corollary 5.6. *Let $\mathcal{U}$ be a basic quantitative equational theory. Then, $T_{\mathcal{U}}(-+E)$ is the free monad on the theory $\mathcal{U}+\mathcal{E}_E$ on* **Met***.*

Similarly, from Theorems 5.4 and 4.2, an analogous result holds also in **CMet**.

Corollary 5.7. *Let $\mathcal{U}$ be a basic continuous equational theory. Then, $\mathbb{C}T_{\mathcal{U}}(-+E)$ is the free monad on the theory $\mathcal{U}+\mathcal{E}_E$ on* **CMet***.*

Example 5.8 (Quantitative pointed covex semilattices)**.** Mio and Vignudelli [32, 33] while reasoning about the algebraic combination of quantitative nondeterminism (*cf.* Section 4.4) and probabilistic choice (*cf.* Section 4.5), studied the category of pointed convex semilattices. In particular, they showed that they are isomorphic to the Eilenberg-Moore algebras for

$\hat{\mathcal{C}}(+\hat{1})$, *i.e.*, the quantitative variant of the monad of *(nonempty) convex sets of pointed probability distributions*.

This monad is just the composition of $\hat{\mathcal{C}}$, the (nonempty) convex sets of probability distribution monad [44], with the termination monad $(- + 1)$. So, as $\hat{\mathcal{C}}$ is presented by the quantitative theory of convex semilattices, their result can be recovered —and possibly further extended on complete metric spaces— as a simple application of Corollary 5.6 and Theorem 5.3.

5.2. **Sum with Interactive Inputs.** Now we consider the sum of generic quantitative effects T with the monads $\tilde{\Sigma}^*$ of contractive operators, of which interactive inputs is a particular instance (*cf.* Section 4.2).

From Theorem 5.3 we know that if T has a quantitative algebraic presentation in terms of a basic theory $\mathcal{U}$, the sum exists, and, when starting with quantitative theories, we know how to describe it. But for the purposes of calculation, it is still convenient to have a more explicit construction of the sum qua monad, and Hyland et al. provided such a construction (*cf.* [20, Theorem 4]), which we recall below for convenience. The key fact used here is that the monad of contractive operators is described as the free monad on an endofunctor with countable rank, namely the contractive signature functor $\tilde{\Sigma}$ given in (4.1).

Theorem 5.9 ([20]). *Given an endofunctor F and a monad T on a category $\mathbf{C}$, if the free monads F^* and $(FT)^*$ exist and are algebraic, then the sum of monads $T + F^*$ exists and is given by a canonical monad structure on the composite $T(FT)^*$.*

As remarked in [20], when $\mathbf{C}$ is locally countably presentable and both T and F have countable rank, then F^* and $(FT)^*$ exist and are algebraic. Moreover, also $T(FT)^*$ has countable rank and so it is also the sum of monads $T + F^*$.

We know that $\mathbf{Met}$ is locally countably presentable and that contractive signature functors $\tilde{\Sigma}$ have countable rank (*cf.* Section 4.2.2). Moreover, as recently proved by Ford et al. [14], any quantitative theory $\mathcal{U}$ induces a monad $T_{\mathcal{U}}$ with countable rank.

Therefore, from the discussion above and by Theorems 5.3 and 4.4, we obtain the following characterisation.

Corollary 5.10. *Let $\mathcal{U}$ be a basic quantitative equational theory. Then, $T_{\mathcal{U}}(\tilde{\Sigma}T_{\mathcal{U}})^*$ is the free monad on the theory $\mathcal{U} + \mathcal{O}(\Sigma)$ on $\mathbf{Met}$.*

As observed in [20], the monad $T(FT)^*$ of Theorem 5.9 is simply another form of the generalised resumptions monad transformer of Cenciarelli and Moggi [11], sending T to $\mu y.T(Fy + -)$. Hence, by the characterisation above and guided by the same observations that lead to [20, Corollary 2], we obtain an analogous transformer at the level of quantitative equational theories as follows.

Corollary 5.11. *Let $\mathcal{U}$ be a basic quantitative equational theory. Then, $\mu y.T_{\mathcal{U}}(\tilde{\Sigma}y + -)$ is the free monad on the theory $\mathcal{U} + \mathcal{O}(\Sigma)$ on $\mathbf{Met}$.*

Similarly, by Theorems 5.4 and 4.5, an analogous result holds also in $\mathbf{CMet}$.

Corollary 5.12. *Let $\mathcal{U}$ be a continuous quantitative equational theory. Then, $\mu y.\mathbb{C}T(\tilde{\Sigma}y + -)$ is the free monad on the theory $\mathcal{U} + \mathcal{O}(\Sigma)$ on $\mathbf{CMet}$.*

Remark 5.13. It is worth remarking that using these ideas one obtains a modular description of the monads of contractive operators. Let Σ_1, Σ_2 be two disjoint signatures of contractive operators. It is clear that $\mathcal{O}(\Sigma_1 \cup \Sigma_2)$ is the same as the sum of theories $\mathcal{O}(\Sigma_1)$ and $\mathcal{O}(\Sigma_2)$. Hence, the sum $\tilde{\Sigma}_1^* + \tilde{\Sigma}_2^*$ is given by the free monad $(\tilde{\Sigma}_1 + \tilde{\Sigma}_2)^*$, where we now mean the pointwise sum of functors.

5.3. **The Algebras of Markov Processes.** In this section we show how to obtain a quantitative equational axiomatization of Markov processes with discounted bisimilarity metric [13] as the composition, via sum, of the following quantitative theories:

(1) *The quantitative theory $\mathcal{B}$ of interpolative barycentric algebras*, used to express probabilistic nondeterminism with Kantorovich metric (Section 4.5);
(2) *The quantitative theory $\mathcal{E}_1$ of exceptions* over $1 = \{*\}$, with $*$ as the only exception type. This will be used to express termination (Section 4.1);
(3) *The quantitative theory of contractive operators* (Section 4.2). In our case, we consider a signature $\Sigma_\diamond = \{\diamond \colon \langle 1, c\rangle\}$ with a unary operator $\diamond$ with contractive factor $c \in (0,1)$. This will be used to axiomatise the transition to a next state with discount factor c.

Formally, we define the *quantitative theory of Markov processes* as

$$\mathcal{U}_{\mathbf{MP}} = \mathcal{B} + \mathcal{E}_1 + \mathcal{O}(\Sigma_\diamond)\,.$$

with signature $\Sigma_{\mathbf{MP}} = \Sigma_{\mathcal{B}} \cup \Sigma_1 \cup \Sigma_\diamond$ given as the disjoint union of those from its component theories. More explicitly,

$$\Sigma_{\mathbf{MP}} = \{+_e \colon 2 \mid e \in [0,1]\} \cup \{\mathsf{raise}_* \colon 0\} \cup \{\diamond \colon \langle 1, c\rangle\}$$

and $\mathcal{U}_{\mathbf{MP}}$ has the following set of axioms

$$
\begin{aligned}
&(\mathsf{B1}) \vdash x +_1 y =_0 x\,,\\
&(\mathsf{B2}) \vdash x +_e x =_0 x\,,\\
&(\mathsf{SC}) \vdash x +_e y =_0 y +_{1-e} x\,,\\
&(\mathsf{SA}) \vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)\,,\ \text{for } e, e' \in [0,1)\,,\\
&(\mathsf{IB}) \{x =_\varepsilon y, x' =_{\varepsilon'} y'\} \vdash x +_e x' =_\delta y +_e y',\ \text{for } \delta \geq e\varepsilon + (1-e)\varepsilon',\\
&(\diamond\text{-}\mathsf{Lip}) \{x =_\varepsilon y\} \vdash \diamond(x) =_\delta \diamond(y)\,,\ \text{for } \delta \geq c\varepsilon\,.
\end{aligned}
$$

Note that, the constant raise_* has no explicit associated axiom since $\mathcal{E}_1$ is the trivial theory, corresponding to that for termination.

Intuitively, $\Sigma_{\mathbf{MP}}$-terms (modulo $=_0$ provability) can be interpreted as equivalence classes of behaviours of Markov processes up-to bisimilarity. The term $t +_e t'$ expresses convex combination of behaviours; raise_* represents termination (or the deadlock behavior); and $\diamond(t)$ expresses the ability of taking a transition to the behaviour t.

5.3.1. *Markov Processes over Metric Spaces.* Following [46, Section 6], we regard Markov processes as coalgebras on the category of metric spaces, and slightly extending their approach to encompass the case when the bisimilarity distance is discounted by a factor $0 < c < 1$.

We consider two variants of Markov processes according to the type of their transition distribution functions[2]:

$$X \longrightarrow \Pi(c \cdot X + 1) \quad \text{in } \mathbf{Met}\,,$$
$$X \longrightarrow \Delta(c \cdot X + 1) \quad \text{in } \mathbf{CMet}\,,$$

where Π and Δ are the functors from Section 4.5, mapping a metric space X to a space of probability measures with Kantorovich metric. The first variant are Markov processes with finitely supported transition probability distributions, commonly regarded as *Markov chains*. The second variant are Markov processes with Randon transition probability distributions. The use of the rescaling functor $(c \cdot -)$ is to expresses that transition functions are c-Lipschitz continuous, with contractive factor $0 \leq c \leq 1$. We will collectively refer these two types of coalgebras structures as c-Markov processes.

In [46], van Breugel et al. characterised the bisimilarity distance on (labelled) Markov processes as the pseudometric induced by the unique homomorphism to the final coalgebra. We will do the same here by replicating their arguments in our specific setting.

Proposition 5.14. *The final coalgebras for* $\Pi(c \cdot - + 1)$ *and* $\Delta(c \cdot - + 1)$ *exist.*

Proof. As the categories **Met** and **CMet** are both complete and accessible (*cf.* Appendices A and B for the formal definitions and proofs), the thesis follows by [46, Theorem 8], by showing that $\Pi(c \cdot - + 1)$ and $\Delta(c \cdot - + 1)$ are accessible functors (more precisely, $\aleph_1$-accessible).

Notice that $\Pi(c \cdot - + 1)$ has a quantitative algebraic presentation in **Met** in terms of the theory $\mathcal{B}^c$ defined as $\mathcal{B} + \mathcal{E}_1$ where the axiom (IB) (*cf.* Section 4.5.1) is replaced by

$$(\mathsf{IB}^c)\, \{x =_\varepsilon y, x' =_{\varepsilon'} y'\} \vdash x +_e x' =_\delta y +_e y', \;\text{ for } \delta \geq c(e\varepsilon + (1-e)\varepsilon')\,,$$

that is, $T_{\mathcal{B}^c + \mathcal{E}_1} \cong \Pi(c \cdot - + 1)$ (the proof follows essentially identically to [30, Theorem 10.5], which implies the isomorphism of monads). As [14] proved that the monads freely generated by a quantitative theory are $\aleph_1$-accessible, we have that the final coalgebra for $\Pi(c \cdot - + 1)$ exists. Moreover, as $\mathbb{C}T_{\mathcal{B}^c + \mathcal{E}_1} \cong \Delta(c \cdot - + 1)$, $\mathbb{C}$ is $\aleph_1$-accessible, and $\aleph_1$-accessibility is closed under composition, we have that also $\Delta(c \cdot - + 1)$ admits a final coalgebra. □

Then, the *c-discounted bisimilarity pseudometric* on a c-Markov process (X, τ) is defined as the function $\mathbf{d}^c \colon X \times X \to [0, \infty]$ given as

$$\mathbf{d}^c(x, x') = d_Z(h(x), h(x'))\,,$$

where $h \colon X \to Z$ is the unique homomorphism to the final c-Markov process (Z, ω).

This distance has a characterisation as the least fixed point of a monotone function on a complete lattice of $[0, \infty]$-valued pseudometrics.

Proposition 5.15. *The c-discounted bisimilarity pseudometric* $\mathbf{d}^c$ *on* (X, τ) *is the* unique *fixed point of the following operator on the complete lattice of extended pseudometrics* d *on* X *with point-wise order* $\sqsubseteq$, *such that* $d \sqsubseteq d_X$,

$$\Psi^c(d)(x, x') = \sup_f \left| \int f \;\mathrm{d}\tau(x) - \int f \;\mathrm{d}\tau(x') \right|\,,$$

with f *ranging over non-expansive positive* 1*-bounded real valued functions on* $c \cdot X + 1$.

[2]Note that the two types of coalgebras we are considering live in two different categories, **Met** and **CMet**.

Proof. Similar to the fixed point characterisation given in [46, Section 6]. The unicity of the fixed point follows by Banach fixed point theorem. Indeed, the set of extended real valued functions on $X \times X$ (which is a superset of the set of extended pseudometrics on X) can be turned into a complete Banach space by means of the sup-norm $||f|| = \sup_{x,x'} |f(x,x')|$ and Ψ^c is a c-contractive operator on it. □

5.3.2. *Quantitative Algebraic Presentation.* Here we relate c-Markov processes and their bisimilarity distance to the free algebras of $\mathcal{U}_{\mathbf{MP}}$, both on **Met** and **CMet**.

On Metric Spaces. We start by characterising the monad $T_{\mathcal{U}_{\mathbf{MP}}}$ on **Met**. We do this in steps, by explaining the contribution of each subtheory in the sum

$$\mathcal{U}_{\mathbf{MP}} = \mathcal{B} + \mathcal{E}_1 + \mathcal{O}(\Sigma_\diamond)\,.$$

(Step 1) First, note that $T_{\mathcal{E}_1} \cong (- + 1)$ is the *maybe monad* (Theorem 4.1). As $\mathcal{B}$ is basic, by Corollary 5.6 and Theorem 4.17, the free monad on $\mathcal{B} + \mathcal{E}_1$ is

$$T_{\mathcal{B}+\mathcal{E}_1} \cong T_{\mathcal{B}}(- + 1) \cong \Pi(- + 1)\,.$$

where $\Pi(- + 1)$ is the *finitely supported sub-distribution monad* with functor assigning to $X \in \mathbf{Met}$ the space of finitely supported Borel sub-probability measures with Kantorovich metric. Thus, $\mathcal{B} + \mathcal{E}_1$ axiomatizes finitely supported sub-probability distributions with Kantorovich metric.

(Step 2) The final step is to sum the above with the theory $\mathcal{O}(\Sigma_\diamond)$. By Corollary 5.11, the free monad on $\mathcal{U}_{\mathbf{MP}} = \mathcal{B} + \mathcal{E}_1 + \mathcal{O}(\Sigma_\diamond)$ is

$$T_{\mathcal{U}_{\mathbf{MP}}} \cong \mu y. T_{\mathcal{B}+\mathcal{E}_1}(c \cdot y + -) \cong \mu y. \Pi(c \cdot y + 1 + -)\,,$$

where we implicitly applied the isomorphisms $c \cdot (A + B) \cong c \cdot A + c \cdot B$ and $1 \cong c \cdot 1$. Explicitly, this means that, the free monad on $\mathcal{U}_{\mathbf{MP}}$ assigns to an arbitrary metric space $X \in \mathbf{Met}$ the *initial solution* to the following functorial equation in **Met**

$$MP_X \cong \Pi(c \cdot MP_X + 1 + X)\,. \tag{5.3}$$

Next we argue that $\mathcal{U}_{\mathbf{MP}}$ axiomatizes the initial c-Markov process on **Met** with c-discounted bisimilarity metric. Let $X = 0$ be the empty metric space (*i.e.*, the initial object in **Met**). Then (5.3) corresponds to the isomorphism on the initial $\Pi(c \cdot - + 1)$-algebra. The isomorphism provides us also with a $\Pi(c \cdot - + 1)$-coalgebra structure on MP_0 which, according to our interpretation, is a c-Markov process (MP_0, τ_0).

The key observation is that the metric on MP_0 is the bisimilarity metric.

