CS208 (Semester 1) Week 7 : Predicate Logic : Natural Deduction

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Predicate Logic : Natural Deduction, Part 1

Upgrading Natural Deduction

Tracking free variables

We are going to prove things like:

$$\vdash \forall x.(p(x) \land q(x)) \to p(x)$$

This will mean we will have proof states like:

$$\cdots \vdash (p(x) \land q(x)) \to p(x)$$

We need to keep track of variables as well as assumed formulas to the left of the $\vdash$ "turnstile".

Judgements

Proving:

$$\underbrace{P_1, x_1, \dots, x_i, P_j, \dots, x_m, P_n}_{\textit{assumptions and variables}} \vdash \underbrace{Q}_{\textit{conclusion}}$$

Focused:

$$\underbrace{P_1, x_1, \dots, x_i, P_j, \dots, x_m, P_n}_{\textit{assumptions and variables}} \ [\underbrace{P}_{\textit{focus}}] \vdash \underbrace{Q}_{\textit{conclusion}}$$

Note:

1. We never focus on a variable, only formulas
2. Each P_j only contains free variables that appear to the *left* of it

Well-scoped terms and formulas

If we have a list of variables and assumptions (a "context"):

$$\Gamma = P_1, x_1, \ldots, x_i, P_j, \ldots, x_m, P_n$$

Γ is the name we're giving to the list

- A formula P is *well-scoped in* Γ if all the free variables of P appear in Γ.
- A term t is *well-scoped in* Γ if all the variables of t appear in Γ.

- All formulas in Γ must be well-scoped by the variables to their left (same condition as previous slide).
- The focus and conclusion must always be well-scoped in Γ.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \to Q(y)$

Well-scoped terms and formulas

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Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \to Q(y)$
   Yes. The variable y is bound in the formula.
2. Context: x Formula: $\forall y.P(y) \to Q(x, y)$

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: x Formula: $\forall y.P(y) \to Q(y)$
   Yes. The variable y is bound in the formula.
2. Context: x Formula: $\forall y.P(y) \to Q(x, y)$
   Yes. The variable y is bound in the formula, and the free variable x is in the context.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \to Q(x, y)$

Well-scoped terms and formulas

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1. Context: *empty* Formula: $\forall y. P(y) \to Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \to Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.
2. Context: *empty* Term: $x + 1$

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y.P(y) \to Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.
2. Context: *empty* Term: $x + 1$
   No. The variable x is free in the term but is not in the context.

Well-scoped terms and formulas

Are the following well-scoped?

1. Context: *empty* Formula: $\forall y. P(y) \to Q(x, y)$
   No. The variable y is bound in the formula, but the free variable x is not in the context.
2. Context: *empty* Term: $x + 1$
   No. The variable x is free in the term but is not in the context.

Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x, y\ [P(x, y)] \vdash Q(x)$$

Well-scoped Judgements

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   Yes. The free variables of the focus and conclusion are x, y, which are in the context.

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Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$x, Q(x), y \; [P(x, y)] \vdash Q(y)$$

Well-scoped Judgements

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1. Is this judgement well-scoped:

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Yes. Each variable appears before (reading left to right) it is used.

Well-scoped Judgements

Is the following well-scoped?

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Well-scoped Judgements

Is the following well-scoped?

1. Is this judgement well-scoped:

$$\forall x.Q(x), y\ [P(x,y)] \vdash Q(y)$$

No. The x in the first $Q(x)$ is OK, but the x in $P(x,y)$ has not been declared in scope.

Summary

1. We started to upgrade Natural Deduction to Predicate Logic
2. We need to manage the *scope* of variables
3. To do so, we add them to the context
4. Variables may only be used by formulas to their right

Predicate Logic : Natural Deduction, Part 2

Substitution

From General to Specific

We will have *general* assumptions like:

$$\forall x.\mathrm{human}(x) \to \mathrm{mortal}(x)$$

And we want to *specialise* (or *instantiate*) to:

$$\mathrm{human}(\mathsf{socrates}()) \to \mathrm{mortal}(\mathsf{socrates}())$$

Substitution

The notation

$$P[x := t]$$

means “replace all *free* occurrences of x in P with t”.

- x is a *variable*
- P is a *formula*
- t is a *term*

But there is a subtlety...

