University of Strathclyde Science

CS208 (Semester 1) Topic 1 : Entailment and Deduction

Dr. Robert Atkey

Computer & Information Sciences

Entailment and Deduction, Part 1

Entailment

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Entailment is a relation between some assumptions:

$$P_1, \ldots, P_n$$

and a conclusion:

$$Q$$

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$$P_1, \ldots, P_n$$

and a conclusion:

$$Q$$

What we want to capture is:

If we assume P_1, ..., P_n are all true, then it is safe to conclude Q.

Is it safe?

If we assume

it is sunny

then is it safe to conclude

it is sunny

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Yes!

Is it safe?

If we assume

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Yes! There are two cases:

1. It is sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{T}$)
2. It isn't sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{F}$)

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Yes! There are two cases:

1. It is sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{T}$)
2. ~~It isn't sunny (i.e., $v(\mathrm{Sunny}) = \mathsf{F}$)~~

But we are assuming "it is sunny", so the second case doesn't matter.

Is it safe?

If we assume

nothing

then is it safe to conclude

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No!

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No! There are two cases:

1. It is sunny (i.e., $v(\text{Sunny}) = \mathsf{T}$)
2. It isn't sunny (i.e., $v(\text{Sunny}) = \mathsf{F}$)

But we are making no assumptions, so either "world" is possible: it might not be sunny.

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

No!

Is it safe?

If we assume

it is raining

then it is safe to conclude:

it is not sunny

No! There are four cases:

1. It is sunny and raining
2. It is sunny and not raining
3. It is not sunny, but is raining
4. It is not sunny and not raining

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Is it safe?

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then it is safe to conclude:

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No! There are four cases:

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Is it safe?

If we assume

it is raining *and* if it is raining it is not sunny

then is it safe to conclude:

it is not sunny

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Yes!. There are two cases:

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Is it safe?

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nothing

then is it safe to conclude:

it is sunny or not sunny

Yes!. There are two cases:

1. It is sunny
2. It is not sunny

In either case the conclusion is true: $A \vee B$ requires at least one of A or B to be true.

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

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If we assume

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Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

Yes! There are four cases:

1. it is sunny, and the moon is made of spaghetti
2. it is not sunny, and the moon is made of spaghetti
3. it is sunny, and the moon is not made of spaghetti
4. it is not sunny, and the moon is not made of spaghetti

Is it safe?

If we assume

it is sunny *and* it is not sunny

then is it safe to conclude:

the moon is made of spaghetti

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Entailment

In general, we have n assumptions $P_1, \ldots, P_n$ and conclusion Q.

We are going to say: $$P_1, \ldots, P_n \models Q$$

Read as $P_1, \ldots, P_n$ *entails* Q

if:

for all "situations" (i.e., valuations)

that make **all** the assumptions P_i true,

the conclusion Q is true.

Entailment

With more symbols

for all valuations v, if, for all i, $[\![P_i]\!]v = \mathsf{T}$, then $[\![Q]\!]v = \mathsf{T}$.

In terms of Semantics

every valuation in all $[\![P_i]\!]$ is also in $[\![Q]\!]$
(in set theory symbols: $([\![P_1]\!] \cap \cdots \cap [\![P_n]\!]) \subseteq [\![Q]\!]$).

Entailment and Deduction, Part 2

Deductive Reasoning

Why have logic(s)?

One reason is to study “arguments”.

- To separate valid and invalid reasoning.
- If we assume P_1, P_2, P_3, then when is it valid to conclude Q?

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One answer is "entailment"

- $P_1, \ldots \models Q$ "is" valid reasoning from assumptions to a conclusion.

Entailment is defined in terms of the semantics of formulas

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This doesn't match how we reason normally.

If we are trying to convince someone, we don't (usually) say:

"let's go through all the combinations of truth values and test each one."

Chains of Inference

Usually, we might say things like:

1. Let's assume that A, B, C are true.
2. If we assume A and B imply D, then D is true.
3. If we assume C and D imply E, then E is true.
4. So, we can conclude E, under the assumptions.

If our reasoning is sound, then we ought to be able to conclude

$$A, B, C, (A \wedge B) \to D, (C \wedge D) \to E \models E$$

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We have a form of modularity

- We don't check the entailment for every possible truth value of A, B, C, D, E ($2^5 = 32$ combinations!)
- We apply individual reasoning *steps* and chain them together.

Semantic Reasoning doesn't scale

In *Propositional Logic*, it is possible (though not always feasible) to check all cases.

- If there are n atomic propositions, check 2^n combinations.
- SAT solvers are good at only checking the ones that matter.
- But there are still Hard Problems that take too long.

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- But there are still Hard Problems that take too long.

Also, later in the course we will study *Predicate Logic*

- Predicate logic allows *universal* statements:

$$\forall x. \forall y.\, x + y = y + x$$

  "For all (numbers) x *and* y*,* $x + y$ *is equal to* $y + x$*"*
- Simply not possible to exhaustively check all numbers.

Deductive Systems

To overcome these problems, we use *deductive systems*.

A **deductive system** is a collection of *rules* for deriving conclusions from assumptions.

- Typically, the rules are "finitely describable"

  (roughly: we can implement them on a computer)

Typically (but not always), we write

$$P_1, \cdots, P_n \vdash Q$$

when we can derive conclusion Q from assumptions $P_1, \cdots, P_n$.