Lemma 5.16. *d_{MP_0} is the c-discounted bisimilarity metric on (MP_0, τ_0).*

Proof. Isomorphisms in **Met** are isometries. Hence, by definition of (MP_0, τ_0) and (5.3)

$$d_{MP_0}(x, x') = \mathcal{K}(d)(\tau_0(x), \tau_0(x'))\,,$$

where d is the metric on $c \cdot MP_0 + 1$. By Kantorovich-Rubinstein Duality [47, Theorem. 5.10]

$$\mathcal{K}(d)(\tau_0(x), \tau_0(x')) = \sup_f \left| \int f \,\mathrm{d}\tau_0(x) - \int f \,\mathrm{d}\tau_0(x') \right|\,,$$

where f ranges over non-expansive functions $f \colon c \cdot X + 1 \to [0, 1]$. Thus, the thesis follows by Proposition 5.15. □

Remark 5.17. For a less abstract description of (MP_0, τ_0), notice that the elements of MP_0 are ground terms over the signature $\Sigma_{\mathbf{MP}}$ modulo $=_0$ provability. One can interpret a term as a pointed (or rooted) acyclic sub-probabilistic Markov chain up-to bisimilarity. For example, the term $\diamond(\diamond(\mathsf{raise}_*) +_{\frac{1}{2}} (\diamond(\diamond(\mathsf{raise}_*)) +_{\frac{1}{2}} \mathsf{raise}_*))$ corresponds to the sub-probabilistic Markov chain below

$$\begin{tikzcd}
& \bullet \arrow[d, "1"'] & \\
& \bullet \arrow[dl, "\frac{1}{2}"'] \arrow[dr, "\frac{1}{4}"] & \\
\bullet & & \bullet \arrow[ll, "1"']
\end{tikzcd}$$

and $\diamond(\mathsf{raise}_*)$ corresponds to the deadlock process, with probability 0 to move to any state.

On Complete Metric Spaces. Now we characterize the monad $\mathbb{C}T_{\mathcal{U}_{\mathbf{MP}}}$ on **CMet**. We do this by following the same steps as for the monad $T_{\mathcal{U}_{\mathbf{MP}}}$ on **Met**.

(Step 1) By Theorem 4.2, $\mathbb{C}T_{\mathcal{E}_1} \cong (- + 1)$ is the *maybe monad*. As $\mathcal{B}$ is continuous, by Corollary 5.7 and Theorem 4.18, the completion of the free monad on $\mathcal{B} + \mathcal{E}_1$ is

$$\mathbb{C}T_{\mathcal{B}+\mathcal{E}_1} \cong \mathbb{C}T_{\mathcal{B}}(- + 1) \cong \Delta(- + 1)\,.$$

where $\Delta(- + 1)$ is the *Radon sub-probability distribution monad* with Kantorovich metric.

(Step 2) In combination with the theory $\mathcal{O}(\Sigma_\diamond)$, by Corollary 5.12, the free completion monad on $\mathcal{U}_{\mathbf{MP}} = \mathcal{B} + \mathcal{E}_1 + \mathcal{O}(\Sigma_\diamond)$ is given by

$$\mathbb{C}T_{\mathcal{U}_{\mathbf{MP}}} \cong \mu y.\mathbb{C}T_{\mathcal{B}+\mathcal{E}_1}(c \cdot y + -) \cong \mu y.\Delta(c \cdot y + 1 + -)\,.$$

This means that, also for the case of complete metric spaces the free monad on $\mathcal{U}_{\mathbf{MP}}$ assigns to any arbitrary metric space $X \in \mathbf{CMet}$ the *initial solution* of the following functorial equation in **CMet**

$$MP_X \cong \Delta(1 + c \cdot MP_X + X)\,. \tag{5.4}$$

Observe that the map $\omega_X \colon MP_X \to \Delta(1 + c \cdot MP_X + X)$ arising from the above isomorphism is a coalgebra structure for the functor $\Delta(1 + c \cdot - + X)$ on **CMet**. Next we show that (MP_X, ω_X) is actually the final coalgebra.

Theorem 5.18 ([45, Section 7])**.** *Every locally contractive endofunctor H on* **CMet** *has a unique fixed point which is both an initial algebra and a final coalgebra for H.*

Recall from Example 2.1 that the internal hom $[X, Y]$ in **CMet** is the set of non-expansive maps from X to Y with point-wise supremum metric $d_{[X,Y]}(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$.

An endofunctor H on **CMet** is *locally c-Lipschitz continuous* if for all $X, Y \in \mathbf{CMet}$, and non-expansive maps $f, g \colon X \to Y$,

$$d_{[HX,HY]}(H(f), H(g)) \le c \cdot d_{[X,Y]}(f, g)\,.$$

H is *locally non-expansive* if it is locally 1-Lipschitz continuous, and *locally contractive* if is locally c-Lipschitz continuous, for some $0 \le c < 1$.

Examples of locally contractive functors are the constant functors and the rescaling functor $(c \cdot -)$, for $0 \le c < 1$. Moreover, locally contractiveness is preserved by products and coproducts and composition; and, if H is locally non-expansive and G is locally contractive, then HG is locally contractive.

Lemma 5.19. *The endofunctor Δ on* **CMet** *is locally non-expansive.*

Proof. We need to check that for all $f, g \in \mathbf{CMet}(X, Y)$,

$$\sup_{x \in X} d_Y(f(x), g(x)) \geq \sup_{\mu \in \Delta X} \mathcal{K}(d_Y)(\Delta f(\mu), \Delta g(\mu))\,. \tag{5.5}$$

Denote by Φ_Y be the set of non-expansive functions $k\colon Y \to [0,1]$, *i.e.*, those functions such that $\forall y, y'. |k(y) - k(y')| \leq d_Y(y, y')$. Then, for any $\mu \in \Delta X$,

$$\begin{aligned}
\mathcal{K}(d_Y)(\Delta f(\mu), \Delta g(\mu)) &= \sup_{k \in \Phi_Y} \left| \int k \,\mathrm{d}\Delta f(\mu) - \int k \,\mathrm{d}\Delta g(\mu) \right| && \text{(Kantorovich duality)} \\
&= \sup_{k \in \Phi_Y} \left| \int k \,\mathrm{d}(\mu \circ f^{-1}) - \int k \,\mathrm{d}(\mu \circ g^{-1}) \right| && \text{(def. } \Delta\text{)} \\
&= \sup_{k \in \Phi_Y} \left| \int k \circ f \,\mathrm{d}\mu - \int k \circ g \,\mathrm{d}\mu \right| && \text{(change of var.)} \\
&= \sup_{k \in \Phi_Y} \left| \int (k \circ f) - (k \circ g) \,\mathrm{d}\mu \right| && \text{(linearity of } \textstyle\int\text{)} \\
&\leq \sup_{k \in \Phi_Y} \int |(k \circ f) - (k \circ g)| \,\mathrm{d}\mu && \text{(subadd. of } |\cdot|\text{)} \\
&\leq \int d_Y \circ \langle f, g \rangle \,\mathrm{d}\mu && (k \text{ non-expansive}) \\
&\leq \int \sup_{x \in X} d_Y(f(x), g(x)) \,\mathrm{d}\mu && \text{(monotonicity of } \textstyle\int\text{)} \\
&= \sup_{x \in X} d_Y(f(x), g(x))\,. && (\mu \text{ probability measure})
\end{aligned}$$

For the generality of $\mu \in \Delta X$, the above inequality implies (5.5). □

Thus, the following holds.

Theorem 5.20. *(MP_X, ω_X) is the final coalgebra for $\Delta(1 + c \cdot Id + X)$ in* **CMet**.

Proof. This is a direct consequence of Theorem 5.18 and Lemma 5.19, since, $1 + c \cdot - + X$ is locally contractive and the composition of a locally contractive functor with a locally non-expansive one is locally contractive. □

Note that, when $X = 0$ is the empty metric space, the coalgebra of this functor correspond to the final c-Markov process we have used in Section 5.3.1 to characterise the c-discounted probabilistic bisimilarity distance. When X is not the empty space, we obtain coalgebraic structures that can be interpreted as Markov process with X-labelled terminal states; one can view the labels in X as describing different kind of termination of the process.

Hence, in the light of Theorem 5.20, we have shown that for the case of complete metric spaces $\mathcal{U}_{\mathbf{MP}}$ axiomatises the c-discounted bisimilarity distance on the final Markov process.

Remark 5.21. While by interpreting the theory $\mathcal{U}_{\mathbf{MP}}$ over **Met** we can only characterise Markov processes that are acyclic, by doing it over **CMet** we obtain an algebraic representation of all bisimilarity classes as the elements of the final coalgebra. Thus, among others, we also recover Markov processes with cyclic structures as the limit of all their finite unfoldings.

6. Tensor of Quantitative Theories

In this section we consider the commutative combination of quantitative theories, their *tensor*, that imposes mutual commutation of the operations from each theory. As such, it specializes the sum of two theories, which is just their unrestrained combination.

The main theoretical result is that the free monad on the tensor of two basic theories corresponds to the categorical tensor of the free monads on the theories (*cf.*, Theorem 6.15).

Our main examples of tensor of quantitative effects are given by the combination of reader and writer quantitative monads with arbitrary quantitative effects (Section 6.3). We conclude the section by showing three nontrivial applications of tensorial combinations of quantitative theories by providing modular axiomatisations of labelled Markov processes (Section 6.4), Mealy machines (Section 6.5), and Markov decision processes (Section 6.6), with their respective bisimilarity distances.

6.1. **Tensor of Strong Monads.** In this section we recall the definition of (categorical) *tensor of strong monads* on a generic symmetric monoidal closed category $\mathbf{V}$. The presentation follows that of Manes [29], which considers only the case for $\mathbf{Set}$ monads.

Fix $\mathbf{V}$ a symmetric monoidal closed category, with monoidal product $\square\colon \mathbf{V}\times\mathbf{V}\to\mathbf{V}$ and internal hom-functor $[-,-]\colon \mathbf{V}\times\mathbf{V}\to\mathbf{V}$. Let v be an object in $\mathbf{V}$. As $\mathbf{V}$ is self-enriched, it has all v-fold *powers* (or v-*powers*) X^v, of any object $X\in\mathbf{V}$, defined as $X^v=[v,X]$ [25]. Moreover, $(-)^v\colon \mathbf{V}\to\mathbf{V}$ is a strong functor with strength $\xi_{X,Y}\colon X\,\square\,Y^v\to(X\,\square\,Y)^v$ obtained by currying

$$v\,\square\,(X\,\square\,Y^v)\xrightarrow{\cong}X\,\square\,(v\,\square\,Y^v)\xrightarrow{X\square ev}X\,\square\,Y\,.$$

Let $F\colon\mathbf{V}\to\mathbf{V}$ be a strong functor with strength t. The v-*power* functor $(-)^v$ is lifted to F-algebras by mapping (A,a) to $(A,a)^v=(A^v,a^v\circ\sigma_A)$, where $\sigma_A\colon FA^v\Rightarrow(FA)^v$ is the strong natural transformation obtained from t by currying

$$v\,\square\,FA^v\xrightarrow{t}F(v\,\square\,A^v)\xrightarrow{Fev}FA\,.$$

We call $(A,a)^v$ the v-power of (A,a). As the definition above is valid for generic objects v in $\mathbf{V}$ we have that F-algebras are closed under powers of $\mathbf{V}$-objects.

Definition 6.1 (Pre-operation of a strong functor)**.** Let $F\colon\mathbf{V}\to\mathbf{V}$ be a strong functor and $v\in\mathbf{V}$. A *v-ary pre-operation of F* is a strong natural transformation of type $(-)^v\Rightarrow F$.

We denote by $\mathcal{O}_F(v)$ the set of v-ary pre-operations of F. An assignment of $g\in\mathcal{O}_F(v)$ to an F-algebra (A,a) is the composite $a^g=a\circ g_A$. We call a^g an *operation* of (A,a).

Proposition 6.2. *Let $(A,a),(B,b)$ be F-algebras of a strong endofunctor F on $\mathbf{V}$ and $f\colon A\to B$ a morphism in $\mathbf{V}$. Then, the following are equivalent:*

(1) *f is a F-homomorphisms from (A,a) to (B,b);*
(2) *For every $v\in\mathbf{V}$ and $g\in\mathcal{O}_F(v)$, $f\circ a^g=b^g\circ f^v$.*

Proof. (1) $\Rightarrow$ (2) follows by definition of a^g, b^g and naturality of g. As for (2) $\Rightarrow$ (1), note that since $\mathbf{V}$ is a symmetric monoidal closed category, we have a 1-1 correspondence between strong and $\mathbf{V}$-enriched endofunctors on $\mathbf{V}$, and also between strong and $\mathbf{V}$-enriched natural transformations [26]. Therefore, by (the weak form of) the enriched Yoneda lemma (*cf.* [25]), there exists a natural bijection between strong natural transformations $g\in\mathcal{O}_F(A)$ and the (generalised) elements of FA, *i.e.*, morphisms of the form $I \to FA$, obtained via the composition

$$I \xrightarrow{id_A} A^A \xrightarrow{g_A} FA\,.$$

Thus, for any $e\colon I \to FA$, there exists $\hat{e} \in \mathcal{O}_F(A)$ such that $\hat{e}_A \circ id_A = e$. Therefore, by naturality of $\hat{e}$, definition of $a^{\hat{e}}$, $b^{\hat{e}}$, and (2), the following diagram commute

$$\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, "id_A"'] \arrow[dd, "e"'] & FA \arrow[rr, "a"] & & A \arrow[dd, "f"] \\
& A^A \arrow[u, "\hat{e}_A"] \arrow[r, "f^A"] \arrow[dl, "\hat{e}_A"'] & B^A \arrow[d, "\hat{e}_B"] & \\
FA \arrow[rr, "Ff"] & & FB \arrow[r, "b"] & B
\end{tikzcd}$$

implying that $f^I \circ a^I = b^I \circ (Ff^I)$. Then (1) follows by the naturality of the isomorphism $V \xrightarrow{\cong} V^I$ (obtained by currying $\lambda\colon I \,\square\, V \xrightarrow{\cong} V$) and the commutativity of the diagram

$$\begin{tikzcd}
FA \arrow[r, "\cong"] \arrow[d, "Ff"'] \arrow[rrr, bend left, "a"] & (FA)^I \arrow[r, "a^I"] \arrow[d, "(Ff)^I"'] & A^I \arrow[d, "f^I"] & A \arrow[l, "\cong"'] \arrow[d, "f"] \\
FB \arrow[r, "\cong"] \arrow[rrr, bend right, "b"'] & (FB)^I \arrow[r, "b^I"] & B^I & B \arrow[l, "\cong"']
\end{tikzcd}$$

□

The above proposition indicates that F-algebras are precisely characterised by their operations. In some situations, depending on the functor F, one gets the same characterisation with much fewer operations. We identify this property with the following definition.

Definition 6.3 (Density)**.** A set $\mathcal{D}$ of pre-operations of a strong functor $F\colon \mathbf{V} \to \mathbf{V}$ is *dense*, if for any F-algebras (A, a), (B, b) and $f\colon A \to B$ in $\mathbf{V}$, the following are equivalent:

(1) f is a F-homomorphisms from (A, a) to (B, b);
(2) For every v-ary pre-operation $g \in \mathcal{D}$, $f \circ a^g = b^g \circ f^v$.

Let F, G be two strong endofunctors on $\mathbf{V}$. A $\langle F, G\rangle$-*bialgebra* is a triple (A, a, b) consisting of an object $A \in \mathbf{V}$ with both a F-algebra structure $a\colon FA \to A$ and a G-algebra structure $b\colon GA \to A$. A morphism of $\langle F, G\rangle$-bialgebras is an arrow that is simultaneously a F- and G-homomorphism. Denote by $\langle F, G\rangle$-**biAlg** the category of $\langle F, G\rangle$-bialgebras.

Proposition 6.4. *Let (A, a, b) be a $\langle F, G\rangle$-bialgebra. The following statements are equivalent:*

(1) *For all $v \in \mathbf{V}$ and $g \in \mathcal{O}_F(v)$, a^g is a G-homomorphism;*
(2) *For all $w \in \mathbf{V}$ and $h \in \mathcal{O}_G(w)$, b^h is a F-homomorphism.*

Diagrammatically:

$$\begin{tikzcd}
GA^v \arrow[r, "\bar{b}"] \arrow[d, "G(a^g)"'] \arrow[dr, phantom, "(1)"] & A^v \arrow[d, "a^g"] \\
GA \arrow[r, "b"] & A
\end{tikzcd}
\quad \textit{iff} \quad
\begin{tikzcd}
FA^w \arrow[r, "\bar{a}"] \arrow[d, "F(b^h)"'] \arrow[dr, phantom, "(2)"] & A^w \arrow[d, "b^h"] \\
FA \arrow[r, "a"] & A
\end{tikzcd}$$

where $(A, a)^w = (A^w, \bar{a})$ and $(A, b)^v = (A^v, \bar{b})$.

In order to prove the above statement, it is convenient to introduce a technical result that will be useful also in later discussions.

