University of Strathclyde Science

CS208 (Semester 1) Topic 4 : Proof for Predicate Logic

Dr. Robert Atkey

Computer & Information Sciences

University of Strathclyde Science

Proof for Predicate Logic, Part 1

Upgrading Natural Deduction

Tracking free variables

We are going to prove things like:

$$\vdash \forall x.(p(x) \wedge q(x)) \rightarrow p(x)$$

This will mean we will have proof states like:

$$\cdots \vdash (p(x) \wedge q(x)) \rightarrow p(x)$$

We need to keep track of variables as well as assumed formulas to the left of the $\vdash$ "turnstile".

Judgements

Proving:

$$\underbrace{P_1, x_1, \ldots, x_i, P_j, \ldots, x_m, P_n}_{\textit{assumptions and variables}} \vdash \underbrace{Q}_{\textit{conclusion}}$$

Focused:

$$\underbrace{P_1, x_1, \ldots, x_i, P_j, \ldots, x_m, P_n}_{\textit{assumptions and variables}} [\underbrace{P}_{\textit{focus}}] \vdash \underbrace{Q}_{\textit{conclusion}}$$

Note:

1. We never focus on a variable, only formulas
2. Each P_j only contains free variables that appear to the *left* of it

Well-scoped terms and formulas

If we have a list of variables and assumptions (a "context"):

$$\Gamma = P_1, x_1, \ldots, x_i, P_j, \ldots, x_m, P_n$$

Γ is the name we're giving to the list

- A formula P is *well-scoped in* Γ if all the free variables of P appear in Γ.
- A term t is *well-scoped in* Γ if all the variables of t appear in Γ.

- All formulas in Γ must be well-scoped by the variables to their left (same condition as previous slide).
- The focus and conclusion must always be well-scoped in Γ.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \rightarrow Q(y)$

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \rightarrow Q(y)$
   Yes. The variable y is bound in the formula.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \rightarrow Q(y)$
   Yes. The variable y is bound in the formula.
2. Context: x Formula: $\forall y.P(y) \rightarrow Q(x, y)$

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \to Q(y)$
   Yes. The variable y is bound in the formula.
2. Context: x Formula: $\forall y.P(y) \to Q(x, y)$
   Yes. The variable y is bound in the formula, and the free variable x is in the context.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \rightarrow Q(x, y)$

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \rightarrow Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \rightarrow Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.
2. Context: *empty* Term: $x + 1$

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \rightarrow Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.
2. Context: *empty* Term: $x + 1$
   No. The variable x is free in the term but is not in the context.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \rightarrow Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.
2. Context: *empty* Term: $x + 1$
   No. The variable x is free in the term but is not in the context.

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x, y\ [P(x, y)] \vdash Q(x)$$

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x, y\ [P(x, y)] \vdash Q(x)$$

Yes. The free variables of the focus and conclusion are x, y, which are in the context.

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x\ [P(x, y)] \vdash Q(x)$$

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x\ [P(x, y)] \vdash Q(x)$$

No. The free variables of the focus and conclusion are x, y, but y is not in the context.

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x, Q(x), y\ [P(x, y)] \vdash Q(y)$$

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x, Q(x), y\ [P(x, y)] \vdash Q(y)$$

Yes. Each variable appears before (reading left to right) it is used.

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$\forall x.Q(x), y\ [P(x, y)] \vdash Q(y)$$

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$\forall x.Q(x), y\ [P(x,y)] \vdash Q(y)$$

No. The x in the first $Q(x)$ is OK, but the x in $P(x,y)$ has not been declared in scope.

Summary

1. We started to upgrade Natural Deduction to Predicate Logic
2. We need to manage the *scope* of variables
3. To do so, we add them to the context
4. Variables may only be used by formulas to their right

University of Strathclyde Science

Proof for Predicate Logic, Part 2

Rules for “Forall”

What does $\forall x.P$ mean?

(assuming a domain of discourse)

University of Strathclyde Science

Answer 1 : it means for all individuals "a", $P[x := a]$ is true.

(we think of "for all" as an infinite conjunction)

What does $\forall x.P$ mean?

(assuming a domain of discourse)

University of Strathclyde Science

Answer 1 : it means for all individuals "a", $P[x := a]$ is true.

(we think of "for all" as an infinite conjunction)

Answer 2 : thinking about proofs:

To *prove* a $\forall x.P$:

- We must prove $P[x := x_0]$ for a *general* x_0.
- The x_0 stands in for any "a" that might be chosen.

