CS208 (Semester 1) Week 5 : Natural Deduction II

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Natural Deduction II, Part 1

Rules for “Or”

“Or” Introduction

$$\frac{\Gamma \vdash Q_1}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Left} \qquad \frac{\Gamma \vdash Q_2}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Right}$$

"Or" Introduction

$$\frac{\Gamma \vdash Q_1}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Left} \qquad \frac{\Gamma \vdash Q_2}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Right}$$

To prove $Q_1 \vee Q_2$, *either* we:

1. prove Q_1, *or*
2. prove Q_2.

Example

$$\dfrac{\dfrac{\overline{A\ [A] \vdash A}\ \textsc{Done}}{A \vdash A}\ \textsc{Use}}{A \vdash A \vee B}\ \textsc{Left}$$

“Or” Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \lor P_2] \vdash Q}\ \textsc{Cases}$$

Γ, P means all the assumptions in Γ, and P

"Or" Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

Γ, P means all the assumptions in Γ, and P

If we are focused on $P_1 \vee P_2$, then:

1. Either P_1 holds, so we have to prove Q assuming P_1; or
2. Either P_2 holds, so we have to prove Q assuming P_2

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$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

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We (the provers) don't know which of P_1 or P_2 is true, so we need to write proofs for both eventualities.

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$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

We (the provers) don't know which of P_1 or P_2 is true, so we need to write proofs for both eventualities.

This is dual to the case for conjunction: for $P_1 \wedge P_2$ *we* had to provide both sides in the introduction rule, but got to choose in the elimination rule.

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vee B, A\ [A] \vdash A}\ \textsc{Done}}{A \vee B, A \vdash A}\ \textsc{Use}}{A \vee B, A \vdash B \vee A}\ \textsc{Right} \qquad \dfrac{\dfrac{\dfrac{}{A \vee B, B\ [B] \vdash B}\ \textsc{Done}}{A \vee B, B \vdash B}\ \textsc{Use}}{A \vee B, B \vdash B \vee A}\ \textsc{Left}}{\dfrac{A \vee B\ [A \vee B] \vdash B \vee A}{A \vee B \vdash B \vee A}\ \textsc{Use}}\ \textsc{Cases}
$$

“False” Introduction

No introduction rule!

“False” Elimination

$$\frac{}{\Gamma\ [\mathsf{F}] \vdash Q}\ \textsc{False}$$

"False" Elimination

$$\frac{}{\Gamma\ [\mathsf{F}] \vdash Q}\ \textsc{False}$$

If we have a false assumption, we can prove anything.

Example

$$\dfrac{\dfrac{\dfrac{}{\mathsf{F}\ [\mathsf{F}] \vdash A \wedge B \wedge C}\ \textsc{False}}{\mathsf{F} \vdash A \wedge B \wedge C}\ \textsc{Use}}{\vdash \mathsf{F} \to (A \wedge B \wedge C)}\ \textsc{Introduce}$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor F, A\ [A] \vdash A}\ \textsc{Done}}{A \lor F, A \vdash A}\ \textsc{Use} \qquad \dfrac{\dfrac{}{A \lor F, F\ [F] \vdash A}\ \textsc{False}}{A \lor F, F \vdash A}\ \textsc{Use}}{A \lor F\ [A \lor F] \vdash A}\ \textsc{Cases}}{\dfrac{A \lor F \vdash A}{\vdash (A \lor F) \to A}\ \textsc{Introduce}}\ \textsc{Use}
$$

Summary

- Rules for "Or":

$$\frac{\Gamma \vdash Q_1}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Left} \qquad \frac{\Gamma \vdash Q_2}{\Gamma \vdash Q_1 \vee Q_2}\ \textsc{Right}$$

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases}$$

- "False" lets us prove anything:

$$\frac{}{\Gamma\ [F] \vdash Q}\ \textsc{False}$$

Natural Deduction II, Part 2

Rules for “Not”

Negation

We could *define* negation:

$$\neg \mathsf{P} \equiv \mathsf{P} \to \mathsf{F}$$

Then we wouldn’t need any rules for it.

Rules for Negation: Introduction

$(\neg P \equiv P \to F)$

$$\frac{\Gamma, P \vdash F}{\Gamma \vdash P \to F}\ \textsc{Introduce}$$

To prove $\neg P$, we prove that P proves false.

Rules for Negation: Elimination

$(\neg P \equiv P \to F)$

$$\dfrac{\Gamma \vdash P \qquad \dfrac{}{\Gamma\ [F] \vdash Q}\ \textsc{False}}{\Gamma\ [P \to F] \vdash Q}\ \textsc{Apply}$$

If we know that $\neg P$ is true, and we can prove P, then we get a contradiction which allows us to prove anything.