Substitution Examples

$$
\begin{aligned}
& (\mathrm{mortal}(x))[x := \mathsf{socrates}()] \\
\implies & \mathrm{mortal}(\mathsf{socrates}())
\end{aligned}
$$

Substitution Examples

$$(\forall y.\mathrm{weatherIs}(\mathsf{d}, y) \to \mathrm{weatherIs}(\mathsf{dayAfter}(\mathsf{d}), y))[\mathsf{d} := \mathsf{tuesday}]$$

$$\implies \forall y.\mathrm{weatherIs}(\mathsf{tuesday}, y) \to \mathrm{weatherIs}(\mathsf{dayAfter}(\mathsf{tuesday}), y)$$

Substitution Examples

$$
\begin{aligned}
& (\exists y.\mathrm{sameElements}(x, y) \land \mathrm{sorted}(y))[x := \mathsf{cons}(z_1, \mathsf{cons}(z_2, \mathsf{nil}))] \\
\Longrightarrow\; & \exists y.\mathrm{sameElements}(\mathsf{cons}(z_1, \mathsf{cons}(z_2, \mathsf{nil})), y) \land \mathrm{sorted}(y)
\end{aligned}
$$

Substitution Examples

$$
\begin{aligned}
& (\forall y. x + y = y + x)[x := z - z] \\
\implies & \forall y. (z - z) + y = y + (z - z)
\end{aligned}
$$

Accidental Name Capture

If we substitute naively, then we produce nonsense:

1. $\exists y.\mathrm{sameElements}(x, y)$
   "there exists a y that has the same elements as x"

2. $(\exists y.\mathrm{sameElements}(x, y))[x := \mathsf{append}(y, [1, 2])]$
   "replace x by the list $\mathsf{append}(y, [1, 2])$*"*

3. $\exists y.\mathrm{sameElements}(\mathsf{append}(y, [1, 2]), y)$
   "there exists a y that has the same elements as $y + [1, 2]$?"

Capture Avoidance

Solution: Rename bound variables

$$
\begin{aligned}
& (\exists y.\mathrm{sameElements}(x, y))[x := \mathsf{append}(y, [1, 2])] \\
\implies & (\exists z.\mathrm{sameElements}(x, z))[x := \mathsf{append}(y, [1, 2])] \\
\implies & \exists z.\mathrm{sameElements}(\mathsf{append}(y, [1, 2]), z)
\end{aligned}
$$

Capture Avoiding Substitution

When working out

$$P[x := t]$$

If any of the variables in t are bound in P then rename them before doing the substitution.

Substitution Examples

1. $P(x, y)[x := y + y]$

Substitution Examples

1. $P(x, y)[x := y + y] \quad = P(y + y, y)$

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3. $(\forall x. P(x, y))[x := y + y]$

Substitution Examples

1. $P(x, y)[x := y + y] \quad = P(y + y, y)$
2. $P(x, y)[y := y + y] \quad = P(x, y + y)$
3. $(\forall x.P(x, y))[x := y + y] \quad = \forall x.P(x, y)$

Substitution Examples

1. $(\forall x.P(x, y))[y := x + x]$

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   Renaming!

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   Renaming!

2. $(\forall x.P(x, y) \to (\exists z.Q(y, z)))[y := z + z]$

Substitution Examples

1. $(\forall x.P(x,y))[y := x + x] \quad = \forall z.P(z, x + x)$

   Renaming!

2. $(\forall x.P(x,y) \to (\exists z.Q(y,z)))[y := z + z]$

   $= \forall x.P(x, z + z) \to (\exists w.Q(z + z, w))$

   Renaming!

Substitution Examples

1. $(\forall x.P(x,y))[y := x + x] \quad = \forall z.P(z, x + x)$

   Renaming!

2. $(\forall x.P(x,y) \to (\exists z.Q(y,z)))[y := z + z]$
   $= \forall x.P(x, z + z) \to (\exists w.Q(z + z, w))$

   Renaming!

3. $(\forall x.P(x,z) \to (\exists z.Q(y,z)))[z := x + x]$

Substitution Examples

1. $(\forall x.P(x,y))[y := x + x] \quad = \forall z.P(z, x + x)$

   Renaming!

2. $(\forall x.P(x,y) \to (\exists z.Q(y,z)))[y := z + z]$
   $= \forall x.P(x, z + z) \to (\exists w.Q(z + z, w))$

   Renaming!

3. $(\forall x.P(x,z) \to (\exists z.Q(y,z)))[z := x + x]$
   $= \forall w.P(w, x + x) \to (\exists z.Q(y,z))$

   Renaming! and no substitution of the final z

Summary

- Substitution

$$P[x := t]$$

  is how we go from the general x to the specific t.
- We need to be careful to rename bound variables to avoid accidental name capture.

