CS208 (Semester 1) Week 4 : Natural Deduction I

Dr. Robert Atkey

Computer & Information Sciences

Natural Deduction I, Part 1

Deductive Reasoning

Why have logic(s)?

One reason is to study "arguments".

- To separate valid and invalid reasoning.
- If we assume P_1, P_2, P_3, then when is it valid to conclude Q?

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One answer is "entailment"

- $P_1, \ldots \models Q$ "is" valid reasoning from assumptions to a conclusion.

Entailment is defined in terms of the semantics of formulas

- $P_1, \ldots \models Q$ if for all valuations v, $[\![P]\!]v = \mathsf{T}$ implies $[\![Q]\!]v = \mathsf{T}$

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This doesn't match how we reason normally.

If we are trying to convince someone, we don't (usually) say:

"let's go through all the combinations of truth values and test each one."

Chains of Inference

Usually, we might say things like:

1. Let's assume that A, B, C are true.
2. If we assume A and B imply D, then D is true.
3. If we assume C and D imply E, then E is true.
4. So, we can conclude E, under the assumptions.

If our reasoning is sound, then we ought to be able to conclude

$$A, B, C, (A \land B) \to D, (C \land D) \to E \models E$$

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We have a form of *modularity*

- We don't check the entailment for every possible truth value of A, B, C, D, E $(2^5 = 32$ combinations!)
- We apply individual reasoning *steps* and chain them together.

Semantic Reasoning doesn't scale

In *Propositional Logic*, it is possible (though not always feasible) to check all cases.

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- SAT solvers are good at only checking the ones that matter.
- But there are still Hard Problems that take too long.

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- But there are still Hard Problems that take too long.

Also, later in the course we will study *Predicate Logic*

- Predicate logic allows *universal* statements:

$$\forall x. \forall y.\, x + y = y + x$$

"For all (numbers) x and y, $x + y$ is equal to $y + x$"

- Simply not possible to exhaustively check all numbers.

Deductive Systems

To overcome these problems, we use *deductive systems.*

A **deductive system** is a collection of *rules* for deriving conclusions from assumptions.

- Typically, the rules are "finitely describable"

  (roughly: we can implement them on a computer)

Typically (but not always), we write

$$P_1, \cdots, P_n \vdash Q$$

when we can derive conclusion Q from assumptions $P_1, \cdots, P_n$.

Soundness and Completeness

Soundness : "Everything that is provable is valid"

$$P_1, \cdots, P_n \vdash Q \qquad \textit{implies} \qquad P_1, \cdots, P_n \models Q$$

(pretty much a requirement to be useful)

Completeness : "Everything that is valid is provable"

$$P_1, \cdots, P_n \models Q \qquad \textit{implies} \qquad P_1, \cdots, P_n \vdash Q$$

(not *essential*, but good to have)

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2. If we remove or alter rules do we get an interesting new logic?

3. We can start to ask questions about the proofs:
   - An entailment $P_1, \cdots, P_n \models Q$ is either valid or invalid. *Meh.*
   - but there may be many proofs (ways of applying the rules).
   - Questions:
     - Do different proofs *mean* different things?
     - Is one proof a simplification of another?
     - Is there information hidden in proofs that we can extract?

Inference Rules

$$\frac{\text{premise}_1 \qquad \cdots \qquad \text{premise}_n}{\text{conclusion}}$$

The idea:

- If we can prove all of premise_1 and … and premise_n; then
- we have a proof of conclusion.

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Rules are organised into *trees* to make *deductions*.

Example

$$\frac{}{\text{bears } \textit{are furry}}\ \textsc{Rule1} \qquad\qquad \frac{}{\text{bears } \textit{make milk}}\ \textsc{Rule2}$$

$$\frac{X \textit{ are furry} \qquad X \textit{ make milk}}{X \textit{ are mammals}}\ \textsc{Rule3}$$

Example

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$$\frac{\text{X } \textit{are furry} \qquad \text{X } \textit{make milk}}{\text{X } \textit{are mammals}}\ \textsc{Rule3}$$

A deduction:

$$\frac{\dfrac{}{\text{bears } \textit{are furry}}\ \textsc{Rule1} \qquad \dfrac{}{\text{bears } \textit{make milk}}\ \textsc{Rule2}}{\text{bears } \textit{are mammals}}\ \textsc{Rule3}$$

Example (cont.)

$$\frac{X\ \textit{are covered in fibres}}{X\ \textit{are furry}}\ \textsc{Rule4}$$

$$\frac{}{\text{coconuts}\ \textit{are covered in fibres}}\ \textsc{Rule5} \qquad \frac{}{\text{coconuts}\ \textit{make milk}}\ \textsc{Rule6}$$

Example (cont.)