Proposition 6.5. *Let (A, a) be a F-algebra of a strong endofunctor F on $\mathbf{V}$. Then, for any $v, w \in \mathbf{V}$ and $g \in \mathcal{O}_F(v)$ the following commute*

$$\begin{tikzcd}
(A^v)^w \arrow[rr, "\chi", "\cong"'] \arrow[dr, "(a^g)^w"'] & & (A^w)^v \arrow[dl, "\bar{a}^g"] \\
& A^w &
\end{tikzcd}$$

where $(A, a)^w = (A^w, \bar{a})$ and χ is the canonical isomorphism.

Proof. By the universality of the counit $ev\colon (w \,\Box\, -) \Rightarrow Id$ of the adjunction $(w \,\Box\, -) \dashv (-)^w$ it suffices to show that the following two diagrams commute:

$$\begin{tikzcd}
& w \,\Box\, (A^v)^w \arrow[dl, "a^g \circ ev"'] \arrow[d, "w \Box (a^g)^w"] \\
A & w \,\Box\, A^w \arrow[l, "ev"]
\end{tikzcd}
\qquad
\begin{tikzcd}
w \,\Box\, (A^v)^w \arrow[r, "w \Box \chi"] \arrow[d, "a^g \circ ev"'] & w \,\Box\, (A^w)^v \arrow[d, "w \Box \bar{a}^g"] \\
A & w \,\Box\, A^w \arrow[l, "ev"]
\end{tikzcd}$$

The diagram to the left commutes by naturality of the counit ev; the one to the right commutes as follows, where ξ and t are respectively the strengths of $(-)^v$ and F

$$\begin{tikzcd}
& w \,\Box\, (A^w)^v \arrow[d, "\xi"] \arrow[ddr, bend left, "w \Box g"] & \\
w \,\Box\, (A^v)^w \arrow[ur, "w \Box \chi"] \arrow[d, "ev"'] & (w \,\Box\, A^w)^v \arrow[dl, "ev^v"'] \arrow[d, "g"] & \\
A^v \arrow[d, "g"'] & F(w \,\Box\, A^w) \arrow[dl, "Fev"'] & w \,\Box\, FA^w \arrow[l, "t"'] \arrow[d, "w \Box \sigma"] \\
FA \arrow[d, "a"'] & & w \,\Box\, (FA)^w \arrow[ll, "ev"'] \arrow[d, "w \Box a^w"] \\
A & & w \,\Box\, A^w \arrow[ll, "ev"']
\end{tikzcd}$$

by naturality of the counit ev; definition of ξ and χ; definition of the law $\sigma\colon F(-)^w \Rightarrow (F-)^w$; definition of a^g, $\bar{a}^g$; by $\bar{a} = a^w \circ \sigma$; and because g is strong. $\square$

Proof. (of Proposition 6.4) (1) $\Rightarrow$ (2) By Proposition 6.2, we prove (2) by showing that for all $v \in \mathbf{V}$ and $g \in \mathcal{O}_F(v)$, $b^h \circ \bar{a}^g = a^g \circ (b^h)^v$. This is shown by the diagram below

$$\begin{tikzcd}
(A^w)^v \arrow[rr, "\bar{a}^g"] \arrow[dr, bend left, "\chi"] \arrow[dd, "(b^h)^v"'] & & A^w \arrow[dd, "b^h"] \\
& (A^v)^w \arrow[ul, bend left, "\chi^{-1}"] \arrow[ur, "(a^g)^w"'] \arrow[dl, "\bar{b}^h"'] & \\
A^v \arrow[rr, "a^g"] & & A
\end{tikzcd}$$

which commutes by Proposition 6.5, (1), definition of a^g, and naturality of g. The implication (2) $\Rightarrow$ (1) is similar. $\square$

Definition 6.6 (Commutative bialgebra)**.** A $\langle F, G\rangle$-bialgebra (A, a, b) is *commutative* if it satisfies either of the equivalent conditions of Proposition 6.4.

In the case the functors F and G admit dense sets of pre-operations, commutativity for their bialgebras can be more conveniently expressed in the following way.

Proposition 6.7. *Let $\mathcal{D}$ and $\mathcal{E}$ be dense sets of pre-operations for F and G, respectively. A $\langle F, G\rangle$-bialgebra (A, a, b) is* commutative *iff it satisfies either of the equivalent conditions:*

(1) *For all $g \in \mathcal{D}$, a^g is a G-homomorphism;*
(2) *For all $h \in \mathcal{E}$, b^h is a F-homomorphism.*

Proof. The equivalence of the statements (1), (2) follows as in Proposition 6.4, by using the density of $\mathcal{D}$ and $\mathcal{E}$ in lieu of Proposition 6.2.

Assume (A, a, b) is a commutative $\langle F, G\rangle$-bialgebra. Then, (1) follows trivially because, $\mathcal{D}$ is a subset of pre-operations of F. For the converse implication, assume (1) and let $h \in \mathcal{O}_G(w)$ for some $w \in \mathbf{V}$. We want to show that

$$\begin{tikzcd} FA^w \arrow[r, "\bar{a}"] \arrow[d, "F(b^h)"'] & A^w \arrow[d, "b^h"] \\ \Sigma A \arrow[r, "a"] & A \end{tikzcd}$$

commutes, where $(A, a)^w = (A^w, \bar{a})$. By density of $\mathcal{D}$, it suffices to show that for all v-ary pre-operation $g \in \mathcal{D}$, $b^h \circ \bar{a}^g = a^g \circ (b^h)^v$. This follows by

$$\begin{tikzcd} (A^w)^v \arrow[rr, "\bar{a}^g"] \arrow[dr, bend left, "\chi"] \arrow[dd, "(b^h)^v"'] & & A^w \arrow[dd, "b^h"] \\ & (A^v)^w \arrow[ul, bend left, "\chi^{-1}"] \arrow[ur, "(a^g)^w"'] \arrow[dl, "\bar{b}^h"'] & \\ A^v \arrow[rr, "a^g"] & & A \end{tikzcd}$$

which commutes by Proposition 6.5, (1), definition of a^g, and naturality of g. $\square$

Let (T, η, μ) be a strong monad on $\mathbf{V}$. Note that, as T is a strong functor and the EM-algebras for T are closed under powers of $\mathbf{V}$-objects, all the results and definitions given in this section extends to EM-algebras for T.

Let T, T' be two strong monads on $\mathbf{V}$. A Eilenberg-Moore $\langle T, T'\rangle$*-bialgebra* is a triple (A, a, a') consisting of an object $A \in \mathbf{V}$ with both a EM T-algebra structure $a \colon TA \to A$ and a EM T'-algebra structure $a' \colon T'A \to A$. We say that a EM $\langle T, T'\rangle$-bialgebra (A, a, b) is *commutative* if it is so as a $\langle T, T'\rangle$-bialgebra for the functors T, T'. We denote by $\mathbf{EM}\langle T, T'\rangle$ the category of EM $\langle T, T'\rangle$-bialgebras and by $\mathbf{EM_c}\langle T, T'\rangle$, the full subcategory of the commutative EM $\langle T, T'\rangle$-bialgebras.

Definition 6.8 (Tensor of monads)**.** If the forgetful functor $\mathbf{EM_c}\langle T, T'\rangle \to \mathbf{V}$ has left adjoint, then the monad induced by the adjunction is the *tensor* of T, T', denoted $T \otimes T'$.

Note that the tensor of monads does not necessarily exist (see [10] for counterexamples). However, when it does $T \otimes T' \cong T' \otimes T$, as the categories of commutative biagebras $\mathbf{EM_c}\langle T, T'\rangle$ and $\mathbf{EM_c}\langle T', T\rangle$ are isomorphic.

6.2. **Tensor of Quantitative Theories.** In this section, we develop the theory for the *tensor* of quantitative equational theories. The main result is that the free monad on the tensor of two theories is the tensor of the monads on the theories. In the proof given, we use the fact that the quantitative theories are *basic*, as this allows us to exploit the correspondence between the algebras of a theory $\mathcal{U}$ and the EM-algebras of the monad $T_{\mathcal{U}}$ (Theorem 3.5).

Let Σ, Σ' be two disjoint signatures. Following Freyd [15] (and [20]), we define the tensor of two quantitative equational theories $\mathcal{U}, \mathcal{U}'$ of respective types Σ and Σ', written $\mathcal{U} \otimes \mathcal{U}'$, as the smallest quantitative theory containing $\mathcal{U}, \mathcal{U}'$ and the quantitative equations

$$\vdash f(g(x_1^1, \dots, x_m^1), \dots, g(x_1^n, \dots, x_m^n)) =_0 g(f(x_1^1, \dots, x_1^n), \dots, f(x_m^1, \dots, x_m^n)) \,, \tag{6.1}$$

for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$, expressing that the operations of one theory commute with the operations of the other.

6.2.1. *Density of Symbolic Pre-operations.* Towards our main result, we identify a dense set of pre-operations for the free monads on quantitative equational theories which, in turn, will give us a simpler characterization for commutative bialgebras for these monads (*cf.* Proposition 6.7).

First notice that any signature functor $\Sigma = \coprod_{f:n\in\Sigma} Id^n$ in $\mathbf{Met}$ is strong, as it is the coproduct of the strong functors $Id^n \cong (-)^{\underline{n}}$, where $\underline{n} \in \mathbf{Met}$ denotes the set $\{1, \dots, n\}$ equipped with the discrete extended metric assigning infinite distance to distinct elements. Moreover, the injections $in_f\colon (-)^{\underline{n}} \Rightarrow \Sigma$ are strong natural transformations, hence they are $\underline{n}$-ary pre-operations of Σ (*cf.* Definition 6.1).

Proposition 6.9. *$\mathcal{S}_\Sigma = \{in_f \mid f\colon n \in \Sigma\}$ is a dense set of pre-operations of Σ.*

Proof. Let $(A, a), (B, b)$ be Σ-algebras in $\mathbf{Met}$ and $h\colon A \to B$ a non-expansive map. We want to prove the equivalence of

(1) f is a Σ-homomorphisms from (A, a) to (B, b);
(2) For every $f\colon n \in \Sigma$, $h \circ a^{in_f} = b^{in_f} \circ h^v$.

(1) $\Rightarrow$ (2) follows by definition of a^{in_f}, b^{in_f} and naturality of $in_f\colon (-)^{\underline{n}} \Rightarrow \Sigma$. The implication (2) $\Rightarrow$ (1) follows by the universality of the coproduct, as $\Sigma = \coprod_{f:n\in\Sigma} Id^n$. $\square$

In the following, the pre-operations in $\mathcal{S}_\Sigma$ will be called *symbolic*, and to simplify the notation, for any $f\colon n \in \Sigma$ and Σ-algebra (A, a), we write a^f instead of a^{in_f}.

Now we turn to study the pre-operations of the monad $T_{\mathcal{U}}$, for a quantitative equational theory $\mathcal{U}$ of type Σ. First notice that the monad $T_{\mathcal{U}}$ is strong, with strength

$$\zeta_{X,Y}\colon X \,\square\, T_{\mathcal{U}} Y \to T_{\mathcal{U}}(X \,\square\, Y)$$

obtained by uncurrying the unique map $h_{X,Y}$ that, by Theorem 3.3, makes the following diagram commute

$$\begin{tikzcd}
Y \arrow[r, "\eta^{\mathcal{U}}_Y"] \arrow[dr, bend right, "\beta_{X,Y}"'] & T_{\mathcal{U}}Y \arrow[d, dashed, "h_{X,Y}"'] & \Sigma T_{\mathcal{U}}Y \arrow[l, "\psi^{\mathcal{U}}_Y"'] \arrow[d, "\Sigma h_{X,Y}"] \\
& (T_{\mathcal{U}}(X \square Y))^X & \Sigma(T_{\mathcal{U}}(X \square Y))^X \arrow[l, "\psi^{\mathcal{U}}_{(T(X\square Y))^X}"']
\end{tikzcd}$$

where $\beta_{X,Y}$ is the currying of $\eta^{\mathcal{U}}_{X\square Y} \colon X \square Y \to T_{\mathcal{U}}(X \square Y)$.

Since a monad is strong iff both its unit and multiplication are strong natural transformations, both $\eta^{\mathcal{U}}$, $\mu^{\mathcal{U}}$ are strong. Moreover, also $\psi^{\mathcal{U}} \colon \Sigma T_{\mathcal{U}} \Rightarrow T_{\mathcal{U}}$ is strong.

Thus any pre-operation $g \in \mathcal{O}_\Sigma(v)$ can be tuned into a pre-operation of $T_{\mathcal{U}}$ as the composite

$$(-)^v \xrightarrow{g} \Sigma \xrightarrow{\Sigma\eta^{\mathcal{U}}} \Sigma T_{\mathcal{U}} \xrightarrow{\psi^{\mathcal{U}}} T_{\mathcal{U}}\,.$$

In particular, when the theory $\mathcal{U}$ is basic, by Theorem 3.5, the above composition allows us to turn any dense set of pre-operations of Σ into a dense set of pre-operations of $T_{\mathcal{U}}$.

Proposition 6.10. *Let $\mathcal{U}$ be a basic quantitative theory of type Σ and $\mathcal{D}$ a dense set of pre-operations of Σ. Then $\{\psi^{\mathcal{U}} \circ \Sigma\eta^{\mathcal{U}} \circ g \mid g \in \mathcal{D}\}$ is a dense set of pre-operations of $T_{\mathcal{U}}$.*

Proof. $(A, a), (B, b)$ be $T_{\mathcal{U}}$-algebras and $h \colon A \to B$ a non-expansive map. We want to prove the equivalence of

(1) h is a $T_{\mathcal{U}}$-homomorphism from (A, a) to (B, b);
(2) For every v-ary pre-operation $g \in \mathcal{D}$, $h \circ a^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ g)} = b^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ g)} \circ h^v$.

(1) $\Rightarrow$ (2) follows by definition of $a^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ g)}$, $b^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ g)}$ and naturality of $\psi^{\mathcal{U}} \circ \Sigma\eta^{\mathcal{U}} \circ g$. For the converse implication, recall that the isomorphism of categories from Theorem 3.5, maps a $T_{\mathcal{U}}$-algebra (A, a) to the Σ-algebra $(A, a \circ \psi^{\mathcal{U}}_A \circ \Sigma\eta^{\mathcal{U}}_A)$ and morphisms essentially to themselves. Thus (2) $\Rightarrow$ (1) follows by density of $\mathcal{D}$ and definition of $a^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ g)}$, $b^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ g)}$. $\square$

When $\mathcal{U}$ is a basic theory, by combining Propositions 6.9 and 6.10, we easily obtain a dense set of pre-operations also for the monad $T_{\mathcal{U}}$.

Corollary 6.11. *$\mathcal{S}_{T_{\mathcal{U}}} = \{\psi^{\mathcal{U}} \circ \Sigma\eta^{\mathcal{U}} \circ in_f \mid f \colon n \in \Sigma\}$ is a dense set of pre-operations of $T_{\mathcal{U}}$, whenever $\mathcal{U}$ is a basic quantitative equational theory.*

Also the pre-operations in $\mathcal{S}_{T_{\mathcal{U}}}$ will be called *symbolic* and we simplify the notation by writing $a^{\langle f\rangle}$ instead of $a^{(\psi^{\mathcal{U}} \circ\Sigma\eta^{\mathcal{U}} \circ in_f)}$, for $f \colon n \in \Sigma$ and $(A, a) \in T_{\mathcal{U}}$-**Alg**.

As an immediate consequence of Corollary 6.11 and Proposition 6.7, we obtain the following simpler characterization for commutative $\langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$-bialgebras.