Predicate Logic : Natural Deduction, Part 3

Rules for “Forall”

What does $\forall x. P$ mean?

(assuming a domain of discourse)

Answer 1 : it means for all individuals "a", $P[x := a]$ is true.

(we think of "for all" as an infinite conjunction)

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Answer 1 : it means for all individuals "a", $P[x := a]$ is true.

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Answer 2 : thinking about proofs:

To *prove* a $\forall x.P$:

- We must prove $P[x := x_0]$ for a *general* x_0.
- The x_0 stands in for any "a" that might be chosen.

To *use* a proof of $\forall x.P$:

- We can *choose* any t we like for x, and get $P[x := t]$

Introduction rule

$$\frac{\Gamma, x_0 \vdash Q[x := x_0]}{\Gamma \vdash \forall x.Q}\ \textsc{Introduce}\forall$$

Introduction rule

$$\frac{\Gamma, x_0 \vdash Q[x := x_0]}{\Gamma \vdash \forall x.Q} \ \textsc{Introduce}\forall$$

"To prove $\forall x.Q$, we prove $Q[x := x_0]$, assuming an arbitrary x_0."

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x, P(x) \land Q(x)\ [P(x)] \vdash P(x)}\ \textsc{Done}}{x, P(x) \land Q(x)\ [P(x) \land Q(x)] \vdash P(x)}\ \textsc{First}}{x, P(x) \land Q(x) \vdash P(x)}\ \textsc{Use}}{x \vdash (P(x) \land Q(x)) \to P(x)}\ \textsc{Introduce}}{\vdash \forall x.(P(x) \land Q(x)) \to P(x)}\ \textsc{Introduce}$$

Elimination

$$\frac{\Gamma\ [P[x := t]] \vdash Q}{\Gamma\ [\forall x.P] \vdash Q}\ \textsc{Instantiate}$$

(side condition: t is well-scoped in Γ)

Elimination

$$\frac{\Gamma\ [P[x := t]] \vdash Q}{\Gamma\ [\forall x.P] \vdash Q}\ \textsc{Instantiate}$$

(side condition: t is well-scoped in Γ)

"If we have P for all x, then we can pick any well-scoped t we like to stand in for it."

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\ [h(\mathsf{s}())] \vdash h(\mathsf{s}())}\,\textsc{Done}}{\Gamma \vdash h(\mathsf{s}())}\,\textsc{Use} \qquad \dfrac{}{\Gamma\ [m(\mathsf{s}())] \vdash m(\mathsf{s}())}\,\textsc{Done}}{\Gamma\ [h(\mathsf{s}()) \to m(\mathsf{s}())] \vdash m(\mathsf{s}())}\,\textsc{Apply}}{\Gamma\ [\forall x. h(x) \to m(x)] \vdash m(\mathsf{s}())}\,\textsc{Instantiate}}{\Gamma \vdash m(\mathsf{s}())}\,\textsc{Use}}{\forall x. h(x) \to m(x) \vdash h(\mathsf{s}()) \to m(\mathsf{s}())}\,\textsc{Introduce}}{\vdash (\forall x. h(x) \to m(x)) \to h(\mathsf{s}()) \to m(\mathsf{s}())}\,\textsc{Introduce}
$$

where $\Gamma = \forall x. h(x) \to m(x), h(\mathsf{s}())$

Summary

- To prove $\forall x.P(x)$, we must prove $P(x_0)$ for a general x_o.
- To use $\forall x.P(X)$, we get to choose the t we use for x.

Predicate Logic : Natural Deduction, Part 4

Rules for “Exists”

What does $\exists x.P$ mean?

(assuming a domain of discourse)

Answer 1 : there is at least one "a" such that $P[x := a]$ is true.

(we think of "exists" as an infinite disjunction)

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(assuming a domain of discourse)

Answer 1 : there is at least one "a" such that $P[x := a]$ is true.

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Answer 2 : thinking about proofs:

To *prove* a $\exists x.P$:

- We must provide a *witness* term t such that $P[x := t]$.

To *use* a proof of $\exists x.P$:

- We have to work with an arbitrary x_0 and all we know is $P[x := x_0]$.

