CS208 (Semester 1) Topic 2 : Proof for Propositional Logic

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ARGUMENTS YARD

Proof for Propositional Logic, Part 1

Natural Deduction

Judgements

We want to deduce *judgements* of the form:

$$P_1, \ldots, P_n \vdash Q$$

meaning:

From assumptions $P_1, \ldots, P_n$, we can prove Q.

Soundness The system will be *sound*, meaning:

$$P_1, \ldots, P_n \vdash Q \text{ provable} \quad \Rightarrow \quad P_1, \ldots, P_n \models Q$$

We will make sure it is sound by checking each rule as we go.

If all the premises are valid entailments, then so is the conclusion

Judgements

The main judgement form is

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With assumptions $P_1, \ldots, P_n$, can prove Q

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We will also use an auxiliary judgement:

$$P_1, \ldots, P_n \; [P] \vdash Q$$

- With assumptions $P_1, \ldots, P_n$, *focusing on* P, can prove Q
- Also “means” $P_1, \ldots, P_n, P \models Q$
- Having a focus is useful for organising proofs

Judgements

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Assumption lists The list of assumptions $P_1, \dots, P_n$ will appear often. So we will shorten it to $\Gamma = P_1, \dots, P_n$.

Basic Rules

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done}$$

Basic Rules

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done}$$

- If we have a focused assumption P, then we can prove P
- (Remember Γ stands for a list of other assumptions)

Basic Rules

$$\frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

Basic Rules

$$\frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

- $P \in \Gamma$ means "P is in Γ".
- If we have a P in our current assumptions, we can focus on it.
- $P \in \Gamma$ is a *side condition*: it is something we check when we apply the rule, not another judgement to be proved.

A first proof

$$A \vdash A$$

A first proof

$$\frac{A\ [A] \vdash A}{A \vdash A}\ \textsc{Use}$$

- First Use the A assumption.

A first proof

$$\dfrac{\overline{A\ [A] \vdash A}\ \textsc{Done}}{A \vdash A}\ \textsc{Use}$$

- First Use the A assumption.
- Then we are Done.

Soundness

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad\qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

Soundness

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad\qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

Done

is sound because assuming P entails P, and extra assumptions make no difference.

Soundness

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad\qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

Done
is sound because assuming P entails P, and extra assumptions make no difference.

Use
is sound because if we assuming P twice entails Q, then it is okay to assume it once.

Rules for connectives

The rule DONE and USE do not mention the connectives.

In Natural Deduction, rules for connectives come in two kinds:

1. **Introduction** rules
   How to *construct* a proof with the connective
2. **Elimination** rules
   How to *use* an assumption with this connective

Rules for connectives

The rule DONE and USE do not mention the connectives.

In Natural Deduction, rules for connectives come in two kinds:

1. **Introduction** rules
   How to *construct* a proof with the connective
2. **Elimination** rules
   How to *use* an assumption with this connective

Very rough analogy: but can be made very precise

1. Introduction rules are like *constructors* for building objects
2. Elimination rules are like *methods* for taking apart objects

“And” Introduction

$$\frac{\Gamma \vdash Q_1 \qquad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \wedge Q_2}\ \textsc{Split}$$

“And” Introduction

$$\frac{\Gamma \vdash Q_1 \qquad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \wedge Q_2}\ \textsc{Split}$$

- To prove $P_1 \wedge P_2$ we have to prove P_1 and P_2
- This rule is often called $\wedge$-Introduction

An example proof

$$\dfrac{\dfrac{}{A, B\ [A] \vdash A}\ \textsc{Done}}{A, B \vdash A}\ \textsc{Use} \quad \dfrac{\dfrac{}{A, B\ [B] \vdash B}\ \textsc{Done}}{A, B \vdash B}\ \textsc{Use}$$

$$\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}\ \textsc{Split}$$

An example proof

$$\dfrac{\dfrac{\dfrac{}{A, B\ [A] \vdash A}\,\textsc{Done}}{A, B \vdash A}\,\textsc{Use} \qquad \dfrac{\dfrac{}{A, B\ [B] \vdash B}\,\textsc{Done}}{A, B \vdash B}\,\textsc{Use}}{A, B \vdash A \wedge B}\,\textsc{Split}$$

To prove $A \wedge B$, we Split into proofs of A and B.
In each case, we Use the corresponding assumption.