To *use* a proof of $\forall x.P$:

- We can *choose* any t we like for x, and get $P[x := t]$

University of Strathclyde Science

Introduction rule

$$\frac{\Gamma, x_0 \vdash Q[x := x_0]}{\Gamma \vdash \forall x.Q} \textsc{Introduce}\forall$$

University of Strathclyde Science

Introduction rule

$$\frac{\Gamma, x_0 \vdash Q[x := x_0]}{\Gamma \vdash \forall x.Q} \textsc{Introduce}\forall$$

"To prove $\forall x.Q$, we prove $Q[x := x_0]$, assuming an arbitrary x_0."

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x, P(x) \wedge Q(x)\ [P(x)] \vdash P(x)}\ \textsc{Done}}{x, P(x) \wedge Q(x)\ [P(x) \wedge Q(x)] \vdash P(x)}\ \textsc{First}}{x, P(x) \wedge Q(x) \vdash P(x)}\ \textsc{Use}}{x \vdash (P(x) \wedge Q(x)) \rightarrow P(x)}\ \textsc{Introduce}}{\vdash \forall x.(P(x) \wedge Q(x)) \rightarrow P(x)}\ \textsc{Introduce}$$

Elimination

$$\frac{\Gamma\ [P[x := t]] \vdash Q}{\Gamma\ [\forall x.P] \vdash Q}\ \textsc{Instantiate}$$

(side condition: t is well-scoped in Γ)

Elimination

$$\frac{\Gamma\ [P[x := t]] \vdash Q}{\Gamma\ [\forall x.P] \vdash Q}\ \textsc{Instantiate}$$

(side condition: t is well-scoped in Γ)

"If we have P for all x, then we can pick any well-scoped t we like to stand in for it."

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\ [h(s())] \vdash h(s())}\ \textsc{Done}}{\Gamma \vdash h(s())}\ \textsc{Use} \qquad \dfrac{}{\Gamma\ [m(s())] \vdash m(s())}\ \textsc{Done}}{\Gamma\ [h(s()) \to m(s()))] \vdash m(s())}\ \textsc{Apply}}{\Gamma\ [\forall x.h(x) \to m(x)] \vdash m(s())}\ \textsc{Instantiate}}{\Gamma \vdash m(s())}\ \textsc{Use}}{\forall x.h(x) \to m(x) \vdash h(s()) \to m(s())}\ \textsc{Introduce}}{\vdash (\forall x.h(x) \to m(x)) \to h(s()) \to m(s())}\ \textsc{Introduce}$$

where $\Gamma = \forall x.h(x) \to m(x), h(s())$

Summary

- To prove $\forall x.P(x)$, we must prove $P(x_0)$ for a general x_0.
- To use $\forall x.P(X)$, we get to choose the t we use for x.

University of Strathclyde Science

Proof for Predicate Logic, Part 3

Rules for “Exists”

What does $\exists x.P$ mean?

(assuming a domain of discourse)

University of Strathclyde Science

Answer 1 : there is at least one "a" such that $P[x := a]$ is true.

(we think of "exists" as an infinite disjunction)

What does $\exists x.P$ mean?

(assuming a domain of discourse)

University of Strathclyde Science

Answer 1 : there is at least one "a" such that $P[x := a]$ is true.

(we think of "exists" as an infinite disjunction)

Answer 2 : thinking about proofs:

To *prove* a $\exists x.P$:

- We must provide a *witness* term t such that $P[x := t]$.

To *use* a proof of $\exists x.P$:

- We have to work with an arbitrary x_0 and all we know is $P[x := x_0]$.

Introduction

$$\frac{\Gamma \vdash P[x := t]}{\Gamma \vdash \exists x.P} \text{ Exists}$$

(side condition: t is well-scoped in Γ)

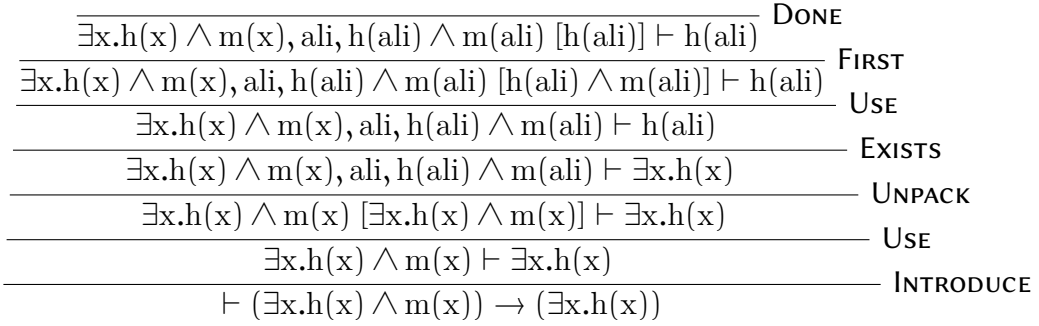
"To prove $\exists x.P$, we have to provide a witness t for x, and show that $P[x := t]$"