Another deduction:

$$\dfrac{\dfrac{\dfrac{}{\text{coconuts } \textit{are covered in fibres}}\ \text{R5}}{\text{coconuts } \textit{are furry}}\ \text{R4} \qquad \dfrac{}{\text{coconuts } \textit{make milk}}\ \text{R6}}{\text{coconuts } \textit{are mammals}}\ \text{R3}$$

Example (cont.)

When *building* deductions, we work bottom up:

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coconuts *are mammals*

1. Write down the conclusion

Example (cont.)

When *building* deductions, we work bottom up:

$$\frac{\text{coconuts } \textit{are furry} \qquad \text{coconuts } \textit{make milk}}{\text{coconuts } \textit{are mammals}}\;\text{R3}$$

1. Write down the conclusion
2. Apply rule RULE3 (X *are mammals* if X *are furry and make milk*)

Example (cont.)

When *building* deductions, we work bottom up:

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1. Write down the conclusion
2. Apply rule Rule3 (X *are mammals* if X *are furry and make milk*)
3. Apply rule Rule4 (X *are furry* if they *are covered in fibres*)

Example (cont.)

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1. Write down the conclusion
2. Apply rule Rule3 (X *are mammals* if X *are furry and make milk*)
3. Apply rule Rule4 (X *are furry* if they *are covered in fibres*)
4. Apply rule Rule5 (an axiom)

Example (cont.)

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5. Apply rule Rule6 (an axiom)

Example (cont.)

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5. Apply rule Rule6 (an axiom)

Summary

- The *why?* of deductive systems.
- Inference rules.
- How to make chains of inference.

Natural Deduction I, Part 2

Natural Deduction

Judgements

We want to deduce *judgements* of the form:

$$P_1, \ldots, P_n \vdash Q$$

meaning:

From assumptions $P_1, \ldots, P_n$, we can prove Q.

Soundness The system will be *sound*, meaning:

$$P_1, \ldots, P_n \vdash Q \text{ provable} \quad \Rightarrow \quad P_1, \ldots, P_n \vDash Q$$

We will make sure it is sound by checking each rule as we go.

If all the premises are valid entailments, then so is the conclusion

Judgements

The main judgement form is

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$$P_1, \ldots, P_n\ [P] \vdash Q$$

- With assumptions $P_1, \ldots, P_n$, *focusing on* P, can prove Q
- Also "means" $P_1, \ldots, P_n, P \vDash Q$
- Having a focus is useful for organising proofs

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Assumption lists The list of assumptions $P_1, \ldots, P_n$ will appear often. So we will shorten it to $\Gamma = P_1, \ldots, P_n$.

Basic Rules

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done}$$

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- If we have a focused assumption P, then we can prove P
- (Remember Γ stands for a list of other assumptions)

Basic Rules

$$\frac{P \in \Gamma \qquad \Gamma\,[P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

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$$\frac{P \in \Gamma \qquad \Gamma\,[P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

- $P \in \Gamma$ means “P is in Γ”.
- If we have a P in our current assumptions, we can focus on it.
- $P \in \Gamma$ is a *side condition*: it is something we check when we apply the rule, not another judgement to be proved.

A first proof

$$A \vdash A$$

A first proof

$$\frac{A\ [A] \vdash A}{A \vdash A}\ \textsc{Use}$$

- First Use the A assumption.

A first proof

$$\dfrac{\dfrac{}{A\ [A] \vdash A}\ \textsc{Done}}{A \vdash A}\ \textsc{Use}$$

- First Use the A assumption.
- Then we are Done.

Soundness

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

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is sound because assuming P entails P, and extra assumptions make no difference.

Soundness

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad\qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

Done
is sound because assuming P entails P, and extra assumptions make no difference.

Use
is sound because if we assuming P twice entails Q, then it is okay to assume it once.

Rules for connectives

The rule Done and Use do not mention the connectives.

In Natural Deduction, rules for connectives come in two kinds:

1. **Introduction** rules
   How to *construct* a proof with the connective
2. **Elimination** rules
   How to *use* an assumption with this connective

Rules for connectives

The rule Done and Use do not mention the connectives.