"And" Elimination

$$\frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad\qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

“And” Elimination

$$\frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

If we are focused on an formula $P_1 \wedge P_2$, we can select either the First or Second component to focus on.

Example proof

$$\dfrac{\dfrac{\overline{A \wedge B\ [B] \vdash B}\ \textsc{Done}}{A \wedge B\ [A \wedge B] \vdash B}\ \textsc{Second}}{A \wedge B \vdash B}\ \textsc{Use}$$

"True" Introduction

$$\frac{}{\Gamma \vdash \top}\ \textsc{True}$$

"True" Introduction

$$\frac{}{\Gamma \vdash \top}\ \textsc{True}$$

- $\top$ is always provable.

"True" Elimination

“True” Elimination

No elimination rule!

Summary

- The judgement forms for (focused) Natural Deduction:

$$P_1, \ldots, P_n \vdash Q \qquad\qquad P_1, \ldots, P_n\ [P] \vdash Q$$

- Rules for Use and Done
- Rules for introducing and eliminating $\wedge$.

Proof for Propositional Logic, Part 2

Rules for "Implies"

“Implies” Introduction

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce}$$

“Implies” Introduction

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce}$$

To prove $P \to Q$, we prove Q under the assumption P.

Example: $A \to A$

$$\dfrac{\dfrac{\dfrac{}{A\ [A] \vdash A}\ \textsc{Done}}{A \vdash A}\ \textsc{Use}}{\vdash A \to A}\ \textsc{Introduce}$$

Example : $(A \wedge B) \rightarrow A$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ [A] \vdash A}\ \textsc{Done}}{A \wedge B\ [A \wedge B] \vdash A}\ \textsc{First}}{A \wedge B \vdash A}\ \textsc{Use}}{\vdash (A \wedge B) \rightarrow A}\ \textsc{Introduce}$$

“Implies” Elimination

$$\frac{\Gamma \vdash P_1 \qquad \Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

“Implies” Elimination

$$\frac{\Gamma \vdash P_1 \qquad \Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

If we have $P_1 \to P_2$ and we can prove P_1, then we have P_2.

Example: $A \to (A \to B) \to B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, A \to B\ [A] \vdash A}\textsc{Done}}{A, A \to B \vdash A}\textsc{Use} \qquad \dfrac{}{A, A \to B\ [B] \vdash B}\textsc{Done}}{A, A \to B\ [A \to B] \vdash B}\textsc{Apply}}{A, A \to B \vdash B}\textsc{Use}}{A \vdash (A \to B) \to B}\textsc{Introduce}}{\vdash A \to (A \to B) \to B}\textsc{Introduce}
$$

The Rules so far

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$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

$$\frac{\Gamma \vdash Q_1 \qquad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \wedge Q_2}\ \textsc{Split} \qquad \frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce} \qquad \frac{\Gamma \vdash P_1 \qquad \Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

Summary

- The rules for Implication

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce} \qquad \frac{\Gamma \vdash P_1 \qquad \Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

Proof for Propositional Logic, Part 3

Rules for “Or”

"Or" Introduction

$$\frac{\Gamma \vdash Q_1}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Left} \qquad\qquad \frac{\Gamma \vdash Q_2}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Right}$$

"Or" Introduction

$$\frac{\Gamma \vdash Q_1}{\Gamma \vdash Q_1 \lor Q_2}\ \textsc{Left} \qquad\qquad \frac{\Gamma \vdash Q_2}{\Gamma \vdash Q_1 \lor Q_2}\ \textsc{Right}$$

To prove $Q_1 \lor Q_2$, *either* we:

1. prove Q_1, *or*
2. prove Q_2.