Corollary 6.12. *Let $\mathcal{U}$, $\mathcal{U}'$ be basic quantitative theories respectively of type Σ, Σ'. A $\langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$-bialgebra (A, a, b) is commutative iff it satisfies either of the equivalent conditions*

(1) *For all $f \colon n \in \Sigma$, $a^{\langle f\rangle}$ is a $T_{\mathcal{U}'}$-homomorphism;*
(2) *For all $g \colon n \in \Sigma'$, $b^{\langle g\rangle}$ is a $T_{\mathcal{U}}$-homomorphism.*

6.2.2. *Tensor of Free Monads on Quantitative Theories.* Let $\mathcal{U}$,$\mathcal{U}'$ be basic quantitative theories respectively of type Σ, Σ'. We show that any model for $\mathcal{U} \otimes \mathcal{U}'$ is a $\langle \mathcal{U} \otimes \mathcal{U}' \rangle$*-bialgebra*: an extended metric space A with both a Σ-algebra structure $a\colon \Sigma A \to A$ satisfying $\mathcal{U}$ and a Σ'-algebra structure $b\colon \Sigma' A \to A$ satisfying $\mathcal{U}'$ and respecting the diagrammatic condition below, for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$

$$\begin{tikzcd} A^{\underline{n}} \arrow[r, "a^f"] & A & A^{\underline{m}} \arrow[l, "b^g"'] \\ (A^{\underline{m}})^{\underline{n}} \arrow[u, "(b^g)^{\underline{n}}"] \arrow[rr, "\chi", "\cong"'] & & (A^{\underline{n}})^{\underline{m}} \arrow[u, "(a^f)^{\underline{m}}"'] \end{tikzcd} \tag{6.2}$$

Formally, we denote by $\mathbb{K}((\Sigma,\mathcal{U}) \otimes (\Sigma',\mathcal{U}'))$ the category of $\langle \mathcal{U} \otimes \mathcal{U}' \rangle$-bialgebras, with morphisms the non-expansive homomorphisms preserving both algebraic structures. Then, the following isomorphism of categories holds.

Proposition 6.13. *$\mathbb{K}(\Sigma + \Sigma',\mathcal{U} \otimes \mathcal{U}') \cong \mathbb{K}((\Sigma,\mathcal{U}) \otimes (\Sigma',\mathcal{U}'))$, for $\mathcal{U}$,$\mathcal{U}'$ basic theories.*

Proof. The isomorphism is given by the pair of functors

$$\begin{tikzcd} \mathbb{K}(\Sigma + \Sigma',\mathcal{U} \otimes \mathcal{U}') \arrow[r, bend left, "H"] & \mathbb{K}((\Sigma,\mathcal{U}) \otimes (\Sigma',\mathcal{U}')) \arrow[l, bend left, "K"] \end{tikzcd}$$

defined, for a $(\Sigma + \Sigma')$-algebra (A, a) satisfying $\mathcal{U} \otimes \mathcal{U}'$ and a $\langle \mathcal{U} \otimes \mathcal{U}' \rangle$-bialgebra (B, b, b'), respectively as

$$H(A, a) = (A, a \circ in_l, a \circ in_r)\,, \qquad\qquad K(B, b, b') = (B, [b, b'])\,,$$

where $[b, b']\colon \Sigma B + \Sigma' B \to B$ is the unique map induced by b and b' by universality of the coproduct. Both functors are identity on morphisms; it is easy to see that a homomorphism in one sense is also a homomorphism in the other.

The pair of functors above is the restriction of the isomorphic pair of functors used in the proof of [3, Proposition 4.1]. Thus, to show H and K are well defined we are just left to deal with checking that the restriction conditions on the subcategories are preserved both ways.

As for H, we prove that whenever $\mathcal{A} = (A, a)$ satisfies the quantitative equation in (6.1), then $(A, a \circ in_l, a \circ in_r)$ satisfies the commutativity of the diagram in (6.2). This follows as, for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$, by definition of algebraic interpretation $(-)^{\mathcal{A}}$, we have

$$f^{\mathcal{A}} = a \circ in_l \circ in_f = (a \circ in_l)^f\,,$$
$$g^{\mathcal{A}} = a \circ in_r \circ in_g = (a \circ in_r)^g\,.$$

Thus, the satisfiability (6.1) coincides with the commutativity of the diagram in (6.2).

For K we need to show that whenever (B, b, b') satisfies the commutativity of the diagram in (6.2), then $\mathcal{A} = (A, [b, b'])$ satisfies (6.1). This follows as, for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$, by definition of algebraic interpretation $(-)^{\mathcal{A}}$, we have

$$f^{\mathcal{A}} = [b, b'] \circ in_l \circ in_f = (b)^f\,,$$
$$g^{\mathcal{A}} = [b, b'] \circ in_r \circ in_g = (b')^g\,.$$

Thus, the commutativity of the diagram in (6.2) coincides with the satisfiability of (6.1). $\square$

Moreover, by adapting the isomorphism of Theorem 3.5 and exploiting the density of symbolic pre-operations (*cf.* Corollary 6.12) the following is also true.

Proposition 6.14. *$\mathbb{K}((\Sigma,\mathcal{U})\otimes(\Sigma',\mathcal{U}'))\cong \mathbf{EM_c}\langle T_{\mathcal{U}},T_{\mathcal{U}'}\rangle$, for $\mathcal{U},\mathcal{U}'$ basic theories.*

Proof. Recall the isomorphism of categories from Theorem 3.5

$$\begin{tikzcd} T_{\mathcal{U}}\text{-}\mathbf{Alg} \arrow[r, bend left, "H"] & \mathbb{K}(\Sigma,\mathcal{U}) \arrow[l, bend left, "K"] \end{tikzcd}$$

mapping morphisms to themselves and on objects acting as follows: for $(A,a)\in T_{\mathcal{U}}\text{-}\mathbf{Alg}$ and $(B,b)\in\mathbb{K}(\Sigma,\mathcal{U})$,

$$H(A,a)=(A,a\circ\psi^{\mathcal{U}}_A\circ\Sigma\eta^{\mathcal{U}}_A)\,, \qquad\qquad K(B,b)=(B,b_\flat)\,,$$

where $\beta_\flat\colon T_{\mathcal{U}}B\to B$ is the unique map that, by Theorem 3.3, satisfies the equations $b_\flat\circ\eta^{\mathcal{U}}_B=id_B$ and $b_\flat\circ\psi^{\mathcal{U}}_B=b\circ\Sigma b_\flat$. (for the details on the proof *cf.* [3, Theorem 4.2]).

Next we show that the obvious point-wise extension of the above functors on the categories of bialgebras $\mathbb{K}((\Sigma,\mathcal{U})\otimes(\Sigma',\mathcal{U}'))$ and $\mathbf{EM_c}\langle T_{\mathcal{U}},T_{\mathcal{U}'}\rangle$ is an isomorphism of categories.

Clearly, since H and K are inverse with each other, so are their point-wise extensions. The only thing we are left to prove is that they are well defined; in particular that the respective commutative conditions are preserved.

Let $(A,a,b)\in\mathbf{EM_c}\langle T_{\mathcal{U}},T_{\mathcal{U}'}\rangle$. We need to show that condition (6.2) is satisfied by $(A,a\circ\psi^{\mathcal{U}}_A\circ\Sigma\eta^{\mathcal{U}}_A,b\circ\psi^{\mathcal{U}}_A\circ\Sigma\eta^{\mathcal{U}}_A)$. Let $(A,b)^{\underline{n}}=(A^{\underline{n}},\bar{b})$. By Corollary 6.12 and Propositions 6.9, 6.10, we have that the bottom square diagram below commutes for all $f\colon n\in\Sigma$ and all $g\colon m\in\Sigma'$, while the top commute by Proposition 6.5:

$$\begin{tikzcd}
(A^{\underline{m}})^{\underline{n}} \arrow[d, "\chi"'] \arrow[dr, bend left, "(b^{\langle g\rangle})^{\underline{n}}"] & \\
(A^{\underline{n}})^{\underline{m}} \arrow[r, "\bar{b}^{\langle g\rangle}"] \arrow[d, "(a^{\langle f\rangle})^{\underline{m}}"'] & A^{\underline{n}} \arrow[d, "a^{\langle f\rangle}"] \\
A^{\underline{m}} \arrow[r, "b^{\langle g\rangle}"] & A
\end{tikzcd}$$

Since $a^{\langle f\rangle}=(a\circ\psi^{\mathcal{U}}_A\circ\Sigma\eta^{\mathcal{U}}_A)^f$ and $b^{\langle g\rangle}=(b\circ\psi^{\mathcal{U}}_A\circ\Sigma\eta^{\mathcal{U}}_A)^g$, the above diagram proves that condition (6.2) holds.

Let $(A,a,b)\in\mathbb{K}((\Sigma,\mathcal{U})\otimes(\Sigma',\mathcal{U}'))$. We need to show that $(A,a_\flat,b_\flat)$ is a $\langle T_{\mathcal{U}},T_{\mathcal{U}'}\rangle$-bialgebra. By Corollary 6.12, it is sufficient to prove that the following diagram commutes for all $g\colon m\in\Sigma'$,

$$\begin{tikzcd}
T_{\mathcal{U}}A^{\underline{m}} \arrow[r, "\overline{a_\flat}"] \arrow[d, "T_{\mathcal{U}}(b_\flat^{\langle g\rangle})"'] & A^{\underline{m}} \arrow[d, "b_\flat^{\langle g\rangle}"] \\
T_{\mathcal{U}}A \arrow[r, "a_\flat"] & A
\end{tikzcd} \tag{6.3}$$

where $(A,a_\flat)^{\underline{m}}=(A^{\underline{m}},\overline{a_\flat})$.

Toward proving (6.3), first notice that the diagram below commutes for all $f\colon n \in \Sigma$ and $g\colon m \in \Sigma'$

$$
\begin{tikzcd}
(A^{\underline{n}})^{\underline{m}} \arrow[rr, "\bar{a}^f"] \arrow[dd, "(b^g)^{\underline{n}}"'] \arrow[dr, bend left, "\chi^{-1}"] & & A^{\underline{m}} \arrow[dd, "b^g"] \\
& (A^{\underline{m}})^{\underline{n}} \arrow[ul, bend left, "\chi"] \arrow[ur, "(a^f)^{\underline{m}}"'] & \\
A^{\underline{n}} \arrow[rr, "a^f"] \arrow[ur, phantom, "(6.2)"] & & A
\end{tikzcd}
\tag{6.4}
$$

for $(A,a)^{\underline{m}} = (A^{\underline{m}}, \bar{a})$ and $(A,b)^{\underline{n}} = (A^{\underline{n}}, \bar{b})$. Indeed, the bottom commutes because (A,a,b) satisfies (6.2), and the top triangle does by Proposition 6.5.

Going back to proving (6.3), by Theorem 3.3, it suffices to show that both $b_\flat^{\langle g\rangle} \circ \overline{a_\flat}$ and $a_\flat \circ T_{\mathcal{U}}(b_\flat^{\langle g\rangle})$ are the (unique) homomorphic extension of a along $b_\flat^g$. This is shown by the following diagrams

$$
\begin{tikzcd}
A^{\underline{m}} \arrow[r, "\eta^{\mathcal{U}}"] \arrow[d, "b_\flat^{\langle g\rangle}"'] & T_{\mathcal{U}}A^{\underline{m}} \arrow[d, "T_{\mathcal{U}}b_\flat^{\langle g\rangle}"] \\
A \arrow[r, "\eta^{\mathcal{U}}"] \arrow[dr, "id"'] & T_{\mathcal{U}}A \arrow[d, "a_\flat"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{\underline{m}} \arrow[r, "\eta^{\mathcal{U}}"] \arrow[dr, "id"'] \arrow[ddr, bend right, "b_\flat^{\langle g\rangle}"'] & T_{\mathcal{U}}A^{\underline{m}} \arrow[d, "\overline{a_\flat}"] \\
& A^{\underline{m}} \arrow[d, "b_\flat^{\langle g\rangle}"] \\
& A
\end{tikzcd}
$$

$$
\begin{tikzcd}
T_{\mathcal{U}}A^{\underline{m}} \arrow[d, "T_{\mathcal{U}}b_\flat^{\langle g\rangle}"'] & \Sigma T_{\mathcal{U}}A^{\underline{m}} \arrow[l, "\psi^{\mathcal{U}}"'] \arrow[d, "\Sigma T_{\mathcal{U}}b_\flat^{\langle g\rangle}"] \\
T_{\mathcal{U}}A \arrow[d, "a_\flat"'] & \Sigma T_{\mathcal{U}}A \arrow[l, "\psi^{\mathcal{U}}"'] \arrow[d, "\Sigma a_\flat"] \\
A & \Sigma A \arrow[l, "a"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& T_{\mathcal{U}}A^{\underline{m}} \arrow[d, "\overline{a_\flat}"'] & \Sigma T_{\mathcal{U}}A^{\underline{m}} \arrow[l, "\psi^{\mathcal{U}}"'] \arrow[d, "\Sigma\overline{a_\flat}"] & \\
\Sigma' T_{\mathcal{U}'}A \arrow[d, "id"] & A^{\underline{m}} \arrow[d, "in_g"] & \Sigma A^{\underline{m}} \arrow[l, "\bar{a}"'] \arrow[d, "\Sigma in_g"'] & \Sigma\Sigma' T_{\mathcal{U}'}A \arrow[d, "id"] \\
\Sigma' T_{\mathcal{U}'}A \arrow[d, "\psi^{\mathcal{U}'}"] & \Sigma' A \arrow[l, "\Sigma' b_\flat"] \arrow[ul, "\Sigma'\eta^{\mathcal{U}'}"'] \arrow[d, "b"] & \Sigma\Sigma' A \arrow[r, "\Sigma\Sigma' b_\flat"'] \arrow[ur, "\Sigma\Sigma'\eta^{\mathcal{U}'}"] \arrow[d, "\Sigma b"] & \Sigma\Sigma' T_{\mathcal{U}'}A \arrow[d, "\psi^{\mathcal{U}'}"] \\
T_{\mathcal{U}'}A \arrow[r, "b_\flat"] & A & \Sigma A \arrow[l, "a"'] & \Sigma T_{\mathcal{U}'} \arrow[l, "\Sigma b_\flat"']
\end{tikzcd}
$$

that commute by definitions of $a_\flat$, $b_\flat$, $b_\flat^{\langle g\rangle}$, b^g; by naturality of $\eta^{\mathcal{U}}$, $\psi^{\mathcal{U}}$; since $(A,a_\flat)^{\underline{m}}$ is a EM $T_{\mathcal{U}}$-algebra; because by Theorem 3.5 $(A,a_\flat)^{\underline{m}} = (A,a)^{\underline{m}}$; and since from Propositions 6.9, 6.10 and (6.4) we have that b^g is a Σ-homomorphism. $\square$

By combining the above two propositions we get the main theorem of this section.

Theorem 6.15. *Let $\mathcal{U}$,$\mathcal{U}'$ be basic quantitative theories. Then, the monad $T_{\mathcal{U}\otimes\mathcal{U}'}$ in* **Met** *is the tensor of monads $T_{\mathcal{U}} \otimes T_{\mathcal{U}'}$.*

Proof. By Propositions 6.13 and 6.14 the forgetful functor from $\mathbf{EM_c}\langle T_{\mathcal{U}}, T_{\mathcal{U}'}\rangle$ to **Met** has a left adjoint and the monad generated by this adjunction is isomorphic to $T_{\mathcal{U}\otimes\mathcal{U}'}$. Thus, by definition of tensor of monads, $T_{\mathcal{U}\otimes\mathcal{U}'} \cong T_{\mathcal{U}} \otimes T_{\mathcal{U}'}$. □

The above results do not require any specific property of **Met**, apart that its morphisms are non-expansive maps. Thus, when the quantitative equational theories are continuous, we can reformulate an alternative version of Theorem 6.15 which is valid in **CMet**.

Theorem 6.16. *Let $\mathcal{U}$,$\mathcal{U}'$ be continuous quantitative theories. Then, $\mathbb{C}T_{\mathcal{U}\otimes\mathcal{U}'}$ in* **CMet** *is the tensor of monads $\mathbb{C}T_{\mathcal{U}} \otimes \mathbb{C}T_{\mathcal{U}'}$.*

Proof. The tensor $\mathcal{U} \otimes \mathcal{U}'$ of continuous theories is also continuous, so that, by Theorem 3.8, the free monad on it in **CMet** is $\mathbb{C}T_{\mathcal{U}\otimes\mathcal{U}'}$. Moreover, by exploiting the universal property of Theorem 3.8, we can refactor the proofs of Propositions 6.13 and 6.14 to obtain the isomorphism $\mathbb{CK}(\Sigma + \Sigma', \mathcal{U} \otimes \mathcal{U}') \cong \mathbf{EM_c}\langle \mathbb{C}T_{\mathcal{U}}, \mathbb{C}T_{\mathcal{U}'}\rangle$. Thus, by definition of tensor of monads, $\mathbb{C}T_{\mathcal{U}\otimes\mathcal{U}'} \cong \mathbb{C}T_{\mathcal{U}} \otimes \mathbb{C}T_{\mathcal{U}'}$. □

6.3. **Tensor with Reader/Writer Effects.** As an example of commutative combination of effects we consider the operation of tensoring a generic quantitative theory with the quantitative reader and writer theories, respectively. Similarly to Hyland et al. [19], we show that these operations corresponds, at the level of monads, to the so called reader and writer monad transformers of Moggi and Cenciarelli [35, 11].

