

CS208 (Semester 1) Week 4 : Natural Deduction I

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One reason is to study "arguments".

- ▶ To separate valid and invalid reasoning.
- If we assume P_1, P_2, P_3 , then when is it valid to conclude Q?



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▶ If we assume P₁, P₂, P₃, then when is it valid to conclude Q?

One answer is "entailment"

P₁,... ⊨ Q "is" valid reasoning from assumptions to a conclusion.

Entailment is defined in terms of the semantics of formulas

►
$$P_1, ... \models Q$$
 if for all valuations v , $\llbracket P \rrbracket v = T$ implies $\llbracket Q \rrbracket v = T$

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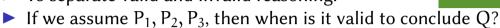
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This doesn't match how we reason normally.

If we are trying to convince someone, we don't (usually) say:

"let's go through all the combinations of truth values and test each one."

Chains of Inference

Usually, we might say things like:

- **1**. Let's assume that A, B, C are true.
- 2. If we assume A and B imply D, then D is true.
- 3. If we assume C and D imply E, then E is true.
- 4. So, we can conclude E, under the assumptions.

If our reasoning is sound, then we ought to be able to conclude

 $A,B,C,(A \wedge B) \rightarrow D,(C \wedge D) \rightarrow E \models E$



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$$A, B, C, (A \land B) \rightarrow D, (C \land D) \rightarrow E \models E$$

We have a form of modularity

- We don't check the entailment for every possible truth value of A, B, C, D, E (2⁵ = 32 combinations!)
- We apply individual reasoning *steps* and chain them together.



Semantic Reasoning doesn't scale



In *Propositional Logic*, it is possible (though not always feasible) to check all cases.

- ▶ If there are n atomic propositions, check 2ⁿ combinations.
- SAT solvers are good at only checking the ones that matter.
- But there are still Hard Problems that take too long.

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- But there are still Hard Problems that take too long.

Also, later in the course we will study *Predicate Logic*

Predicate logic allows *universal* statements:

$$\forall x. \forall y. x + y = y + x$$

"For all (numbers) x and y, x + y is equal to y + x"
▶ Simply not possible to exhaustively check all numbers.

Deductive Systems



To overcome these problems, we use *deductive systems*.

A **deductive system** is a collection of *rules* for deriving conclusions from assumptions.

Typically, the rules are "finitely describable"

(roughly: we can implement them on a computer)

Typically (but not always), we write

$$P_1, \cdots, P_n \vdash Q$$

when we can derive conclusion Q from assumptions P_1, \dots, P_n .

Soundness and Completeness



Soundness : "Everything that is provable is valid"

$$P_1, \cdots, P_n \vdash Q$$
 implies $P_1, \cdots, P_n \models Q$

(pretty much a requirement to be useful)

Completeness : "Everything that is valid is provable"

 $P_1, \cdots, P_n \models Q \qquad \textit{implies} \qquad P_1, \cdots, P_n \vdash Q$

(not *essential*, but good to have)

Advantages of Deductive Systems



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2. If we remove or alter rules do we get an interesting new logic?

- 3. We can start to ask questions about the proofs:
 - An entailment $P_1, \dots, P_n \models Q$ is either valid or invalid. Meh.
 - but there may be many proofs (ways of applying the rules).
 - Questions:
 - Do different proofs *mean* different things?
 - Is one proof a simplification of another?
 - Is there information hidden in proofs that we can extract?

Inference Rules



$\frac{\text{premise}_1 \cdots \text{premise}_n}{\text{conclusion}}$

The idea:

- ▶ If we can prove all of premise₁ and ... and premise_n; then
- we have a proof of conclusion.

Inference Rules



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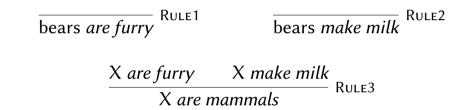
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Rules are organised into trees to make deductions.

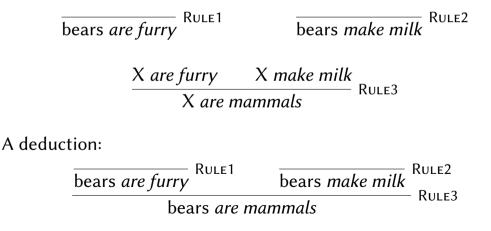
Example





Example





Example (cont.)



$\frac{X \text{ are covered in fibres}}{X \text{ are furry}} RULE4$

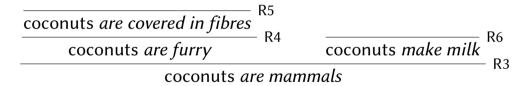
coconuts are covered in fibres RULE5

coconuts make milk RULE6

Example (cont.)