Example

$$\dfrac{\dfrac{\overline{A\ [A] \vdash A}\ \textsc{Done}}{A \vdash A}\ \textsc{Use}}{A \vdash A \vee B}\ \textsc{Left}$$

"Or" Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

Γ, P means all the assumptions in Γ, and P

"Or" Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

Γ, P means all the assumptions in Γ, and P

If we are focused on $P_1 \vee P_2$, then:

1. Either P_1 holds, so we have to prove Q assuming P_1; or
2. Either P_2 holds, so we have to prove Q assuming P_2

"Or" Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

“Or” Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \lor P_2] \vdash Q}\ \textsc{Cases}$$

We (the provers) don’t know which of P_1 or P_2 is true, so we need to write proofs for both eventualities.

"Or" Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

We (the provers) don't know which of P_1 or P_2 is true, so we need to write proofs for both eventualities.

This is dual to the case for conjunction: for $P_1 \wedge P_2$ *we* had to provide both sides in the introduction rule, but got to choose in the elimination rule.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor B, A\ [A] \vdash A}\ \text{Done}}{A \lor B, A \vdash A}\ \text{Use}}{A \lor B, A \vdash B \lor A}\ \text{Right} \qquad \dfrac{\dfrac{\dfrac{}{A \lor B, B\ [B] \vdash B}\ \text{Done}}{A \lor B, B \vdash B}\ \text{Use}}{A \lor B, B \vdash B \lor A}\ \text{Left}}{A \lor B\ [A \lor B] \vdash B \lor A}\ \text{Cases}}{A \lor B \vdash B \lor A}\ \text{Use}$$

“False” Introduction

No introduction rule!

"False" Elimination

$$\frac{}{\Gamma\ [F] \vdash Q}\ \textsc{False}$$

"False" Elimination

$$\frac{}{\Gamma\ [F] \vdash Q}\ \textsc{False}$$

If we have a false assumption, we can prove anything.

Example

$$\cfrac{\cfrac{\overline{\mathsf{F}\ [\mathsf{F}] \vdash A \wedge B \wedge C}\ \textsc{False}}{\mathsf{F} \vdash A \wedge B \wedge C}\ \textsc{Use}}{\vdash \mathsf{F} \to (A \wedge B \wedge C)}\ \textsc{Introduce}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \lor F, A\ [A] \vdash A}\ \textsc{Done}}{A \lor F, A \vdash A}\ \textsc{Use} \qquad \dfrac{\overline{A \lor F, F\ [F] \vdash A}\ \textsc{False}}{A \lor F, F \vdash A}\ \textsc{Use}}{A \lor F\ [A \lor F] \vdash A}\ \textsc{Cases}}{A \lor F \vdash A}\ \textsc{Use}}{\vdash (A \lor F) \to A}\ \textsc{Introduce}$$

Summary

- Rules for "Or":

$$\frac{\Gamma \vdash Q_1}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Left} \qquad \frac{\Gamma \vdash Q_2}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Right}$$

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

- "False" lets us prove anything:

$$\frac{}{\Gamma\ [F] \vdash Q}\ \textsc{False}$$

Proof for Propositional Logic, Part 4

Rules for “Not”

Negation

We could *define* negation:

$$\neg P \equiv P \to F$$

Then we wouldn’t need any rules for it.

Rules for Negation: Introduction

$(\neg P \equiv P \to F)$

$$\frac{\Gamma, P \vdash F}{\Gamma \vdash P \to F}\ \textsc{Introduce}$$

To prove $\neg P$, we prove that P proves false.

Rules for Negation: Elimination

$(\neg P \equiv P \to F)$

$$\dfrac{\Gamma \vdash P \qquad \dfrac{}{\Gamma\ [F] \vdash Q}\ \textsc{False}}{\Gamma\ [P \to F] \vdash Q}\ \textsc{Apply}$$

If we know that $\neg P$ is true, and we can prove P, then we get a contradiction which allows us to prove anything.