Reader Monad Transformer. Let T be a strong monad with strength t and E a finite set.

The strength t gives rise to a distributive law of the monad T over the monad $(-)^{\underline{E}}$

$$\lambda_X : TX^{\underline{E}} \Rightarrow (TX)^{\underline{E}}$$

obtained by currying $T ev_X^{\underline{E}} \circ t_{\underline{E}, X^{\underline{E}}}$. As distributive laws induce a notion of monad composition [7], Moggi's *reader monad transformer*

$$T \mapsto (T-)^{\underline{E}}$$

is also available in **Met**. The following says that we can recover this monad transformer as the operation of tensoring with the reader monad.

Theorem 6.17 (Tensoring with Reader Monad)**.** *Let T be a strong monad. Then, $T \otimes (-)^{\underline{E}}$ exists and is given as the monad composition $(T-)^{\underline{E}}$.*

Proof. Recall that the composite $(T-)^{\underline{E}}$ is the monad that arises from the adjunction with the forgetful functor λ-**biAlg** → **Met**, where λ-**biAlg** denotes the full subcategory of EM $\langle T, (-)^{\underline{E}}\rangle$-bialgebras (A, a, b) satisfying the commutativity of the diagram

$$\begin{tikzcd} TA \arrow[r, "a"] & A & A^{\underline{E}} \arrow[l, "b"'] \\ T(A^{\underline{E}}) \arrow[u, "Tb"] \arrow[rr, "\lambda"] & & (TA)^{\underline{E}} \arrow[u, "(a)^{\underline{E}}"'] \end{tikzcd} \tag{6.5}$$

The bialgebras satisfying (6.5) are called, λ-bialgebras for the law $\lambda\colon T(-^{\underline{E}}) \Rightarrow (T-)^{\underline{E}}$ (see e.g., [7]). We show that the category of λ-bialgebras is identical to the category of commutative $\langle T \otimes (-)^{\underline{E}}\rangle$-bialgebras, that is, that the commutativity of the diagram above corresponds to either one of the equivalent conditions from Proposition 6.4.

One direction is easy, as if we assume (A, a, b) to be a commutative $\langle T \otimes (-)^{\underline{E}}\rangle$-bialgebra, then (6.5) is just the instantiation of (2) from Proposition 6.4 for $h = id \in \mathcal{O}_{(-)^{\underline{E}}}(\underline{E})$ as, by definition of $\underline{E}$-power algebra, $(A, a)^{\underline{E}} = (A^{\underline{E}}, (a)^{\underline{E}} \circ \lambda_A)$.

For the converse direction, assume (6.5) holds and let $g \in \mathcal{O}_T(v)$, for some $v \in \mathbf{Met}$. Then, asking that a^g is a $(-)^{\underline{E}}$-homomorphism (*i.e.*, condition (1) from Proposition 6.4) corresponds to the commutativity of the following diagram, as $(A, b)^v = (A^v, b^v \circ \sigma_A)$ and $(A, a)^{\underline{E}} = (A^{\underline{E}}, (a)^{\underline{E}} \circ \lambda_A)$:

$$\begin{tikzcd}
(A^v)^{\underline{E}} \arrow[r, "\sigma"] \arrow[d, "g^{\underline{E}}"'] & (A^{\underline{E}})^v \arrow[r, "b^v"] \arrow[d, "g"'] & A^v \arrow[d, "g"'] \\
(TA)^{\underline{E}} \arrow[dr, bend right, "a^{\underline{E}}"'] & T(A^{\underline{E}}) \arrow[r, "Tb"] \arrow[d, "(a)^{\underline{E}}\circ\lambda"'] & TA \arrow[d, "a"] \\
 & A^{\underline{E}} \arrow[r, "b"] & A
\end{tikzcd}$$

The bottom-left square is (6.5), so commutes by hypothesis; the top-right square commutes by naturality of g; and finally, the left square commutes by Proposition 6.5 as, by definitions of the strengths of $(-)^v$ and $(-)^{\underline{E}}$, $\sigma\colon (A^v)^{\underline{E}} \Rightarrow (A^{\underline{E}})^v$ coincides with the canonical isomorphism (denoted as χ in Proposition 6.5).

Therefore, as the two categories of bialgebras coincide, by definition of tensor of monads, $T \otimes (-)^{\underline{E}} = (T-)^{\underline{E}}$. $\square$

By using the above result in combination with Theorem 6.15, we obtain an analogous transformer at the level of quantitative equational theories as follows.

Corollary 6.18. *Let $\mathcal{U}$ be a basic quantitative equational theory. Then, $(T_{\mathcal{U}}-)^{\underline{E}}$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{R}$ in* $\mathbf{Met}$.

Moreover, as $\mathcal{R}$ is a continuous theory, by Theorems 4.9, 6.15, and 6.17, we obtain the following variant of the quantitative reader theory transformer on continuous theories.

Corollary 6.19. *Let $\mathcal{U}$ be a continuous quantitative theory. Then, $(\mathbb{C}T_{\mathcal{U}}-)^{\underline{E}}$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{R}$ in* $\mathbf{CMet}$.

Writer Monad Transformer. Let T be a strong monad with strength t and $(\Lambda, *, 0)$ a monoid structure with $\Lambda \in \mathbf{Met}$, unit $0 \in \Lambda$, and non-expansive multiplication $*\colon \Lambda \times \Lambda \to \Lambda$.

The strength t gives rise to a canonical distributive law of the monad $(\Lambda \,\square\, -)$ over T as

$$t_{\Lambda,-}\colon (\Lambda \,\square\, T-) \Rightarrow T(\Lambda \,\square\, -)\,.$$

So the composite $T(\Lambda \,\square\, -)$ acquires a canonical monad structure via the above distributive law [7], and we obtain the following version of Moggi's *writer monad transformer* in $\mathbf{Met}$:

$$T \mapsto T(\Lambda \,\square\, -)\,.$$

Hyland et al. In [20] observed that Moggi's writer monad transformer can be equivalently recovered as the operation of tensoring with the writer monad.

Theorem 6.20 (Tensoring with Writer Monad [20, Theorem 12]). *Let T be a strong monad with countable rank. Then, the monad composition $T(\Lambda \,\square\, -)$ is given as $T \otimes (\Lambda \,\square\, -)$.*

As any quantitative theory $\mathcal{U}$ induces a monad $T_{\mathcal{U}}$ with countable rank (*cf.* Ford et al. [14]), by combining the above with Theorems 6.15 and 4.10, we get an analogous transformer at the level of quantitative equational theories as follows:

Corollary 6.21. *Let $\mathcal{U}$ be a basic quantitative theory. Then, $T_{\mathcal{U}}(\Lambda \,\square\, -)$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{W}$ in* **Met***.*

As $\mathcal{W}$ is also a continuous quantitative theory, by similar arguments as before, we obtain the following variant of quantitative writer theory transformer on continuous theories.

Corollary 6.22. *Let $\mathcal{U}$ be a continuous quantitative theory. Then, $\mathbb{C}T_{\mathcal{U}}(\Lambda \,\square\, -)$ is the free monad on the theory $\mathcal{U} \otimes \mathcal{W}$ in* **CMet***.*

6.4. **The Algebras of Labeled Markov Processes.** In this section we provide a quantitative equational axiomatization of labelled Markov processes with their discounted bisimilarity metric [46, Section 6].

6.4.1. *Labelled Markov Processes over Metric Spaces.* Let A be a finite set of action labels. As in [46, Section 6], we regard A-labelled Markov processes over extended metric spaces as coalgebras on the category of metric spaces. In detail, we consider two variants of labelled Markov processes:

$$X \longrightarrow \Pi(c \cdot X + 1)^{\underline{A}} \quad \text{in } \mathbf{Met}\,,$$
$$X \longrightarrow \Delta(c \cdot X + 1)^{\underline{A}} \quad \text{in } \mathbf{CMet}\,,$$

where Π and Δ are the functors from Section 4.5, mapping a metric space X to a space of probability measures with Kantorovich metric. We will collectively refer these coalgebras as labelled c-Markov processes.

Similarly to Section 5.3.1, the use of the rescaling functor $(c \cdot -)$ is to encompass the case where the probabilistic bisimilarity distance is discounted by a factor $0 < c < 1$. This will not change the essence of the results from [46] that are used in this section to characterise the probabilistic bisimilarity metric.

In [46], van Breugel et al. characterised the bisimilarity distance on labelled Markov processes as the pseudometric induced by the unique homomorphism to the final coalgebra. Specifically, the *c-discounted bisimilarity pseudometric* on a labelled c-Markov process (X, τ) is obtained as the function $\mathbf{d}^c \colon X \times X \to [0, 1]$ given as

$$\mathbf{d}^c(x, x') = d_Z(h(x), h(x'))\,,$$

where $h \colon X \to Z$ is the unique homomorphism to the final labelled c-Markov process (Z, ω).

This distance has a characterisation as the least fixed point of a monotone function on a complete lattice of 1-bounded pseudometrics.

Proposition 6.23 ([46, Theorem 40]). *The c-discounted bisimilarity pseudometric $\mathbf{d}^c$ on (X, τ) is the* unique *fixed point of the following operator on the complete lattice of extended pseudometrics d on X with point-wise order $\sqsubseteq$, such that $d \sqsubseteq d_X$,*

$$\Psi^c(d)(x, x') = \sup_{a \in A} \sup_{f} \left| \int f \; \mathrm{d}\tau(x)(a) - \int f \; \mathrm{d}\tau(x')(a) \right| \,,$$

with f ranging over non-expansive positive 1*-bounded real valued functions on* $c \cdot X + 1$*.*

6.4.2. *Quantitative Algebraic Presentation.* We provide a quantitative equational theory that axiomatizes (the monad of) A-labelled Markov processes with c-discounted bisimilarity metric. We do this by extending the axiomatization of (unlabelled) Markov processes from Section 5.3 with a new "reading" operator used to describe the reaction to the choice of a label from a finite set A of action labels. As expected, the reading operations will be axiomatised by the theory $\mathcal{R}_A$ of reading computations (*cf.* Section 4.3.1).

Formally, for $A = \{a_1, \dots, a_n\}$ we define the quantitative theory of labelled Markov processes as the following combination of quantitative theories,

$$\mathcal{U}_{\mathbf{LMP}} = ((\mathcal{B} + \mathcal{E}_1) \otimes \mathcal{R}_A) + \mathcal{O}(\Sigma_\diamond)\,.$$

with signature $\Sigma_{\mathbf{MP}} = \Sigma_{\mathcal{B}} \cup \Sigma_1 \cup \Sigma_{\mathcal{R}_A} \cup \Sigma_\diamond$ given as the disjoint union of those from its component theories. Explicitly,

$$\Sigma_{\mathbf{LMP}} = \{+_e \colon 2 \mid e \in [0,1]\} \cup \{\mathsf{raise}_* \colon 0\} \cup \{\mathsf{r} \colon |A|\} \cup \{\diamond \colon \langle 1, c\rangle\}$$

theory $\mathcal{U}_{\mathbf{LMP}}$ is given by the following set of axioms

$$\begin{aligned}
(\mathsf{B1})&\ \vdash x +_1 y =_0 x\,,\\
(\mathsf{B2})&\ \vdash x +_e x =_0 x\,,\\
(\mathsf{SC})&\ \vdash x +_e y =_0 y +_{1-e} x\,,\\
(\mathsf{SA})&\ \vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)\,,\ \text{for } e, e' \in [0,1)\,,\\
(\mathsf{IB})&\ \{x =_\varepsilon y, x' =_{\varepsilon'} y'\} \vdash x +_e x' =_\delta y +_e y',\ \text{for } \delta \geq e\varepsilon + (1-e)\varepsilon',\\
(\mathsf{Idem})&\ \vdash x =_0 \mathsf{r}(x, \dots, x)\,,\\
(\mathsf{Diag})&\ \vdash \mathsf{r}(x_{1,1}, \dots, x_{n,n}) =_0 \mathsf{r}(\mathsf{r}(x_{1,1}, \dots, x_{1,n}), \dots, \mathsf{r}(x_{n,1}, \dots, x_{n,n}))\\
(\mathsf{Com})&\ \vdash \mathsf{r}(x_1 +_e y_1, \dots, x_n +_e y_n) =_0 \mathsf{r}(x_1, \dots, x_n) +_e \mathsf{r}(y_1, \dots, y_n)\,,\\
(\diamond\text{-}\mathsf{Lip})&\ \{x =_\varepsilon y\} \vdash \diamond(x) =_\delta \diamond(y)\,,\ \text{for } \delta \geq c\varepsilon\,.
\end{aligned}$$

Note that, the constant raise_* has no explicit associated axiom since $\mathcal{E}_1$ is the trivial theory and (Idem) already implies the commutativity axiom required by tensoring with $\mathcal{R}_A$.

Intuitively, $\Sigma_{\mathbf{LMP}}$-terms (modulo $=_0$ provability) should be interpreted as equivalence classes of behaviours of labelled Markov processes up-to bisimilarity. The term $t +_e t'$ expresses convex combination of behaviours; raise_* represents termination; $\mathsf{r}(t_1, \dots, t_n)$ is used to express that t_i is the selected behaviour after the choice of the action label $a_i \in A$; and $\diamond(t)$ expresses the ability of taking a transition to the behaviour t.

On Metric Spaces. We characterise the monad $T_{\mathcal{U}_{\mathbf{LMP}}}$ on **Met** in steps, by explaining the contribution of the different theories in

$$\mathcal{U}_{\mathbf{LMP}} = ((\mathcal{B} + \mathcal{E}_1) \otimes \mathcal{R}_A) + \mathcal{O}(\Sigma_\diamond)\,.$$

(Step 1) As shown in Section 5.3.2, $T_{\mathcal{B}+\mathcal{E}_1}$ is the *finitely supported sub-distribution monad*

$$T_{\mathcal{B}+\mathcal{E}_1} \cong \Pi(- + 1)\,.$$

Thus, $\mathcal{B} + \mathcal{E}_1$ axiomatizes finitely supported sub-distributions with Kantorovich metric.

(Step 2) By Theorem 6.15 and Corollary 6.18, we further get the monad isomorphism

$$T_{(\mathcal{B}+\mathcal{E}_1)\otimes\mathcal{R}_A} \cong \Pi(1+-) \otimes (-)^{\underline{A}} \cong (\Pi(1+-))^{\underline{A}},$$

saying that tensoring with the theory $\mathcal{R}_A$ of reading computations corresponds to axiomatically adding the capability of reacting to the choice of an action label.

(Step 3) The final step is to sum the above with the theory $\mathcal{O}(\Sigma_\diamond)$. Then, by Corollary 5.11, the monad on $\mathcal{U}_{\mathbf{LMP}}$ is

$$T_{\mathcal{U}_{\mathbf{LMP}}} \cong \mu y. T_{(\mathcal{B}+\mathcal{E}_1)\otimes\mathcal{R}_A}(c \cdot y + -) \cong \mu y. \Pi(c \cdot y + 1 + -)^{\underline{A}},$$

where we implicitly applied the isomorphisms $c \cdot (A + B) \cong c \cdot A + c \cdot B$ and $1 \cong c \cdot 1$.

Explicitly, this means that, the free monad on $\mathcal{U}_{\mathbf{LMP}}$ assigns to an arbitrary metric space $X \in \mathbf{Met}$ the *initial solution* of the following functorial equation in $\mathbf{Met}$

$$LMP_X \cong (\Pi(c \cdot LMP_X + 1 + X))^{\underline{A}}.$$

In particular, when $X = 0$ is the empty metric space (*i.e.*, the initial object in $\mathbf{Met}$) the above corresponds to the isomorphism on the initial $(\Pi(c\cdot - + 1))^{\underline{A}}$-algebra. The isomorphism gives us also a $(\Pi(c\cdot - + 1))^{\underline{A}}$-coalgebra structure $\tau_0 \colon LMP_0 \to (\Pi(c \cdot LMP_0 + 1))^{\underline{A}}$ on LMP_0.

The key observation is that the metric of LMP_0 is the bisimilarity metric.