In Natural Deduction, rules for connectives come in two kinds:

1. **Introduction** rules
   How to *construct* a proof with the connective
2. **Elimination** rules
   How to *use* an assumption with this connective

Very rough analogy: but can be made very precise

1. Introduction rules are like *constructors* for building objects
2. Elimination rules are like *methods* for taking apart objects

"And" Introduction

$$\frac{\Gamma \vdash Q_1 \qquad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \wedge Q_2}\ \textsc{Split}$$

“And” Introduction

$$\frac{\Gamma \vdash Q_1 \qquad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \wedge Q_2}\ \textsc{Split}$$

- To prove $P_1 \wedge P_2$ we have to prove P_1 and P_2
- This rule is often called $\wedge$-Introduction

An example proof

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$$\dfrac{\dfrac{\overline{A, B\ [A] \vdash A}\ \textsc{Done}}{A, B \vdash A}\ \textsc{Use} \qquad \dfrac{\overline{A, B\ [B] \vdash B}\ \textsc{Done}}{A, B \vdash B}\ \textsc{Use}}{A, B \vdash A \wedge B}\ \textsc{Split}$$

An example proof

$$\dfrac{\dfrac{\dfrac{}{A, B\ [A] \vdash A}\ \textsc{Done}}{A, B \vdash A}\ \textsc{Use} \qquad \dfrac{\dfrac{}{A, B\ [B] \vdash B}\ \textsc{Done}}{A, B \vdash B}\ \textsc{Use}}{A, B \vdash A \wedge B}\ \textsc{Split}$$

To prove $A \wedge B$, we Split into proofs of A and B.
In each case, we Use the corresponding assumption.

"And" Elimination

$$\frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

"And" Elimination

$$\frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

If we are focused on an formula $P_1 \wedge P_2$, we can select either the FIRST or SECOND component to focus on.

Example proof

$$\dfrac{\dfrac{\dfrac{}{A \wedge B\ [B] \vdash B}\ \textsc{Done}}{A \wedge B\ [A \wedge B] \vdash B}\ \textsc{Second}}{A \wedge B \vdash B}\ \textsc{Use}$$

“True” Introduction

$$\frac{}{\Gamma \vdash \top}\ \textsc{True}$$

"True" Introduction

$$\dfrac{}{\Gamma \vdash \top}\ \textsc{True}$$

- $\top$ is always provable.

“True” Elimination

“True” Elimination

No elimination rule!

Summary

- The judgement forms for (focused) Natural Deduction:

$$P_1, \ldots, P_n \vdash Q \qquad\qquad P_1, \ldots, P_n \; [P] \vdash Q$$

- Rules for Use and Done
- Rules for introducing and eliminating $\wedge$.

Natural Deduction I, Part 3

Rules for "Implies"

“Implies” Introduction

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce}$$

"Implies" Introduction

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}\ \textsc{Introduce}$$

To prove $P \rightarrow Q$, we prove Q under the assumption P.

Example: $A \to A$

$$\dfrac{\dfrac{\dfrac{}{A\ [A] \vdash A}\ \textsc{Done}}{A \vdash A}\ \textsc{Use}}{\vdash A \to A}\ \textsc{Introduce}$$

Example : $(A \wedge B) \rightarrow A$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ [A] \vdash A}\ \textsc{Done}}{A \wedge B\ [A \wedge B] \vdash A}\ \textsc{First}}{A \wedge B \vdash A}\ \textsc{Use}}{\vdash (A \wedge B) \rightarrow A}\ \textsc{Introduce}$$

“Implies” Elimination

$$\frac{\Gamma \vdash P_1 \qquad \Gamma\,[P_2] \vdash Q}{\Gamma\,[P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

"Implies" Elimination

$$\frac{\Gamma \vdash P_1 \qquad \Gamma\,[P_2] \vdash Q}{\Gamma\,[P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

If we have $P_1 \to P_2$ and we can prove P_1, then we have P_2.

Example: $A \to (A \to B) \to B$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, A \to B\ [A] \vdash A}\,\textsc{Done}}{A, A \to B \vdash A}\,\textsc{Use} \qquad \dfrac{}{A, A \to B\ [B] \vdash B}\,\textsc{Done}}{A, A \to B\ [A \to B] \vdash B}\,\textsc{Apply}}{A, A \to B \vdash B}\,\textsc{Use}}{A \vdash (A \to B) \to B}\,\textsc{Introduce}}{\vdash A \to (A \to B) \to B}\,\textsc{Introduce}$$

The Rules so far

$$\frac{}{\Gamma\ [P] \vdash P}\ \textsc{Done} \qquad \frac{P \in \Gamma \qquad \Gamma\ [P] \vdash Q}{\Gamma \vdash Q}\ \textsc{Use}$$

$$\frac{\Gamma \vdash Q_1 \qquad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \wedge Q_2}\ \textsc{Split} \qquad \frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce} \qquad \frac{\Gamma \vdash P_1 \qquad \Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

Summary

- The rules for Implication

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q}\ \textsc{Introduce} \qquad \frac{\Gamma \vdash P_1 \qquad \Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \to P_2] \vdash Q}\ \textsc{Apply}$$

Natural Deduction I, Part 4

Using the Interactive Editor