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{human}(\mathsf{socrates}())\ [\mathrm{human}(\mathsf{socrates}())] \vdash \mathrm{human}(\mathsf{socrates}())}\ \textsc{Done}}{\mathrm{human}(\mathsf{socrates}()) \vdash \mathrm{human}(\mathsf{socrates}())}\ \textsc{Use}}{\mathrm{human}(\mathsf{socrates}()) \vdash \exists \mathrm{x}.\mathrm{human}(\mathrm{x})}\ \textsc{Exists}}{\vdash \mathrm{human}(\mathsf{socrates}()) \rightarrow (\exists \mathrm{x}.\mathrm{human}(\mathrm{x}))}\ \textsc{Introduce}$$

Elimination

$$\frac{\Gamma, x_0, P[x := x_0] \vdash Q}{\Gamma\ [\exists x.P] \vdash Q}\ \textsc{Unpack}$$

"To use $\exists x.P$, we get some arbitrary x_0 that we know $P[x := x_0]$ about."


$\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali)\ [h(ali)] \vdash h(ali)$
Done
$\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali)\ [h(ali) \wedge m(ali)] \vdash h(ali)$
First
$\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali) \vdash h(ali)$
Use
$\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali) \vdash \exists x.h(x)$
Exists
$\exists x.h(x) \wedge m(x)\ [\exists x.h(x) \wedge m(x)] \vdash \exists x.h(x)$
Unpack
$\exists x.h(x) \wedge m(x) \vdash \exists x.h(x)$
Use
$\vdash (\exists x.h(x) \wedge m(x)) \rightarrow (\exists x.h(x))$
Introduce

Comparing $\wedge$ and $\forall$

Introduction

$$\frac{\Gamma \vdash P_1 \qquad \Gamma \vdash P_2}{\Gamma \vdash P_1 \wedge P_2}\ \textsc{Split} \qquad\qquad \frac{\Gamma, x_0 \vdash P[x := x_0]}{\Gamma \vdash \forall x.P}\ \forall\text{-I}$$

For $\wedge$, we have to prove P_i, no matter what i is. For $\forall$, we have to prove $P[x := x_0]$, no matter what x_0 is.

Comparing $\wedge$ and $\forall$

Elimination

$$\frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

$$\frac{\Gamma\ [P[x := t]] \vdash Q}{\Gamma\ [\forall x.P] \vdash Q}\ \textsc{Instantiate}$$

For $\wedge$, we choose 1 or 2. For $\forall$, we choose t.

Comparing $\vee$ and $\exists$

Introduction

$$\frac{\Gamma \vdash P_1}{\Gamma \vdash P_1 \vee P_2}\ \textsc{Left} \qquad \frac{\Gamma \vdash P_2}{\Gamma \vdash P_1 \vee P_2}\ \textsc{Right} \qquad \frac{\Gamma \vdash P[x := t]}{\Gamma \vdash \exists x.P}\ \textsc{Exists}$$

For $\vee$, we choose which of 1 or 2 we want. For $\exists$, we choose the witnessing term t.

Comparing $\vee$ and $\exists$

Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases} \qquad \frac{\Gamma, x_0, P[x := x_0] \vdash Q}{\Gamma\ [\exists x.P] \vdash Q}\ \textsc{Unpack}$$

For $\vee$, we must deal with 1 or 2. For $\exists$, we must cope with any x_0.

Summary

- To prove $\exists x.P(x)$ we must give a witness t and prove $P(t)$.
- To use $\exists x.P(X)$ we get to assume there is some y and $P(y)$.

University of Strathclyde Science

Proof for Predicate Logic, Part 4

Rules for Equality

What is Equality?

$$t_1 = t_2$$

What is Equality?

Some properties:

1. *Reflexivity:* for all x, $x = x$
2. *Symmetry:* for all x and y, if $x = y$ then $y = x$
3. *Transitivity:* for all x, y and z, if $x = y$ and $y = z$, then $x = z$

What is Equality?

Some properties:

1. *Reflexivity:* for all x, $x = x$
2. *Symmetry:* for all x and y, if $x = y$ then $y = x$
3. *Transitivity:* for all x, y and z, if $x = y$ and $y = z$, then $x = z$

Any binary relation that satisfies these properties is called an *equivalence relation*.