Specialised Rules for Negation

Introduction:

$$\frac{\Gamma, P \vdash F}{\Gamma \vdash \neg P}\ \textsc{Not-Intro}$$

Elimination:

$$\frac{\Gamma \vdash P}{\Gamma\ [\neg P] \vdash Q}\ \textsc{Not-Elim}$$

Example: $(A \lor B) \to \neg A \to B$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{A \lor B, \neg A, A\ [A] \vdash A}\ \textsc{Done}
          }{A \lor B, \neg A, A \vdash A}\ \textsc{Use}
        }{A \lor B, \neg A, A\ [\neg A] \vdash B}\ \neg\text{-}\textsc{Elim}
      }{A \lor B, \neg A, A \vdash B}\ \textsc{Use}
      \quad
      \dfrac{
        \dfrac{}{A \lor B, \neg A, B\ [B] \vdash B}\ \textsc{Done}
      }{A \lor B, \neg A, B \vdash B}\ \textsc{Use}
    }{A \lor B, \neg A\ [A \lor B] \vdash B}\ \textsc{Cases}
  }{
    \dfrac{A \lor B, \neg A \vdash B}{A \lor B \vdash \neg A \to B}\ \textsc{Introduce}
  }\ \textsc{Use}
}{\vdash (A \lor B) \to \neg A \to B}\ \textsc{Introduce}
$$

Summary

- Negation can be defined in terms of Implication and False
- Nicer to have specific rules:

$$\frac{\Gamma, P \vdash F}{\Gamma \vdash \neg P} \qquad \frac{\Gamma \vdash P}{\Gamma\ [\neg P] \vdash Q}$$

Proof for Propositional Logic, Part 5

Soundness & Completeness & Philosophy

Soundness and Completeness

Soundness : "Everything that is provable is valid":

$$P_1, \ldots, P_n \vdash Q \quad \Rightarrow P_1, \ldots, P_n \models Q$$

I've tried, informally, to convince you of this for each rule. If each rule is sound, then the whole system is sound.

Soundness and Completeness

Soundness : "Everything that is provable is valid":

$$P_1, \dots, P_n \vdash Q \quad \Rightarrow P_1, \dots, P_n \vDash Q$$

I've tried, informally, to convince you of this for each rule. If each rule is sound, then the whole system is sound.

Completeness : "Everything that is provable is valid":

$$P_1, \dots, P_n \vDash Q \quad \Rightarrow P_1, \dots, P_n \vdash Q$$

Does this property hold of the system so far?

Failure of Completeness

Recall that this entailment is valid:

$$\models A \vee \neg A$$

Can we prove this?

Failure of Completeness

Recall that this entailment is valid:

$$\vDash A \vee \neg A$$

Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Failure of Completeness

Recall that this entailment is valid:

$$\models A \vee \neg A$$

Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

1. Apply Use to use an assumption.

Failure of Completeness

Recall that this entailment is valid:

$$\models A \vee \neg A$$

Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

1. Apply Use to use an assumption. *No assumptions!*

Failure of Completeness

Recall that this entailment is valid:

$$\models A \lor \neg A$$

Can we prove this? Is there a proof of $\vdash A \lor \neg A$?

Have three options:

1. Apply Use to use an assumption. *No assumptions!*
2. Apply Left and try to prove $\vdash A$,

Failure of Completeness

Recall that this entailment is valid:

$$\vDash A \vee \neg A$$

Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

1. Apply USE to use an assumption. *No assumptions!*
2. Apply LEFT and try to prove $\vdash A$, *but this can't be provable, by soundness!*

Failure of Completeness

Recall that this entailment is valid:

$$\models A \vee \neg A$$

Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

1. Apply USE to use an assumption. *No assumptions!*
2. Apply LEFT and try to prove $\vdash A$, *but this can't be provable, by soundness!*
3. Apply RIGHT and try to prove $\vdash \neg A$,

Failure of Completeness

Recall that this entailment is valid:

$$\vDash A \lor \neg A$$

Can we prove this? Is there a proof of $\vdash A \lor \neg A$?