Another deduction:



Example (cont.)

When *building* deductions, we work bottom up:



Example (cont.)

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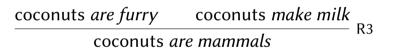
coconuts are mammals

1. Write down the conclusion

Example (cont.)



When *building* deductions, we work bottom up:

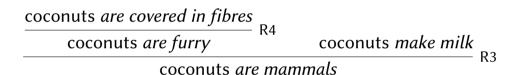


- 1. Write down the conclusion
- 2. Apply rule RULE3 (X are mammals if X are furry and make milk)

Example (cont.)



When *building* deductions, we work bottom up:

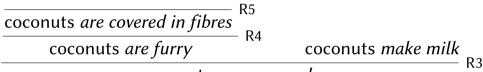


- 1. Write down the conclusion
- 2. Apply rule RULE3 (X are mammals if X are furry and make milk)
- 3. Apply rule RULE4 (X are furry if they are covered in fibres)

Example (cont.)



When *building* deductions, we work bottom up:



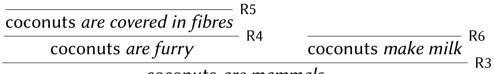
coconuts are mammals

- 1. Write down the conclusion
- 2. Apply rule RULE3 (X are mammals if X are furry and make milk)
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- 4. Apply rule RULE5 (an axiom)

Example (cont.)



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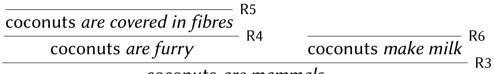
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- 4. Apply rule RULE5 (an axiom)
- 5. Apply rule RULE6 (an axiom)

Example (cont.)



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coconuts are mammals

- 1. Write down the conclusion
- 2. Apply rule RULE3 (X are mammals if X are furry and make milk)
- 3. Apply rule RULE4 (X are furry if they are covered in fibres)
- 4. Apply rule RULE5 (an axiom)
- 5. Apply rule RULE6 (an axiom)

Summary



- ► The *why*? of deductive systems.
- Inference rules.
- ► How to make chains of inference.



Natural Deduction I, Part 2 Natural Deduction

Judgements

We want to deduce *judgements* of the form:

$$P_1, \ldots, P_n \vdash Q$$

meaning:

From assumptions P_1, \ldots, P_n , we can prove Q.

Soundness The system will be *sound*, meaning:

$$P_1, \dots, P_n \vdash Q \text{ provable } \Rightarrow P_1, \dots, P_n \models Q$$

We will make sure it is sound by checking each rule as we go. If all the premises are valid entailments, then so is the conclusion



Judgements

The main judgement form is

 $P_1, \ldots, P_n \vdash Q$ With assumptions P_1, \ldots, P_n , can prove Q



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 $P_1,\ldots,P_n\vdash Q$

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We will also use an auxiliary judgement:

 $P_1, \ldots, P_n [P] \vdash Q$

- \cdot With assumptions $P_1,\ldots,P_n,$ focusing on P, can prove Q
- $\cdot \text{ Also ``means'' } P_1, \dots, P_n, P \models Q$
- \cdot Having a focus is useful for organising proofs



Natural Deduction I, Part 2: Natural Deduction

Judgements

The main judgement form is

$$P_1, \ldots, P_n \vdash Q$$

We will also use an auxiliary judgement:

 P_1,\ldots,P_n $[P] \vdash Q$



Natural Deduction I, Part 2: Natural Deduction

Judgements

The main judgement form is

 $P_1, \ldots, P_n \vdash Q$

We will also use an auxiliary judgement:

 $P_1,\ldots,P_n[P]\vdash Q$

Assumption lists The list of assumptions P_1, \ldots, P_n will appear often. So we will shorten it to $\Gamma = P_1, \ldots, P_n$.



Basic Rules



$$\overline{\Gamma\left[P\right]\vdash P}^{\text{ Done}}$$

Basic Rules



$$\overline{\Gamma\left[P\right]\vdash P}^{\text{ Done}}$$

If we have a focused assumption P, then we can prove P (Becamber Γ stands for a list of other assumptions)

• (Remember Γ stands for a list of other assumptions)

Basic Rules



$$\frac{P \in \Gamma \quad \Gamma \left[P \right] \vdash Q}{\Gamma \vdash Q} \text{ Use }$$

Basic Rules



 $\frac{P \in \Gamma \quad \Gamma \left[P \right] \vdash Q}{\Gamma \vdash Q} \text{ Use }$

- ▶ $P \in \Gamma$ means "P is in Γ ".
- If we have a P in our current assumptions, we can focus on it.
- P ∈ Γ is a *side condition*: it is something we check when we apply the rule, not another judgement to be proved.