Lemma 6.24. *d_{LMP_0} is the c-discounted probabilistic bisimilarity metric on (LMP_0, τ_0).*

Proof. Similar to Lemma 5.16. □

On Complete Metric Spaces. Since all the quantitative theories considered are continuous, we can replicate the same steps also while interpreting the theory $\mathcal{U}_{\mathbf{LMP}}$ over complete metric spaces, obtaining the monad

$$\mathbb{C}T_{\mathcal{U}_{\mathbf{LMP}}} \cong \mu y. \Delta(c \cdot y + 1 + -)^{\underline{A}}.$$

By following similar arguments to Section 5.3.2, one can prove that the the functorial equation $LMP_X \cong \Delta(c \cdot LMP_X + 1 + X)^{\underline{A}}$ has a unique solution. By applying the monad above on $X = 0$ we recover the carrier of the final $(\Delta(c \cdot - + 1))^{\underline{A}}$-coalgebra, equipped with c-discounted probabilistic bisimilarity metric.

6.5. **The Algebras of Mealy Machines.** In a similar spirit to the axiomatization of labelled Markov processes, here we provide a quantitative axiomatization of Mealy machines with their (coalgebraically defined) discounted bisimilarity metric.

6.5.1. *Mealy machines over Metric Spaces.* Informally, *Mealy machines* are deterministic automata with outputs. Formally, they are tuples (X, I, Λ, t, o) consisting of a set of states X, a finite set $I = \{i_1, \ldots, i_n\}$ of *inputs*, a set Λ of *outputs*, a *transition function* $t \colon X \times I \to X$, and an *output function* $o \colon X \times I \to \Lambda$.

These structures are clearly $\mathbf{Set}$ coalgebras for the functor $(\Lambda \times -)^I$ [41, 42]. In order to give a coalgebraic definition of a bisimilarity metric for Mealy machines, we will interpret them as coalgebras (X, τ) on categories of metric spaces. Specifically

$$\tau \colon X \longrightarrow (c \cdot X \,\square\, \Lambda)^{\underline{I}} \quad \text{in } \mathbf{Met}/\mathbf{CMet},$$

where $0 < c < 1$ and we assume Λ to be a complete metric space of outputs. The rescaling functor $(c \cdot -)$ is used to obtain a discounted bisimilarity distance. When we want to emphasize the rôle of the discount factor we call these coalgebras c-Mealy machines.

Similarly to [46], we define the the *c-discounted bisimilarity pseudometric* on a c-Mealy machine (X, τ) as the pseudometric induced by the unique homomorphism to the final coalgebra. That is,

$$\mathbf{d}^c(x, x') = d_Z(h(x), h(x'))\,,$$

where $h\colon X \to Z$ is the unique homomorphism to the final c-Mealy machine (Z, ω).

A concrete characterisation of the final c-Mealy machine can be obtained as in [41]. We don't repeat the argument here as it is not necessary for our technical development, which requires only its existence.

This distance has a characterisation as the least fixed point of a monotone function on a complete lattice of $[0, \infty]$-valued pseudometrics.

Proposition 6.25. *The c-discounted bisimilarity pseudometric $\mathbf{d}^c$ on (X, τ) is the* unique *fixed point of the following operator on the complete lattice of extended pseudometrics d on X with point-wise order $\sqsubseteq$, such that $d \sqsubseteq d_X$,*

$$\Psi^c(d)(x, x') = \sup_{i \in I} \left(c \cdot d(x_i, x'_i) + d_\Lambda(\lambda_i, \lambda'_i) \right)\,,$$

where $\tau(i)(x) = (x_i, \lambda_i)$ and $\tau(i)(x') = (x'_i, \lambda'_i)$.

Proof. The unicity of the fixed point follows by Banach fixed point theorem. Indeed, the set of extended real valued functions on $X \times X$ (which is a superset of the set of extended pseudometrics on X) can be turned into a complete Banach space by means of the sup-norm $||f|| = \sup_{x,x'} |f(x, x')|$ and Ψ^c is a c-contractive operator on it. Moreover, $\mathbf{d}^c = \lim_{n\to\infty} (\Psi^c)^n(\mathbf{0})$, where $\mathbf{0}$ is the constantly 0 pseudometric. Since Ψ^c is a monotone operator, $(\Psi^c)^n(\mathbf{0}) \sqsubseteq (\Psi^c)^{n+1}(\mathbf{0})$. Moreover, Ψ^c maps pseudometrics into pseudometrics. As pseudometrics are closed under point-wise suprema, $\mathbf{d}^c$ is a pseudometric. $\square$

6.5.2. *Quantitative Algebraic Presentation.* Next we provide a quantitative equational theory that axiomatizes (the monad of) Mealy machines with c-discounted bisimilarity metric. As we did already in the previous sections we will do this by combining simpler theories via of sum and tensor. The basic theories we use are:

(1) *The quantitative theory $\mathcal{R}_I$ of reading computations* will be used to axiomatize the reaction to the choice of an input symbol $i \in I$ (*cf.* Section 4.3.1);
(2) *The quantitative theory $\mathcal{W}_\Lambda$ of writing computations* will be used to describe the action of outputing a symbol $\alpha \in \Lambda$. (*cf.* Section 4.3.2). In our axiomatic interpretation, we assume Λ to have a monoid structure and outputs to be recorded in a "output tape" by means writing operations.
(3) *The quantitative theory of contractive operators $\mathcal{O}(\Sigma_\diamond)$* with signature $\Sigma_\diamond = \{\diamond\colon \langle 1, c\rangle\}$ will be used to axiomatise the transition to a next state with discounting factor $0 < c < 1$ (*cf.* Section 4.2).

Formally, for a finite set of inputs $I = \{i_1, \ldots, i_n\}$ and complete metric space Λ of outputs with monoid structure $(\Lambda, 0, *)$, we define the quantitative theory of Mealy machines as the following combination of quantitative theories,

$$\mathcal{U}_{\mathbf{MM}} = (\mathcal{R}_I \otimes \mathcal{W}_\Lambda) + \mathcal{O}(\Sigma_\diamond)\,.$$

with signature $\Sigma_{\mathbf{MM}} = \Sigma_{\mathcal{R}_I} \cup \Sigma_{\mathcal{W}_\Lambda} \cup \Sigma_\diamond$ given as the disjoint union of those from its component theories. Explicitly,

$$\Sigma_{\mathbf{MM}} = \{\mathsf{r}\colon |I|\} \cup \{\mathsf{w}_\alpha\colon 1 \mid \alpha \in \Lambda\} \cup \{\diamond\colon \langle 1, c\rangle\}$$

and the theory $\mathcal{U}_{\mathbf{MM}}$ is given by the following axioms

$$\begin{aligned}
&(\mathsf{Idem}) \vdash x =_0 \mathsf{r}(x, \dots, x)\,,\\
&(\mathsf{Diag}) \vdash \mathsf{r}(x_{1,1}, \dots, x_{n,n}) =_0 \mathsf{r}(\mathsf{r}(x_{1,1}, \dots, x_{1,n}), \dots, \mathsf{r}(x_{n,1}, \dots, x_{n,n}))\\
&(\mathsf{Zero}) \vdash x =_0 \mathsf{w}_0(x)\,,\\
&(\mathsf{Mult}) \vdash \mathsf{w}_\alpha(\mathsf{w}_{\alpha'}(x)) =_0 \mathsf{w}_{\alpha * \alpha'}(x)\,,\\
&(\mathsf{Diff})\, \{x =_\varepsilon x'\} \vdash \mathsf{w}_\alpha(x) =_\delta \mathsf{w}_{\alpha'}(x')\,,\ \text{for } \delta \geq d_\Lambda(\alpha, \alpha') + \varepsilon\,,\\
&(\mathsf{Com}) \vdash \mathsf{r}(\mathsf{w}_\alpha(x_1), \dots, \mathsf{w}_\alpha(x_n)) =_0 \mathsf{w}_\alpha(\mathsf{r}(x_1, \dots, x_n))\,,\\
&(\diamond\text{-}\mathsf{Lip})\, \{x =_\varepsilon y\} \vdash \diamond(x) =_\delta \diamond(y)\,,\ \text{for } \delta \geq c\varepsilon\,.
\end{aligned}$$

Intuitively, $\Sigma_{\mathbf{MM}}$-terms (modulo $=_0$ provability) should be interpreted as equivalence classes of behaviours of Mealy machines up-to bisimilarity. The term $\mathsf{r}(t_1, \dots, t_n)$ is used to express that t_k is the selected behaviour after reading input $i_k \in I$; $\mathsf{w}_\alpha(t)$ is the term expressing behaviour of writing the output $\alpha \in \Lambda$ in the output tape; and $\diamond(t)$ expresses the ability of taking a transition to the behaviour t.

On Metric Spaces. We characterise the monad $T_{\mathcal{U}_{\mathbf{MM}}}$ on **Met** in steps, by explaining the contribution of the different theories in $\mathcal{U}_{\mathbf{MM}}$.

(Step 1) As shown in Section 4.3.1, $T_{\mathcal{R}_I}$ is the *reader monad*

$$T_{\mathcal{R}_I} \cong (-)^{\underline{I}}\,.$$

Thus, $\mathcal{R}_I$ axiomatizes the space of functions with domain the set I.

(Step 2) By Theorem 6.15 and Corollary 6.21 (equivalently, Corollary 6.18), we further get the monad isomorphisms

$$T_{\mathcal{R}_I \otimes \mathcal{W}_\Lambda} \cong (-)^{\underline{I}} \otimes (- \mathbin{\square} \Lambda) \cong (- \mathbin{\square} \Lambda)^{\underline{I}}\,,$$

saying that tensoring with the theory $\mathcal{W}_\Lambda$ of writing computations corresponds to axiomatically adding the capability of writing an output symbol after reading an input action.

(Step 3) By summing the above theories with the theory $\mathcal{O}(\Sigma_\diamond)$, by Corollary 5.11, we get that the free monad on $\mathcal{U}_{\mathbf{MM}}$ is

$$T_{\mathcal{U}_{\mathbf{MM}}} \cong \mu y. T_{\mathcal{R}_I \otimes \mathcal{W}_\Lambda}(c \cdot y + -) \cong \mu y. ((c \cdot y + -) \mathbin{\square} \Lambda)^{\underline{I}}\,.$$

Explicitly, the free monad on $\mathcal{U}_{\mathbf{MM}}$ assigns to an arbitrary metric space $X \in \mathbf{Met}$ the *initial solution* of the following functorial equation in **Met**

$$MM_X \cong (c \cdot MM_X + X) \mathbin{\square} \Lambda)^{\underline{I}}\,.$$

In particular, when $X = 0$ is the empty metric space the above corresponds to the isomorphism of the initial $(c \cdot - \mathbin{\square} \Lambda)^{\underline{I}}$-algebra. From this we recover a $(c \cdot - \mathbin{\square} \Lambda)^{\underline{I}}$-coalgebra structure $\tau_0\colon MM_0 \to (c \cdot MM_0 \mathbin{\square} \Lambda)^{\underline{I}}$ on MM_0, whence a c-Mealy machine.

Lemma 6.26. *d_{MM_0} is the c-discounted probabilistic bisimilarity metric on (MM_0, τ_0).*

Proof. Similar to Lemma 5.16. ∎

On Complete Metric Spaces. As the quantitative theories considered are continuous, we can replicate the same steps also while interpreting the theory $\mathcal{U}_{\mathbf{MM}}$ over complete metric spaces, obtaining the monad

$$\mathbb{C}T_{\mathcal{U}_{\mathbf{MM}}} \cong \mu y.((c \cdot y + -) \square \Lambda)^{\underline{I}}.$$

By following similar arguments to Section 5.3.2, one can prove that the the functorial equation $MM_X \cong (c \cdot MM_X + X) \square \Lambda)^{\underline{I}}$ has a unique solution in **CMet**. Hence, by applying the monad above on $X = 0$ we recover the carrier of the final $(c \cdot - \square \Lambda)^{\underline{I}}$-coalgebra, equipped with c-discounted probabilistic bisimilarity metric.

6.6. **The Algebras of Markov Decision Processes with Rewards.** In this section we provide a quantitative equational axiomatization of Markov decision processes with rewards and their (coalgebraically defined) discounted bisimilarity metric. The axiomatization is obtained by extending that of labelled Markov processes from Section 6.4 by adding the ability to record the rewards associated with a specific probabilistic decision.

6.6.1. *Markov Decision Processes over Metric Spaces.* Informally, Markov decision processes are labelled Markov processes where each choice of action label (decision) is associated with a probabilistic reward. Formally, as in [46], we regard them as coalgebras on the category of extended metric spaces. In detail, we consider two variants of Markov decision processes:

$$X \longrightarrow \Pi(c \cdot X \square \mathbb{R})^{\underline{A}} \quad \text{in } \mathbf{Met}\,,$$
$$X \longrightarrow \Delta(c \cdot X \square \mathbb{R})^{\underline{A}} \quad \text{in } \mathbf{CMet}\,,$$

where Π and Δ are the functors from Section 4.5. For convenience, the rescaling functor $(c \cdot -)$ is used to account of a discount factor on the bisimilarity metric and the functor $(- \square \mathbb{R})$ is to give a metric interpretation to the combination with the reward structure.

Remark 6.27. In [40] a Markov decision process is defined as a tuple $(S, p(\cdot|s,a), r(s,a))$ with a Markov kernel $p \colon S \times A \to \Delta(S)$ and randomised reward function $r \colon S \times A \to \Delta(\mathbb{R})$. Our coalgebraic representation is the natural generalisation over metric spaces, where the randomness of the Markov kernel and reward function is combined as a probability measure on $(c \cdot S \square \mathbb{R})$, by regarding $\mathbb{R}$ and S as extended metric spaces.

Similar to Section 5.3.1, one can show that the final coalgebra for the functors $\Pi(c \cdot - \square \mathbb{R})^{\underline{A}}$ in **Met** and $\Delta(c \cdot - \square \mathbb{R})^{\underline{A}}$ in **CMet** exists, thus we define the c-discounted probabilistic bisimilarity distance on a Markov decision process (X, τ) as the pseudometric

$$\mathbf{d}^c(x, x') = d_Z(h(x), h(x'))$$

induced by the unique homomorphism $h \colon X \to Z$ to the final coalgebra.

Also in this time, the probabilistic bisimilarity distance can be given a fixed point characterization.

Proposition 6.28. *The c-discounted bisimilarity pseudometric $\mathbf{d}^c$ on (X, τ) is the* unique *fixed point of the following operator on the complete lattice of extended pseudometrics d on X with point-wise order $\sqsubseteq$, such that $d \sqsubseteq d_X$,*

$$\Psi^c(d)(x, x') = \sup_{a \in A} \sup_{f} \left| \int f \ \mathrm{d}\tau(x)(a) - \int f \ \mathrm{d}\tau(x')(a) \right| ,$$

with f ranging over non-expansive positive 1-bounded real valued functions on $c \cdot X \square \mathbb{R}$.

6.6.2. *Quantitative Algebraic Presentation.* We provide a quantitative axiomatization of Markov decision processes with rewards equipped with discounted bisimilarity metric. As the construction is similar to Section 6.4, we avoid repeating the details of each step of the monad characterization.

Let $A = \{a_1, \ldots, a_n\}$ be a finite set of actions and $(\mathbb{R}, +, 0)$ be the standard monoid structure on the reals. We define the quantitative theory $\mathcal{U}_{\mathbf{MDP}}$ of Markov decision processes with real-valued rewards as the following combination of quantitative theories,

$$\mathcal{U}_{\mathbf{MDP}} = ((\mathcal{B} \otimes \mathcal{W}_{\mathbb{R}}) \otimes \mathcal{R}_A) + \mathcal{O}(\Sigma_\diamond)\,.$$

with signature $\Sigma_{\mathbf{MDP}} = \Sigma_{\mathcal{B}} \cup \Sigma_{\mathcal{W}_{\mathbb{R}}} \cup \Sigma_{\mathcal{R}_A} \cup \Sigma_\diamond$ given as the disjoint union of those from its component theories.

On Metric Spaces and Complete Metric Spaces. Similarly to what we have done in for labelled Markov processes, we relate Markov decision processes and their c-discounted probabilistic bisimilarity pseudometric with the free monads on the theory $\mathcal{U}_{\mathbf{MDP}}$ in **Met** and **CMet**.