Soundness and Completeness

Soundness : "Everything that is provable is valid"

$$P_1, \cdots, P_n \vdash Q \qquad \textit{implies} \qquad P_1, \cdots, P_n \vDash Q$$

(pretty much a requirement to be useful)

Completeness : "Everything that is valid is provable"

$$P_1, \cdots, P_n \vDash Q \qquad \textit{implies} \qquad P_1, \cdots, P_n \vdash Q$$

(not *essential*, but good to have)

Advantages of Deductive Systems

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1. We can write computer programs to check our proofs, even when talking about infinitely many things.

2. If we remove or alter rules do we get an interesting new logic?

3. We can start to ask questions about the proofs:
   - An entailment $P_1, \cdots, P_n \models Q$ is either valid or invalid. *Meh.*
   - but there may be many proofs (ways of applying the rules).
   - Questions:
     - Do different proofs *mean* different things?
     - Is one proof a simplification of another?
     - Is there information hidden in proofs that we can extract?

Inference Rules

$$\frac{\mathrm{premise}_1 \qquad \cdots \qquad \mathrm{premise}_n}{\mathrm{conclusion}}$$

The idea:

- If we can prove all of $\mathrm{premise}_1$ and ... and $\mathrm{premise}_n$; then
- we have a proof of $\mathrm{conclusion}$.

Inference Rules

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The idea:

- If we can prove all of premise_1 and ... and premise_n; then
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We might have zero premises,
in which case the conclusion requires no proof ("is an axiom").

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in which case the conclusion requires no proof ("is an axiom").

Rules are organised into *trees* to make *deductions*.

Example

$$\frac{}{\text{bears } \textit{are furry}}\ \textsc{Rule1} \qquad \frac{}{\text{bears } \textit{make milk}}\ \textsc{Rule2}$$

$$\frac{\text{X } \textit{are furry} \qquad \text{X } \textit{make milk}}{\text{X } \textit{are mammals}}\ \textsc{Rule3}$$

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A deduction:

$$\frac{\dfrac{}{\text{bears } \textit{are furry}}\ \textsc{Rule1} \qquad \dfrac{}{\text{bears } \textit{make milk}}\ \textsc{Rule2}}{\text{bears } \textit{are mammals}}\ \textsc{Rule3}$$

Example (cont.)

$$\frac{X \textit{ are covered in fibres}}{X \textit{ are furry}}\ \textsc{Rule4}$$

$$\frac{}{\text{coconuts } \textit{are covered in fibres}}\ \textsc{Rule5} \qquad \frac{}{\text{coconuts } \textit{make milk}}\ \textsc{Rule6}$$

Example (cont.)

Another deduction:

$$\dfrac{\dfrac{\dfrac{}{\text{coconuts } \textit{are covered in fibres}}\,\text{R5}}{\text{coconuts } \textit{are furry}}\,\text{R4} \qquad \dfrac{}{\text{coconuts } \textit{make milk}}\,\text{R6}}{\text{coconuts } \textit{are mammals}}\,\text{R3}$$

Example (cont.)

When *building* deductions, we work bottom up:

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When *building* deductions, we work bottom up:

coconuts *are mammals*

1. Write down the conclusion

Example (cont.)

When *building* deductions, we work bottom up:

$$\dfrac{\text{coconuts } \textit{are furry} \qquad \text{coconuts } \textit{make milk}}{\text{coconuts } \textit{are mammals}}\ \text{R3}$$

1. Write down the conclusion
2. Apply rule Rule3 (X *are mammals* if X *are furry and make milk*)

Example (cont.)

When *building* deductions, we work bottom up:

$$\dfrac{\dfrac{\text{coconuts } \textit{are covered in fibres}}{\text{coconuts } \textit{are furry}}\ \text{R4} \qquad \text{coconuts } \textit{make milk}}{\text{coconuts } \textit{are mammals}}\ \text{R3}$$

1. Write down the conclusion
2. Apply rule Rule3 (X *are mammals* if X *are furry and make milk*)
3. Apply rule Rule4 (X *are furry* if they *are covered in fibres*)

Example (cont.)

When *building* deductions, we work bottom up:

$$\dfrac{\dfrac{\dfrac{}{\text{coconuts } \textit{are covered in fibres}}\,\mathrm{R5}}{\text{coconuts } \textit{are furry}}\,\mathrm{R4} \qquad \text{coconuts } \textit{make milk}}{\text{coconuts } \textit{are mammals}}\,\mathrm{R3}$$

1. Write down the conclusion
2. Apply rule Rule3 (X *are mammals* if X *are furry and make milk*)
3. Apply rule Rule4 (X *are furry* if they *are covered in fibres*)
4. Apply rule Rule5 (an axiom)

Example (cont.)

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2. Apply rule Rule3 (X *are mammals* if X *are furry and make milk*)
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4. Apply rule Rule5 (an axiom)
5. Apply rule Rule6 (an axiom)

Example (cont.)

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4. Apply rule Rule5 (an axiom)
5. Apply rule Rule6 (an axiom)

Summary

- The *why?* of deductive systems.
- Inference rules.
- How to make chains of inference.