A first proof



$A \vdash A$

A first proof



$$\frac{A \ [A] \vdash A}{A \vdash A} \ \mathsf{Use}$$

► First Use the A assumption.

A first proof



$$\frac{\overline{A \ [A] \vdash A}}{A \vdash A} \frac{\text{Done}}{\text{Use}}$$

- ► First Use the A assumption.
- ► Then we are Done.

Soundness

Г



$$\frac{P \in \Gamma \quad \Gamma[P] \vdash Q}{\Gamma \vdash Q} \text{ Use}$$

Soundness



$$\frac{\Gamma[P] \vdash P}{\Gamma[P] \vdash P} \text{ Done } \qquad \frac{P \in \Gamma \quad \Gamma[P] \vdash Q}{\Gamma \vdash Q} \text{ Use }$$

Done

is sound because assuming P entails P, and extra assumptions make no difference.

Soundness



$$\frac{\Gamma[P] \vdash P}{\Gamma[P] \vdash P} \text{ Done } \qquad \frac{P \in \Gamma \quad \Gamma[P] \vdash Q}{\Gamma \vdash Q} \text{ Use }$$

Done

is sound because assuming P entails P, and extra assumptions make no difference.

Use

is sound because if we assuming P twice entails Q, then it is okay to assume it once.

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Rules for connectives



The rule DONE and Use do not mention the connectives.

In Natural Deduction, rules for connectives come in two kinds:

1. Introduction rules

How to construct a proof with the connective

2. Elimination rules

How to use an assumption with this connective

Rules for connectives



The rule DONE and USE do not mention the connectives.

In Natural Deduction, rules for connectives come in two kinds:

1. Introduction rules

How to construct a proof with the connective

2. Elimination rules

How to use an assumption with this connective

Very rough analogy: but can be made very precise

- 1. Introduction rules are like constructors for building objects
- 2. Elimination rules are like *methods* for taking apart objects

"And" Introduction



 $\frac{\Gamma \vdash Q_1 \quad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \land Q_2} \text{ Split}$

"And" Introduction

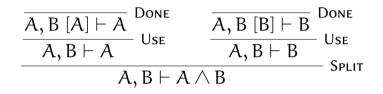


$$\frac{\Gamma \vdash Q_1 \quad \Gamma \vdash Q_2}{\Gamma \vdash Q_1 \land Q_2} \text{ Split}$$

To prove P₁ \(\Lambda\) P₂ we have to prove P₁ and P₂
 This rule is often called \(\lambda\)-INTRODUCTION



An example proof





An example proof



To prove $A \land B$, we Split into proofs of A and B. In each case, we Use the corresponding assumption. Natural Deduction I, Part 2: Natural Deduction **"And" Elimination**

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$$\frac{\Gamma \; [\mathsf{P}_1] \vdash Q}{\Gamma \; [\mathsf{P}_1 \land \mathsf{P}_2] \vdash Q} \; \mathsf{First}$$

$$\frac{\Gamma [P_2] \vdash Q}{\Gamma [P_1 \land P_2] \vdash Q} \text{ Second}$$

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"And" Elimination

$$\frac{\Gamma \ [\mathsf{P}_1] \vdash Q}{\Gamma \ [\mathsf{P}_1 \land \mathsf{P}_2] \vdash Q} \ \mathsf{First} \qquad \qquad \frac{\Gamma \ [\mathsf{P}_2] \vdash Q}{\Gamma \ [\mathsf{P}_1 \land \mathsf{P}_2] \vdash Q} \ \mathsf{Second}$$

If we are focused on an formula $P_1 \wedge P_2$, we can select either the First or Second component to focus on.



Example proof

"True" Introduction





"True" Introduction





► T is always provable.

"True" Elimination



"True" Elimination



No elimination rule!

Summary



The judgement forms for (focused) Natural Deduction:

$$P_1,\ldots,P_n\vdash Q \qquad \qquad P_1,\ldots,P_n [P]\vdash Q$$

- Rules for Use and Done
- Rules for introducing and eliminating \wedge .



"Implies" Introduction



$$\frac{\Gamma\!\!,\mathsf{P}\vdash Q}{\Gamma\vdash\mathsf{P}\rightarrow Q} \text{ Introduce}$$

"Implies" Introduction

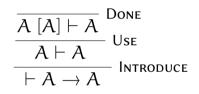


$$\frac{\Gamma, \mathsf{P} \vdash Q}{\Gamma \vdash \mathsf{P} \rightarrow Q} \text{ Introduce}$$

To prove $P \rightarrow Q,$ we prove Q under the assumption P.