What is Equality?

The **special** property of equality is the following:

> If $s = t$, then everything that is true about s is true about t.

What is Equality?

The **special** property of equality is the following:

If $s = t$, then everything that is true about s is true about t.

(and vice versa. but do we need to say this?)

What is Equality?

The **special** property of equality is the following:

> If $s = t$, then everything that is true about s is true about t.

(and vice versa. but do we need to say this?)

Gottfried Leibniz (co-inventor of Calculus) took this as the *definition* of equality.

What is Equality?

University of Strathclyde Science

With more symbols:

$$\text{If } t_1 = t_2, \text{ then for all P, if } P[x \mapsto t_1] \text{ then } P[x \mapsto t_2]$$

What is Equality?

All we will need is:

1. Reflexivity: for every term t, $t = t$
2. Substitution: $t_1 = t_2$ and $P[x \mapsto t_1]$ implies $P[x \mapsto t_2]$

Amazingly, this is enough!

Symmetry

University of Strathclyde Science

To prove that $x = y$ implies $y = x$:

1. We know that $x = x$ by reflexivity
2. So we use our assumption to replace the first x by y to get $y = x$.

Transitivity

To prove that $x = y$ and $y = z$ implies $x = z$:

1. Substitute the second assumption in the first to get $x = z$.

Rules for Equality: Introduction

$$\frac{}{\Gamma \vdash t = t}\ \textsc{Reflexivity}$$

Every term is equal to itself.

Rules for Equality: Elimination

$$\frac{\Gamma \vdash P[x \mapsto t_2]}{\Gamma\ [t_1 = t_2] \vdash P[x \mapsto t_1]}\ \textsc{Subst}$$

If we know that $t_1 = t_2$ then we can replace t_1 with t_2 in the goal. This is substitution backwards: if we know $P[x \mapsto t_2]$ and $t_1 = t_2$, then we know $P[x \mapsto t_1]$.

Example: Symmetry

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x, y, x = y \vdash y = y}\ \textsc{Reflexivity}}{x, y, x = y\ [x = y] \vdash y = x}\ \textsc{Subst}}{x, y, x = y \vdash y = x}\ \textsc{Use}}{x, y \vdash x = y \to y = x}\ \textsc{Introduce}}{x \vdash \forall y.x = y \to y = x}\ \textsc{Introduce}}{\vdash \forall x.\forall y.x = y \to y = x}\ \textsc{Introduce}$$

Example: Transitivity

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x, y, z, x = y, y = z\ [y = z] \vdash y = z}\ \textsc{Done}}{x, y, z, x = y, y = z \vdash y = z}\ \textsc{Use}}{x, y, z, x = y, y = z\ [x = y] \vdash x = z}\ \textsc{Subst}}{x, y, z, x = y, y = z \vdash x = z}\ \textsc{Use}}{x, y, z, x = y \vdash y = z \to x = z}\ \textsc{Introduce}}{x, y, z \vdash x = y \to y = z \to x = z}\ \textsc{Introduce}}{x, y \vdash \forall z.x = y \to y = z \to x = z}\ \textsc{Introduce}}{x \vdash \forall y.\forall z.x = y \to y = z \to x = z}\ \textsc{Introduce}}{\vdash \forall x.\forall y.\forall z.x = y \to y = z \to x = z}\ \textsc{Introduce}$$

Rewriting

1. Subst can be quite tricky to use because we have to give a formula P such that $P[x \mapsto t_1]$ is the one we start with, and $P[x \mapsto t_2]$ is the one we end up with.
2. Usually, we want to replace *every* occurrence of t_1 with t_2. We write this as:

$$P\{t_1 \mapsto t_2\}$$

Rewriting

$$\frac{\Gamma \vdash P\{t_1 \mapsto t_2\}}{\Gamma\ [t_1 = t_2] \vdash P}\ \textsc{Rewrite}{\rightarrow}$$

If we have $t_1 = t_2$ then we can replace t_1 with t_2 everywhere.

Rewriting

For convenience:

$$\frac{\Gamma \vdash P\{t_2 \mapsto t_1\}}{\Gamma\ [t_1 = t_2] \vdash P}\ \textsc{Rewrite}\leftarrow$$

If we have $t_1 = t_2$ then we can replace t_1 with t_2 everywhere.

Summary

Equality is characterised by two principles:

1. Everything is equal to itself (*reflexivity*)
2. If $s = t$, then everything that is true about s is true about t.