Specialised Rules for Negation

Introduction:

$$\frac{\Gamma, P \vdash F}{\Gamma \vdash \neg P}\ \textsc{Not-Intro}$$

Elimination:

$$\frac{\Gamma \vdash P}{\Gamma\ [\neg P] \vdash Q}\ \textsc{Not-Elim}$$

Example: $(A \lor B) \to \neg A \to B$

$$\dfrac{}{A \lor B, \neg A, A\ [A] \vdash A}\ \textsc{Done}$$

$$\dfrac{A \lor B, \neg A, A\ [A] \vdash A}{A \lor B, \neg A, A \vdash A}\ \textsc{Use}$$

$$\dfrac{A \lor B, \neg A, A \vdash A}{A \lor B, \neg A, A\ [\neg A] \vdash B}\ \neg\text{-}\textsc{Elim}$$

$$\dfrac{A \lor B, \neg A, A\ [\neg A] \vdash B}{A \lor B, \neg A, A \vdash B}\ \textsc{Use}$$

$$\dfrac{}{A \lor B, \neg A, B\ [B] \vdash B}\ \textsc{Done}$$

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$$\dfrac{A \lor B, \neg A, A \vdash B \qquad A \lor B, \neg A, B \vdash B}{A \lor B, \neg A\ [A \lor B] \vdash B}\ \textsc{Cases}$$

$$\dfrac{A \lor B, \neg A\ [A \lor B] \vdash B}{A \lor B, \neg A \vdash B}\ \textsc{Use}$$

$$\dfrac{A \lor B, \neg A \vdash B}{A \lor B \vdash \neg A \to B}\ \textsc{Introduce}$$

$$\dfrac{A \lor B \vdash \neg A \to B}{\vdash (A \lor B) \to \neg A \to B}\ \textsc{Introduce}$$

Summary

- Negation can be defined in terms of Implication and False
- Nicer to have specific rules:

$$\frac{\Gamma, P \vdash F}{\Gamma \vdash \neg P} \qquad \frac{\Gamma \vdash P}{\Gamma\ [\neg P] \vdash Q}$$

Natural Deduction II, Part 3

Examples in the Interactive Editor

Natural Deduction II, Part 4

Soundness & Completeness & Philosophy

Soundness and Completeness

Soundness : "Everything that is provable is valid":

$$P_1, \ldots, P_n \vdash Q \quad \Rightarrow P_1, \ldots, P_n \models Q$$

I've tried, informally, to convince you of this for each rule. If each rule is sound, then the whole system is sound.

Soundness and Completeness

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Completeness : "Everything that is provable is valid":

$$P_1, \ldots, P_n \vDash Q \quad \Rightarrow P_1, \ldots, P_n \vdash Q$$

Does this property hold of the system so far?

Failure of Completeness

Recall that this entailment is valid:

$$\models A \vee \neg A$$

Can we prove this?

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3. Apply Right and try to prove $\vdash \neg A$,

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Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

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Failure of Completeness

Recall that this entailment is valid:

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Can we prove this? Is there a proof of $\vdash A \vee \neg A$?

Have three options:

1. Apply Use to use an assumption. *No assumptions!*
2. Apply Left and try to prove $\vdash A$, *but this can't be provable, by soundness!*
3. Apply Right and try to prove $\vdash \neg A$, *but this can't be provable, by soundness!*

So the system so far is **not** complete, with respect to our semantics.

Fixing completeness

We could add the following rule:

$$\frac{\Gamma, P \vdash Q \qquad \Gamma, \neg P \vdash Q}{\Gamma \vdash Q}\ \textsc{ExcludedMiddle}$$

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This lets us prove $\vdash A \vee \neg A$.

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This lets us prove $\vdash A \vee \neg A$.

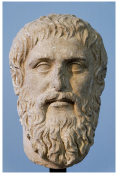
It is *sound*, but is it a good idea?

Some Philosophy of Mathematics

Where do mathematical objects live?

(objects include numbers, shapes, functions, propositions, proofs, ...)

"Platonism"



- Objects exist "out there", independently of us.
- There is a universal notion of "truth".
  - Every proposition is either true or false, even if *we* can't see why.

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“Intuitionism”

(L.E.J. Brouwer, 1900/10/20s)

- Objects exist as constructions within our heads.
- Including proofs of propositions
  - We convince ourselves of the truth of a proposition by constructing evidence for it.

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Evidence based Interpretation

(Instead of saying P □ Q is true when...)

Evidence of...	is
T	there always evidence of T
F	there is no evidence of F
$P \wedge Q$	evidence of P and evidence of Q
$P \vee Q$	evidence of P or evidence of Q
$P \to Q$	a process converting evidence of P into evidence of Q

Evidence for Negation

We define $\neg P = P \to F$.

- evidence of $\neg P$ is a process converting evidence of P to evidence of F
- but there is no evidence of F
- so there can be no evidence of P.

Excluded Middle?

In two valued (T, F) logic, *excluded middle* is valid for any P:

$$P \lor \neg P$$

The proof of validity (via truth tables) makes no commitment to which one is actually true.

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The proof of validity (via truth tables) makes no commitment to which one is actually true.

However, in terms of evidence, we have to *construct* either

1. evidence of P, or
2. evidence of $\neg P$.

For an arbitrary proposition P, this seems unlikely.

Failure of Excluded Middle

For instance, if x is a real number (has an arbitrarily long decimal expansion), then, in terms of evidence

$$(x = 0) \vee \neg(x = 0)$$

asks us to determine whether x is 0.

But there is no process to do this in finite time.

(Another example: does this Turing Machine halt?)

Intuitionistic Logic

Intuitionistic Logic is the similar to "Classical" Logic, except that it does not include the Law of Excluded Middle $P \vee \neg P$ for all propositions P.

Note: this does not mean that $\neg(P \vee \neg P)$ is provable. There may be some Ps for which $P \vee \neg P$ holds.

(For example, $(x = 0) \vee \neg(x = 0)$ when x is an integer)

Summary

- The system was have so far is *sound* but not *complete*
- We can make it complete by adding a rule for *excluded middle*:

$$P \vee \neg P$$

- But should we? What does Logic mean?