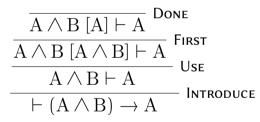
Example: $A \rightarrow A$





Example :
$$(A \land B) \rightarrow A$$





"Implies" Elimination



$$\frac{\Gamma \vdash P_1 \quad \Gamma \ [P_2] \vdash Q}{\Gamma \ [P_1 \rightarrow P_2] \vdash Q} \text{ Apply}$$

"Implies" Elimination



$$\frac{\Gamma \vdash P_1 \quad \Gamma \ [P_2] \vdash Q}{\Gamma \ [P_1 \rightarrow P_2] \vdash Q} \text{ Apply}$$

If we have $P_1 \rightarrow P_2$ and we can prove P_1 , then we have P_2 .

Example:
$$A \rightarrow (A \rightarrow B) \rightarrow B$$



$$\begin{array}{c} \overline{\textbf{A},\textbf{A} \rightarrow \textbf{B} [\textbf{A}] \vdash \textbf{A}} & \text{Done} \\ \hline \textbf{A},\textbf{A} \rightarrow \textbf{B} [\textbf{A}] \vdash \textbf{A} & \text{Use} & \hline \textbf{A},\textbf{A} \rightarrow \textbf{B} [\textbf{B}] \vdash \textbf{B} & \text{Done} \\ \hline \textbf{A},\textbf{A} \rightarrow \textbf{B} \vdash \textbf{A} & \textbf{B} [\textbf{B}] \vdash \textbf{B} & \textbf{Apply} \\ \hline \textbf{A},\textbf{A} \rightarrow \textbf{B} [\textbf{A} \rightarrow \textbf{B}] \vdash \textbf{B} & \textbf{Use} \\ \hline \textbf{A},\textbf{A} \rightarrow \textbf{B} \vdash \textbf{B} & \textbf{Use} \\ \hline \textbf{A},\textbf{A} \rightarrow \textbf{B} \vdash \textbf{B} & \textbf{Introduce} \\ \hline \textbf{A} \vdash (\textbf{A} \rightarrow \textbf{B}) \rightarrow \textbf{B} & \textbf{Introduce} \\ \hline \textbf{A} \rightarrow (\textbf{A} \rightarrow \textbf{B}) \rightarrow \textbf{B} & \textbf{Introduce} \\ \hline \textbf{A} \rightarrow (\textbf{A} \rightarrow \textbf{B}) \rightarrow \textbf{B} & \textbf{Introduce} \end{array}$$

The Rules so far



$$\begin{array}{c} \hline \Gamma \left[P \right] \vdash P \end{array} \text{ Done } \qquad \begin{array}{c} P \in \Gamma \quad \Gamma \left[P \right] \vdash Q \\ \hline \Gamma \vdash Q \end{array} \text{ Use } \\ \hline \hline \Gamma \vdash Q_1 \quad \Gamma \vdash Q_2 \\ \hline \Gamma \vdash Q_1 \land Q_2 \end{array} \text{ Split } \qquad \begin{array}{c} \Gamma \left[P_1 \right] \vdash Q \\ \hline \Gamma \left[P_1 \land P_2 \right] \vdash Q \end{array} \text{ First } \begin{array}{c} \Gamma \left[P_2 \right] \vdash Q \\ \hline \Gamma \left[P_1 \land P_2 \right] \vdash Q \end{array} \text{ Second } \\ \hline \begin{array}{c} \hline \Gamma \left[P_1 \land P_2 \right] \vdash Q \end{array} \text{ First } \begin{array}{c} \Gamma \left[P_2 \right] \vdash Q \\ \hline \Gamma \left[P_1 \land P_2 \right] \vdash Q \end{array} \text{ Second } \\ \hline \begin{array}{c} \hline \Gamma \left[P_1 \land P_2 \right] \vdash Q \end{array} \text{ Apply } \end{array}$$

Summary



The rules for Implication

$$rac{{ar \Gamma},{ extsf{P}}dash Q}{{ar \Gamma}dash extsf{P} o Q}$$
 Introduce

$$\frac{\Gamma \vdash P_1 \quad \Gamma \ [P_2] \vdash Q}{\Gamma \ [P_1 \rightarrow P_2] \vdash Q} \text{ Apply }$$



Using the Interactive Editor