Have three options:

1. Apply Use to use an assumption. *No assumptions!*
2. Apply Left and try to prove $\vdash A$, *but this can't be provable, by soundness!*
3. Apply Right and try to prove $\vdash \neg A$, *but this can't be provable, by soundness!*

Failure of Completeness

Recall that this entailment is valid:

$$\vDash A \vee \neg A$$

Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

1. Apply Use to use an assumption. *No assumptions!*
2. Apply Left and try to prove $\vdash A$, *but this can't be provable, by soundness!*
3. Apply Right and try to prove $\vdash \neg A$, *but this can't be provable, by soundness!*

So the system so far is **not** complete, with respect to our semantics.

Fixing completeness

We could add the following rule:

$$\frac{\Gamma, P \vdash Q \qquad \Gamma, \neg P \vdash Q}{\Gamma \vdash Q}\ \textsc{ExcludedMiddle}$$

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To prove Q, pick any proposition P and say "either P or ¬P".

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To prove Q, pick any proposition P and say "either P or $\neg$P".

This lets us prove $\vdash A \vee \neg A$.

Fixing completeness

We could add the following rule:

$$\frac{\Gamma, P \vdash Q \qquad \Gamma, \neg P \vdash Q}{\Gamma \vdash Q}\ \textsc{ExcludedMiddle}$$

To prove Q, pick any proposition P and say "either P or $\neg$P".

This lets us prove $\vdash A \vee \neg A$.

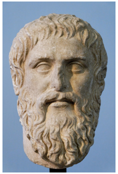
It is *sound*, but is it a good idea?

Some Philosophy of Mathematics

Where do mathematical objects live?

(objects include numbers, shapes, functions, propositions, proofs, ...)

"Platonism"



- Objects exist "out there", independently of us.
- There is a universal notion of "truth".
  - Every proposition is either true or false, even if *we* can't see why.

Image: By Copy of Silanion, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=7831217`

“Intuitionism”

(L.E.J. Brouwer, 1900/10/20s)

- Objects exist as constructions within our heads.
- Including proofs of propositions
  - We convince ourselves of the truth of a proposition by constructing evidence for it.

Image: By Source (WP:NFCC#4), Fair use, `https://en.wikipedia.org/w/index.php?curid=39567913`

Evidence based Interpretation

(Instead of saying P □ Q is true when...)

Evidence of...	is
T	there always evidence of T
F	there is no evidence of F
$P \wedge Q$	evidence of P and evidence of Q
$P \vee Q$	evidence of P or evidence of Q
$P \to Q$	a process converting evidence of P into evidence of Q

Evidence for Negation

We define $\neg P = P \to F$.

- evidence of $\neg P$ is a process converting evidence of P to evidence of F
- but there is no evidence of F
- so there can be no evidence of P.

Excluded Middle?

In two valued (T, F) logic, *excluded middle* is valid for any P:

$$P \vee \neg P$$

The proof of validity (via truth tables) makes no commitment to which one is actually true.

Excluded Middle?

In two valued (T, F) logic, *excluded middle* is valid for any P:

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The proof of validity (via truth tables) makes no commitment to which one is actually true.

However, in terms of evidence, we have to *construct* either

1. evidence of P, or
2. evidence of $\neg$P.

For an arbitrary proposition P, this seems unlikely.

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Failure of Excluded Middle

For instance, if x is a real number (has an arbitrarily long decimal expansion), then, in terms of evidence

$$(x = 0) \vee \neg(x = 0)$$

asks us to determine whether x is 0.

But there is no process to do this in finite time.

(Another example: does this Turing Machine halt?)

Intuitionistic Logic

Intuitionistic Logic is the similar to "Classical" Logic, except that it does not include the Law of Excluded Middle $P \vee \neg P$ for all propositions P.

Note: this does not mean that $\neg(P \vee \neg P)$ is provable. There may be some Ps for which $P \vee \neg P$ holds.

(For example, $(x = 0) \vee \neg(x = 0)$ when x is an integer)

Summary

- The system was have so far is *sound* but not *complete*
- We can make it complete by adding a rule for *excluded middle*:

$$P \vee \neg P$$

- But should we? What does Logic mean?