The only step that changes in the characterisation of the monad $T_{\mathcal{U}_{\mathbf{MDP}}}$ in **Met**, regards the combination of theories $\mathcal{B} \otimes \mathcal{W}_{\mathbb{R}}$, which is dealt using Corollary 6.21. Thus, similarly to Section 6.4 we get

$$T_{\mathcal{U}_{\mathbf{MDP}}} = T_{((\mathcal{B}\otimes\mathcal{W}_{\mathbb{R}})\otimes\mathcal{R}_A)+\mathcal{O}(\Sigma_\diamond)} \cong \mu y.\Pi((c \cdot y + -) \,\square\, \mathbb{R})^{\underline{A}}\,.$$

The metric on the initial solution for the functorial fixed point definition corresponds to the c-discounted probabilistic bisimilarity (pseudo)metric on its coalgebra structure.

Similar considerations apply also when interpreting the theories in the category **CMet** of complete metric spaces, as the argument follows without issues because $\mathbb{R}$ a complete metric space. Thus we obtain the following characterisation for the monad:

$$\mathbb{C}T_{\mathcal{U}_{\mathbf{LMP}}} \cong \mu y.\Delta((c \cdot y + -) \,\square\, \mathbb{R})^{\underline{A}}\,.$$

As the fixed point solution in **CMet** is unique, $\mathbb{C}T_{\mathcal{U}_{\mathbf{LMP}}}0$ is an algebraic characterization of the final $\Delta((c \cdot -) \,\square\, \mathbb{R})^{\underline{A}}$-coalgebra with probabilistic bisimilarity metric.

7. Conclusions

We studied the disjoint and commutative combinations of quantitative effects, respectively as the sum and tensor of their quantitative equational theories. The key results are Theorems 5.3 and 6.15, asserting that the sum and tensor of two quantitative theories corresponds to the categorical sum and tensor, respectively, of their free monads. In addition to these general results, we provide quantitative analogues Moggi's monad transformers for exceptions, resumption, reader, and writer.

We illustrate the applicability of our theoretical development with the axiomatisations four coalgebraic bisimilarity metrics: for Markov processes, labeled Markov processes, Mealy machines, and Markov decision processes. Apart from the intrinsic interest in their quantitative equational presentation as effects, what is particularly pleasant is the systematic compositional way with which one can obtain quantitative axiomatisations of different variants of coalgebraic structures by just combining theories as new basic ingredients.

An example that escapes our compositional treatment via sum and tensor is the combination of probabilities and non-determinism as illustrated in [33]. A possible future work in this direction is to extend the combination of theories with another operator: the distributive tensor (see [21, Section 6]). Following an intuition similar to Cheng [12], we claim that these correspond in a suitable way to Garner's weak distributive law [16]. Our claim seems promising in the light of the work [17, 9] which consider equational axiomatisations combining probabilities and non-determinism.

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Appendix A. Limits and Colimits of Extended Metric Spaces

Limits and colimits in the category **Met** of extended metric spaces and non-expansive maps are defined similarly to those in **Set**, at least for the part of their underlying set. Some care, though, should to taken in the definition of the distance function.

From an abstract point of view, the reason is that the forgetful functor $U\colon \mathbf{Met} \to \mathbf{Set}$, sending an extended metric space (X, d_X) to its underlying set X, is faithful (*i.e.*, homsets are mapped injectively), and has as left adjoint the functor $\mathit{Disc}\colon \mathbf{Set} \to \mathbf{Met}$ that assigns to each set X the *discrete* extended metric space $\underline{X}$ giving distance ∞ to every distinct pair of elements. Therefore U preserves all limits which may exist in **Met** (this is *why* the underlying set of product spaces is the cartesian product of their underlying sets).

Moreover, **Met** is a full reflective subcategory of **PMet**, the category of extended pseudometric spaces (*i.e.*, a relaxation of extended metric spaces where different elements $x \neq y$ can be assigned distance $d(x,y) = 0$), with reflection mapping a pseudometric space (X, d_X) into its quotient modulo the equivalence $x \cong y$ iff $d_X(x,y) = 0$. Since **PMet** is cocomplete with colimits constructed similarly to **Set** also **Met** is cocomplete and its colimits are just simple quotiented versions of those in **Set**.

Although the abstract argument above is enough to prove completeness and cocompleteness of **Met**, in the proof below we give a direct concrete construction of its limits and colimits.

Proposition A.1. **Met** *is a complete and cocomplete category.*

Proof. Let $D\colon \mathcal{I} \to \mathbf{Met}$ be a small diagram, and let $D(i) = (X_i, d_i)$, for each object $i \in \mathcal{I}$. Let $U\colon \mathbf{Met} \to \mathbf{Set}$ be the standard forgetful functor, sending (X, d_X) to X. Clearly, also $UD\colon \mathcal{I} \to \mathbf{Set}$ is a small diagram. We show completeness and cocompleteness separately:

Completeness: Let $(f_i\colon L \to X_i)_{i \in \mathcal{I}}$ be the limit cone to UD. We define $d_L\colon L \times L \to [0, \infty]$ as follows, for arbitrary $x, y \in L$

$$d_L(x,y) = \sup_{i \in \mathcal{I}} d_i(f_i(x), f_i(y)),$$

and claim that this is an extended metric[3].

Let $x, y, z \in L$. Identity of indiscernible follows by

$$\begin{aligned}
d_L(x,y) = 0 &\iff \sup_{i \in \mathcal{I}} d_i(f_i(x), f_i(y)) = 0 && \text{(def. } d_L\text{)}\\
&\iff \forall i \in \mathcal{I},\ d_i(f_i(x), f_i(y)) = 0 && (d_i \text{ positive})\\
&\iff \forall i \in \mathcal{I},\ f_i(x) = f_i(y) && (d_i \text{ metric})\\
&\iff x = y, && ((f_i)_{i \in \mathcal{I}} \text{ limit cone})
\end{aligned}$$

symmetry by

$$\begin{aligned}
d_L(x,y) &= \sup_{i \in \mathcal{I}} d_i(f_i(x), f_i(y)) && \text{(def. } d_L\text{)}\\
&= \sup_{i \in \mathcal{I}} d_i(f_i(y), f_i(x)) && (d_i \text{ metric})\\
&= d_L(y,x), && \text{(def. } d_L\text{)}
\end{aligned}$$

[3]Note that the definition of d_L makes sense since the supremum exists in $[0, \infty]$; this would not be true for standard (finite) metrics taking values in $[0, \infty)$.

and triangular inequality by

$$\begin{aligned} d_L(x,z) + d_L(z,y) &= \sup_{i\in\mathcal{I}} d_i(f_i(x), f_i(z)) + \sup_{i\in\mathcal{I}} d_i(f_i(z), f_i(y)) && \text{(def. } d_L\text{)} \\ &\geq \sup_{i\in\mathcal{I}} \big(d_i(f_i(x), f_i(z)) + d_i(f_i(z), f_i(y))\big) && \text{(sup)} \\ &\geq \sup_{i\in\mathcal{I}} d_i(f_i(x), f_i(y)) && (d_i \text{ metric}) \\ &= d_L(x,y)\,. && \text{(def. } d_L\text{)} \end{aligned}$$

With this metric all f_i are non-expansive functions. Indeed we have, for all $i \in \mathcal{I}$ and $x, y \in L$

$$d_i(f_i(x), f_i(y)) \leq \sup_{i\in\mathcal{I}} d_i(f_i(x), f_i(y)) = d_L(x,y)\,.$$

Since the forgetful functor $U\colon \mathbf{Met} \to \mathbf{Set}$ is faithful, the non-expansiveness of the maps f_i implies that $(f_i\colon (L, d_L) \to (X_i, d_i))_{i\in\mathcal{I}}$ is a cone to D. Next we show that this is actually the limiting cone.

Let $(h_i\colon (H, d_H) \to (X_i, d_i))_{i\in\mathcal{I}}$ be a cone to D. Then $(h_i\colon H \to X_i)_{i\in\mathcal{I}}$ is a cone to UD. Since $(f_i\colon L \to X_i)_{i\in\mathcal{I}}$ is the limit cone to UD, there exists a unique function $g\colon H \to L$ in $\mathbf{Set}$ satisfying $f_i \circ g = h_i$, for all $i \in \mathcal{I}$. We finish our proof by showing that g is a non-expansive function. By non-expansiveness of the h_i's we have that, for all $i \in \mathcal{I}$ and $a, b \in H$, $d_i(h_i(a), h_i(b)) \leq d_H(a, b)$, and thus also

$$\begin{aligned} d_L(g(a), g(b)) &= \sup_{i\in\mathcal{I}} d_i(f_i(g(a)), f_i(g(b))) && \text{(def. } d_L\text{)} \\ &= \sup_{i\in\mathcal{I}} d_i(h_i(a), h_i(b)) && (f_i \circ g = h_i) \\ &\leq d_H(a,b)\,. && (h_i \text{ non-expansive}) \end{aligned}$$

Thus we conclude that $(f_i\colon (L, d_L) \to (X_i, d_i))_{i\in\mathcal{I}}$ is a limit cone to D.

Cocompleteness: Let $(f_i\colon X_i \to L)_{i\in\mathcal{I}}$ be the colimit cocone to UD. We define $d_L\colon L \times L \to [0,\infty]$, for arbitrary $x, y \in L$, as follows:

$$d_L(x,y) = \sup_{d\in M_L} d(x,y)\,,$$

where M_L is the set of all extended pseudometrics d on L making all f_i's non-expansive functions $f_i\colon (X_i, d_i) \to (L, d)$. We claim that this is an extended pseudometric. Since all $d \in M_L$ are pseudometrics, we can derive immediately that $d_L(x,x) = 0$ and $d_L(x,y) = d_L(y,x)$, for all $x, y \in L$. Moreover, for all $x, y, z \in L$, we have

$$\begin{aligned} d_L(x,z) + d_L(z,y) &= \sup_{d\in M_L} d(x,z) + \sup_{d\in M_L} d(z,y) && \text{(def. } d_L\text{)} \\ &\geq \sup_{d\in M_L} d(x,z) + d(z,y) && \text{(sup)} \\ &\geq \sup_{d\in M_L} d(x,y) && (d \text{ metric}) \\ &= d_L(x,y)\,. && \text{(def. } d_L\text{)} \end{aligned}$$

Moreover, for all $i \in \mathcal{I}$ and $x, y \in X_i$

$$\begin{aligned} d_i(f_i(x), f_i(y)) &\leq \sup_{d\in M_L} d(x,y) && \text{(def. } M_L\text{)} \\ &= d_L(x,y)\,. && \text{(def. } d_L\text{)} \end{aligned}$$

Thus, all the functions f_i are non-expansive w.r.t. the pseudometric d_L.

Now we turn the extended pseudometric space (L, d_L) into an extended metric space (C, d_C) by taking the quotient modulo the equivalence $x \cong y$ iff $d_L(x, y) = 0$. The extended metric $d_C : C \times C \to [0, \infty]$ is given by

$$d_C([x], [y]) = d_L(x, y)$$

for all $x, y \in L$, where $[\cdot] : L \to C$ denote the quotient map w.r.t. $\cong$. Note that d_C is well-defined because by triangular inequality of d_L the definition above is independent from the choice of the representative x of the $\cong$-equivalence class $[x]$ in C.

From the non-expansiveness of the maps f_i we have that also $[\cdot] \circ f_i$ are non expansive. Thus, since the forgetful functor $U : \mathbf{Met} \to \mathbf{Set}$ is faithful, $([\cdot] \circ f_i : (X_i, d_i) \to (C, d_C))_{i \in \mathcal{I}}$ is a cocone to D in $\mathbf{Met}$. Next we show that this is actually the colimiting cocone.

Let $(h_i : (X_i, d_i) \to (H, d_H))_{i \in \mathcal{I}}$ be a cocone to D. Then $(h_i : X_i \to H)_{i \in \mathcal{I}}$ is a cocone to UD. Since $(f_i : X_i \to L)_{i \in \mathcal{I}}$ is the colimit cocone to UD, there exists a unique function $g : L \to H$ in $\mathbf{Set}$ satisfying $g \circ f_i = h_i$, for all $i \in \mathcal{I}$. We prove that g is non-expansive w.r.t. the pseudometric d_L. Let $d_g : L \times L \to [0, \infty]$ be defined as $d_g(x, y) = d_H(g(x), g(y))$. It is easy to see that this is an extended pseudometric on L. Moreover, for all $i \in \mathcal{I}$ and $x', y' \in X_i$ we have

$$\begin{aligned} d_g(f_i(x'), f_i(y')) &= d_H(g(f_i(x')), g(f_i(y'))) && (\text{def. } d_g) \\ &= d_H(h_i(x'), h_i(y')) && (g \circ f_i = h_i) \\ &\leq d_i(x', y') \,. && (h_i \text{ non-expansive}) \end{aligned}$$

Thus $d_g \in M_L$. Using this we observe that, for all $x, y \in L$

$$d_L(x, y) = \sup_{d \in M_L} d(x, y) \geq d_g(x, y) = d_H(g(x), g(y)) \,. \tag{A.1}$$

Let $g'([x]) = g(x)$, for all $x \in L$. By (A.1) and the fact that d_H is a metric, $g(x) = g(y)$ whenever $x \cong y$, hence the $g' : C \to H$ is a well-defined function on C. Moreover, by definition of d_C and (A.1), g' is also non-expansive as a map $g' : (C, d_C) \to (H, d_H)$ in $\mathbf{Met}$.

Clearly, $g' \circ [\cdot] \circ f_i = h_i$, for all $i \in \mathcal{I}$. Assume that there exists another map $g'' : C \to H$ such that $g'' \circ [\cdot] \circ f_i = h_i$, for all $i \in \mathcal{I}$. By the universal property of g, $g' \circ [\cdot] = g = g'' \circ [\cdot]$, thus $g'([x]) = g''([x])$, for all $x \in L$.

Thus $([\cdot] \circ f_i : (X_i, d_i) \to (C, d_C))_{i \in \mathcal{I}}$ is the colimit cocone to D. □

The situation is very similar when we consider the full subcategory **CMet** of complete extended metric spaces. Indeed, **CMet** is closed under limits, which are defined as in **Met**. Moreover, as **CMet** is a reflective subcategory of **Met**, with reflection the Cauchy completion functor, we have that also **CMet** is cocomplete, with colimits constructed as in **Met** and completed via Cauchy completion.

Proposition A.2. **CMet** *is a complete and cocomplete category.* □

Appendix B. Extended Metric Spaces are Locally Countably Presentable

Let λ be a regular infinite cardinal (i.e., one that is not cofinal to any smaller cardinal). A small category is called λ-*filtered* if any subcategory of less than λ morphisms has a cocone in it. When $\lambda = \aleph_0$, the term *finitely* filtered (or simply, filtered) is most commonly used, and *countably* filtered in the case $\lambda = \aleph_1$.

Example B.1. Let $\aleph_0$ denote the skeleton of the category of finite sets and all functions between them. Then $\aleph_0$ is finitely filtered, but not countably filtered. While the skeleton category $\aleph_1$ of all countable sets is countably filtered.

A diagram is λ-*filtered* if its domain is λ-filtered, and a colimit is λ-*filtered* when it is the colimit of a λ-filtered diagram. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called λ-*accessible* if its domain $\mathcal{C}$ has λ-filtered colimits and F preserves them[4].

An object X of a small category $\mathcal{C}$ is λ-*presentable* if its hom-functor

$$\mathcal{C}(X, -)\colon \mathcal{C} \to \mathbf{Set}$$

is λ-accessible. Explicitly, X is λ-presentable iff for each λ-filtered colimit cocone $(c_i\colon D(i) \to C)_{i\in\mathcal{I}}$ of a λ-filtered diagram $D\colon \mathcal{I} \to \mathcal{C}$, and each morphism $f\colon X \to C$, there exists $i \in \mathcal{I}$ such that

- f factorizes through c_i, *i.e.*, $f = c_i \circ g$ for some $g\colon X \to D(i)$, and
- the factorization is essentially unique in the sense that if $f = c_i \circ g = c_i \circ g'$, then $D(i \to j) \circ g = D(i \to j) \circ g'$, for some $j \in \mathcal{I}$.

Definition B.2 (Accessibility and Local Presentability)**.** A category $\mathcal{C}$ is λ-*accessible* if

- it has all λ-filtered colimits;
- there is a set $\mathcal{C}_\lambda$ of λ-presentable objects such that every object is a λ-filtered colimit of objects of $\mathcal{C}_\lambda$.

It is *locally λ-presentable* if, moreover, it has all small colimits (*i.e.*, it is cocomplete).

A category is said *accessible* (resp. *locally presentable*) if it is λ-accessible (resp. λ-locally presentable) for some regular infinite cardinal λ. In the case $\lambda = \aleph_0$, we speak about *locally finitely presentable category*, and for $\lambda = \aleph_1$ about *locally countably presentable category.*

Example B.3. The category **Set** is locally finitely presentable with finitely presentable objects precisely the finite sets (for $\mathbf{Set}_{\aleph_0}$ we can choose the set of all natural numbers). The category $\omega\mathbf{CPO}$ of cpo's (*i.e.*, posets with joints of all increasing ω-chains) and ω-continuous functions is not locally finitely presentable, however, it is locally countably presentable with countably presentable objects precisely the countable cpo's (for $\omega\mathbf{CPO}_{\aleph_1}$ we can choose the set of all countable ordinals with standard partial order).

Next we focus our attention to the category **Met**. In turn we prove that

(1) the only finitely presentable objects in **Met** are the finite discrete spaces, with distances either 0 or ∞ (Proposition B.4);
(2) **Met** is locally countably presentable, with countably presentable objects precisely the countable spaces (Lemma B.7 and Theorem B.8)

[4]In some literature $\aleph_0$-accessible functors are said *of finite rank* and and $\aleph_1$-accessible functors *of countable rank*. This is the terminology preferred by John Power in his seminal work about enriched Lavwere theories.

Note that a direct consequence of (1) is that **Met** is *not* locally finitely presentable, since filtered colimits of discrete spaces are discrete.

The proofs of these results are immediate adaptations of [2] which shows that the category of 1-bounded pseudometric spaces with non-expansive maps is locally countably presentable.

Proposition B.4. *Finitely presentable objects in* **Met** *are finite and discrete.*

Proof. Note that every extended metric space is a colimit of the filtered diagram obtained by taking all its finite subspaces and their inclusion maps. Let (X, d_X) be a finitely presentable object in **Met**. Then the identity must factorize through the inclusion of one of the finite subspaces. Thus, X must be finite.

For each positive integer $n > 0$, define the function $d_n \colon X \times X \to [0, \infty]$ as

$$d_n(x, y) = \left(1 + \frac{1}{n}\right) \cdot d_X(x, y)\,,$$

where $\infty \cdot r = \infty$ for any $r \in [0, \infty)$. Clearly, all d_n's are extended metrics. Consider the ω-chain of spaces (X, d_n) with identities as connecting maps. This is a countably filtered diagram with colimit cocone $(id_X \colon (X, d_n) \to (X, d_X))_{n>0}$. Since (X, d_X) is finitely presentable, the identity $id_X \colon (X, d_X) \to (X, d_X)$ must factorize through the a colimit map $id_X \colon (X, d_n) \to (X, d_X)$ for some positive integer $n > 0$. But the distances in (X, d_X) which are strictly between 0 and ∞ are increased by d_n. So (X, d_X) must discrete. □

Proposition B.5. *Let* $(f_i \colon D(i) \to (C, d_C))_{i \in \mathcal{I}}$ *be the colimit cocone to a small diagram* $D \colon \mathcal{I} \to$ **Met** *and* $D(i) = (X_i, d_i)$*, for each* $i \in \mathcal{I}$*. If* D *is filtered,* C *is the quotient set of* $\coprod_{i \in \mathcal{I}} X_i$ *under the equivalence*

$$in_i(x) \sim in_{i'}(x') \quad \textit{iff there exists } j \in \mathcal{I} \textit{ with } D(i \to j)(x) = D(i' \to j)(x')\,,$$

where $in_i \colon X_i \to \coprod_{i \in I} X_i$ *are the canonical injections into the coproduct and* f_i *assigns to each* $x \in X_i$ *the equivalence class of* $in_i(x)$*, for all* $i \in \mathcal{I}$*.*

Proof. We prove that $\sim$ is an equivalence relation. Reflexivity and symmetry are trivially satisfied. Transitivity follows by the fact that $\mathcal{I}$ is filtered. Indeed, let $in_i(x) \sim in_{i'}(x')$ and $in_{i'}(x') \sim in_{i''}(x'')$. Then there exist $j, j' \in \mathcal{I}$ such that $D(i \to j)(x) = D(i' \to j)(x')$ and $D(i' \to j')(x') = D(i'' \to j')(x'')$. Since $\mathcal{I}$ is filtered, there exists $j'' \in \mathcal{I}$ above i, i', i'', j, and j' such that

$$\begin{gathered} D(i \to j'') = D(j \to j'') \circ D(i \to j)\,, \\ D(i' \to j'') = D(j \to j'') \circ D(i' \to j) = D(j' \to j'') \circ D(i' \to j')\,, \text{ and} \\ D(i'' \to j'') = D(j' \to j'') \circ D(i'' \to j')\,. \end{gathered}$$

From this we derive $D(i \to j'')(x) = D(i'' \to j'')(x'')$, i.e., $in_i(x) \sim in_{i''}(x'')$.

Let $U \colon$ **Met** $\to$ **Set** denote the forgetful functor into **Set**. By the construction of colimits in **Met** (see Proposition A.1) it suffices to prove that $(f_i \colon X_i \to (\coprod_{i \in I} X_i)/_\sim)_{i \in \mathcal{I}}$ is a colimit cocone to UD in **Set**.

For convenience, let $X = (\coprod_{i \in I} X_i)/_\sim$ and, for each $i \in \mathcal{I}$, let $[in_i(x)]$ denote the equivalence class of $in_i(x)$. We first prove that $(f_i \colon X_i \to X)_{i \in \mathcal{I}}$ is a cocone to UD. Let $i \to j \in \mathcal{I}$. Then from $D(i \to j)(x) = D(id_j)(D(i \to j)(x))$ for all $x \in X_i$, we have $in_i(x) \sim in_j(D(i \to j)(x))$. Therefore, for all $x \in X_i$

$$f_i(x) = [in_i(x)] = [in_j(x)(D(i \to j)(x))] = f_j(D(i \to j)(x))\,.$$

Hence $f_i = f_j \circ D(i \to j)$ for all $i, j \in \mathcal{I}$. Now we prove that $(f_i \colon X_i \to X)_{i \in \mathcal{I}}$ is a colimit. Let $(h_i \colon X_i \to H)_{i \in \mathcal{I}}$ be a cocone to UD. Define $g \colon X \to H$ for arbitrary $x \in X_i$ as follows:

$$g([in_i(x)]) = \gamma(in_i(x)) \,,$$

where $\gamma \colon \coprod_{i \in I} X_i \to H$ is the unique map such that $\gamma \circ in_i = h_i$, for all $i \in \mathcal{I}$. Note that g is well defined. Indeed, if $D(i \to j)(x) = D(i' \to j)(x')$ for some $j \in \mathcal{I}$, we have that

$$\begin{aligned} g([in_i(x)]) &= \gamma(in_i(x)) && \text{(def. } g) \\ &= h_i(x) && (\gamma \circ in_i = h_i) \\ &= h_j(D(i \to j)(x)) && ((h_i \colon X_i \to H)_i \text{ cocone to } UD) \\ &= h_j(D(i' \to j)(x')) \\ &= h_{i'}(x') && ((h_i \colon X_i \to H)_i \text{ cocone to } UD) \\ &= \gamma(in_{i'}(x')) && (\gamma \circ in_{i'} = h_{i'}) \\ &= g([in_{i'}(x')]) \,. && \text{(def. } g) \end{aligned}$$

Let $i \in \mathcal{I}$. Then, for all $x \in X_i$

$$\begin{aligned} g \circ f_i(x) &= g([in_i(x)]) && \text{(def. } f_i) \\ &= \gamma(in_i(x)) && \text{(def. } g) \\ &= h_i(x) \,. && (\gamma \circ in_i = h_i) \end{aligned}$$

Thus $g \circ f_i = h_i$ for all $i \in \mathcal{I}$. Assume now that there exists $g' \colon X \to H$ such that $g' \circ f_i = h_i$ for all $i \in \mathcal{I}$. Then, for all $x \in X_i$

$$\begin{aligned} g'([in_i(x)]) &= g'(f_i(x)) && \text{(def. } f_i) \\ &= h_i(x) && (g' \circ f_i = h_i) \\ &= g(f_i(x)) && (g \circ f_i = h_i) \\ &= g([in_i(x)]) \,. && \text{(def. } f_i) \end{aligned}$$

Thus $g' = g$. □

Proposition B.6. *Let $(f_i \colon D(i) \to (C, d_C))_{i \in \mathcal{I}}$ be the colimit cocone to a small diagram $D \colon \mathcal{I} \to \mathbf{Met}$ and $D(i) = (X_i, d_i)$, for each $i \in \mathcal{I}$. If D is filtered, then, for all $x, y \in C$*

$$d_C(x, y) = \inf\{d_i(x', y') \mid i \in \mathcal{I}, f_i(x') = x \,, \textit{and } f_i(y') = y\} \,.$$

Proof. Assume D is filtered. As shown in Proposition A.1,

$$d_C(x, y) = \sup_{d \in M_C} d(x, y) \,, \tag{B.1}$$

where M_C is the set of all extended pseudometrics d' on C making all the functions $f_i \colon (X_i, d_i) \to (C, d')$ non-expansive. Let $d \colon C \times C \to [0, \infty]$ be

$$d(x, y) = \inf\{d_i(x', y') \mid i \in \mathcal{I}, f_i(x') = x \,, \text{and } f_i(y') = y\} \,. \tag{B.2}$$

Since D is filtered, then also UD is so, where $U \colon \mathbf{Met} \to \mathbf{Set}$ is the obvious forgetful functor to $\mathbf{Set}$. Since $\mathbf{Set}$ is locally finitely representable, then for any finite subset $\{x_1, \dots, x_n\} \subseteq C$, there exist $i \in \mathcal{I}$ and $\{x'_1, \dots, x'_n\} \subseteq X_i$ such that $f_i(x'_j) = x_j$, for all $0 \leq j \leq n$. In particular, this implies that for any $x, y \in C$, the infimum in (B.2) never ranges over an empty set.

Let $x, y \in C$. We prove $d_C(x, y) \leq d(x, y)$ and $d_C(x, y) \geq d(x, y)$ separately.

- By non-expansivity of the maps $f_i\colon (X_i, d_i) \to (C, d_C)$, for any $i \in \mathcal{I}$ such that $f_i(x') = x$ and $f_i(y') = y$ for some $x', y' \in X_i$, we have
$$d_i(x', y') \geq d_C(f_i(x'), f_i(y')) = d_C(x, y)\,.$$
Thus $d_C(x, y) \leq \inf\{d_i(x', y') \mid i \in \mathcal{I}\,, f_i(x') = x\,, \text{and } f_i(y') = y\}$. By (B.2), $d_C(x, y) \leq d(x, y)$.
- We prove $d \in M_L$. We start by showing that d is a pseudometric on C. Since all d_i are extended metrics, we immediately derive that $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in C$. Moreover, for all $x, y, z \in C$:
$$\begin{aligned} d(x, y) &= \inf\{d_i(x', y') \mid i \in \mathcal{I}\,, f_i(x') = x\,, \text{and } f_i(y') = y\} \\ &\leq \inf\{d_i(x', z') + d_i(z', y') \mid i \in \mathcal{I}\,, f_i(x') = x\,, f_i(y') = y\,, f_i(z') = z\} \\ &\leq d(x, z) + d(z, y)\,, \end{aligned}$$
where the last inequality follows by the fact that D is filtered, hence for all $j, k \in \mathcal{I}$ there exists $i \in \mathcal{I}$ and non-expansive maps $D(j \to i)$, $D(k \to i)$ such that $f_j = f_i \circ D(j \to i)$ and $f_k = f_i \circ D(k \to i)$. Therefore, d is a pseudometric. The non-expansiveness of the maps $f_j\colon (X_j, d_j) \to (C, d)$, for all $j \in \mathcal{I}$ follows immediately by (B.2):
$$d(f_j(x'), f_j(y')) = \inf\{d_i(x', y') \mid i \in \mathcal{I} \text{ and } x', y' \in X_j\} \leq d_j(x', y')\,.$$
Thus, $d \in M_L$. Therefore, by (B.1) we have $d_C(x, y) \geq d(x, y)$. □

Lemma B.7. $(X, d_X) \in \mathbf{Met}$ *is countably presentable iff it is countable.*

Proof. Every extended metric space (X, d_X) is a countably filtered colimit of its countable subspaces. If (X, d_X) is countably presentable, then the identity id_X must factorize through the inclusion of one of the countable subspaces of (X, d_X). Thus, (X, d_X) is countable.

Conversely, let (X, d_X) be a countable space, and let $(f_i\colon D(i) \to (C, d_C))_{i \in \mathcal{I}}$ be the colimit cocone to a countably filtered diagram $D\colon \mathcal{I} \to \mathbf{Met}$. Any morphism $h\colon (X, d_X) \to (C, d_C)$ factorizes through the image $h(X) \subseteq (C, d_C)$. Note that $h(X)$ is countable space because X is so.

For each $i \in \mathcal{I}$, let $D(i) = (X_i, d_i)$. Since D is filtered, by Proposition B.6, for any $x, y \in h(X)$,
$$d_C(x, y) = \inf\{d_i(x', y') \mid i \in \mathcal{I}\,, f_i(x') = x\,, \text{and } f_i(y') = y\}\,. \tag{B.3}$$
Thus, for any $n \in \mathbb{N}$ there exist $j_n \in \mathcal{I}$ and $x_n, y_n \in X_{j_n}$ such that
$$f_{j_n}(x_n) = x\,,\ f_{j_n}(y_n) = y\,,\ \text{and } d_{j_n}(x_n, y_n) \leq d_C(x, y) + \frac{1}{n+1}\,. \tag{B.4}$$
Since D is countably filtered, by (B.4) and Proposition B.5, there exist $j_{x,y} \in \mathcal{I}$ and connecting morphisms $D(j_n \to j_{x,y})\colon X_{j_n} \to X_{j_{x,y}}$, mapping all x_n to a single element in $\bar{x}_y \in X_{j_{x,y}}$, i.e., for all $n \in \mathbb{N}$
$$D(j_n \to j_{x,y})(x_n) = \bar{x}_y \quad \text{and} \quad f_{j_{x,y}}(\bar{x}_y) = x\,. \tag{B.5}$$
From (B.5) and the fact that $h(X)$ is countable, by Proposition B.5, there exists $j \in \mathcal{I}$ and connecting morphisms such that, for all $x, y \in h(X)$.
$$D(j_{x,y} \to j)(\bar{x}_y) = \bar{x} \quad \text{and} \quad f_j(\bar{x}) = x\,. \tag{B.6}$$

From the construction above, we can define the map $g\colon X \to X_j$ as follows:
$$g(x) = \overline{h(x)}\,,$$

where $\overline{h(x)}$ is the element in X_j satisfying (B.6). Next we prove that g is non-expansive. Assume by contradiction that, $d_X(x,y) < d_j(g(x),g(y))$ for some $x,y \in X$. Then there exists $n \in \mathbb{N}$ such that $d_X(x,y) + \frac{1}{n+1} \leq d_j(g(x),g(y))$, and by non-expansivity of h

$$d_C(h(x),h(y)) + \frac{1}{n+1} \leq d_j(g(x),g(y)) = d_j(\overline{h(x)}, \overline{h(y)})\,. \tag{B.7}$$

But, by (B.6) $f_j(\overline{h(x)}) = h(x)$ and $f_j(\overline{h(y)}) = h(y)$. Thus (B.7) contradicts (B.3). Consequently, for all $x,y \in X$, $d_X(x,y) \leq d_j(g(x),g(y))$, i.e., g is non-expansive.

By definition of g and (B.6) we immediately obtain that $h = f_j \circ g$. Hence h factorizes through f_j. By Proposition B.5 the factorization is essentially unique. Indeed, whenever $y,y' \in X_i$ fulfil $f_i(y) = f_i(y')$, then there exists $j \in \mathcal{I}$ such that $D(i \to j)(y) = D(i' \to j)(y')$. □

Theorem B.8. **Met** *is a locally countably presentable category.*

Proof. By Proposition A.1, **Met** is cocomplete. Every extended metric space is the countably filtered colimit of its countable subspaces. By Lemma B.7 the countable spaces are countably presentable. Hence for $\mathbf{Met}_{\aleph_1}$ one can take the set of objects of the skeleton category of the full subcategory of all countable extended metric spaces, or equivalently, the set of all countable ordinals endowed with an extended metric space. □