Introduction

$$\frac{\Gamma \vdash P[x := t]}{\Gamma \vdash \exists x.P}\ \textsc{Exists}$$

(side condition: t is well-scoped in Γ)

"To prove $\exists x.P$, we have to provide a witness t for x, and show that $P[x := t]$"

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{human}(\mathsf{socrates}())\ [\mathrm{human}(\mathsf{socrates}())] \vdash \mathrm{human}(\mathsf{socrates}())}\ \textsc{Done}}{\mathrm{human}(\mathsf{socrates}()) \vdash \mathrm{human}(\mathsf{socrates}())}\ \textsc{Use}}{\mathrm{human}(\mathsf{socrates}()) \vdash \exists x.\mathrm{human}(x)}\ \textsc{Exists}}{\vdash \mathrm{human}(\mathsf{socrates}()) \to (\exists x.\mathrm{human}(x))}\ \textsc{Introduce}$$

Elimination

$$\frac{\Gamma, x_0, P[x := x_0] \vdash Q}{\Gamma\ [\exists x.P] \vdash Q}\ \textsc{Unpack}$$

"To use $\exists x.P$, we get some arbitrary x_0 that we know $P[x := x_0]$ about."

$$\dfrac{}{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali)\ [h(ali)] \vdash h(ali)}\ \textsc{Done}$$

$$\dfrac{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali)\ [h(ali)] \vdash h(ali)}{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali)\ [h(ali) \wedge m(ali)] \vdash h(ali)}\ \textsc{First}$$

$$\dfrac{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali)\ [h(ali) \wedge m(ali)] \vdash h(ali)}{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali) \vdash h(ali)}\ \textsc{Use}$$

$$\dfrac{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali) \vdash h(ali)}{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali) \vdash \exists x.h(x)}\ \textsc{Exists}$$

$$\dfrac{\exists x.h(x) \wedge m(x), ali, h(ali) \wedge m(ali) \vdash \exists x.h(x)}{\exists x.h(x) \wedge m(x)\ [\exists x.h(x) \wedge m(x)] \vdash \exists x.h(x)}\ \textsc{Unpack}$$

$$\dfrac{\exists x.h(x) \wedge m(x)\ [\exists x.h(x) \wedge m(x)] \vdash \exists x.h(x)}{\exists x.h(x) \wedge m(x) \vdash \exists x.h(x)}\ \textsc{Use}$$

$$\dfrac{\exists x.h(x) \wedge m(x) \vdash \exists x.h(x)}{\vdash (\exists x.h(x) \wedge m(x)) \to (\exists x.h(x))}\ \textsc{Introduce}$$

Comparing $\wedge$ and $\forall$

Introduction

$$\frac{\Gamma \vdash P_1 \qquad \Gamma \vdash P_2}{\Gamma \vdash P_1 \wedge P_2}\ \textsc{Split} \qquad\qquad \frac{\Gamma, x_0 \vdash P[x := x_0]}{\Gamma \vdash \forall x.P}\ \forall\text{-I}$$

For $\wedge$, we have to prove P_i, no matter what i is. For $\forall$, we have to prove $P[x := x_0]$, no matter what x_0 is.

Comparing $\wedge$ and $\forall$

Elimination

$$\frac{\Gamma\ [P_1] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{First} \qquad \frac{\Gamma\ [P_2] \vdash Q}{\Gamma\ [P_1 \wedge P_2] \vdash Q}\ \textsc{Second}$$

$$\frac{\Gamma\ [P[x := t]] \vdash Q}{\Gamma\ [\forall x.P] \vdash Q}\ \textsc{Instantiate}$$

For $\wedge$, we choose 1 or 2. For $\forall$, we choose t.

Comparing $\lor$ and $\exists$

Introduction

$$\frac{\Gamma \vdash P_1}{\Gamma \vdash P_1 \lor P_2}\ \textsc{Left} \qquad \frac{\Gamma \vdash P_2}{\Gamma \vdash P_1 \lor P_2}\ \textsc{Right} \qquad \frac{\Gamma \vdash P[x := t]}{\Gamma \vdash \exists x.P}\ \textsc{Exists}$$

For $\lor$, we choose which of 1 or 2 we want. For $\exists$, we choose the witnessing term t.

Comparing $\vee$ and $\exists$

Elimination

$$\frac{\Gamma, P_1 \vdash Q \qquad \Gamma, P_2 \vdash Q}{\Gamma\ [P_1 \vee P_2] \vdash Q}\ \textsc{Cases} \qquad\qquad \frac{\Gamma, x_0, P[x := x_0] \vdash Q}{\Gamma\ [\exists x.P] \vdash Q}\ \textsc{Unpack}$$

For $\vee$, we must deal with 1 or 2. For $\exists$, we must cope with any x_0.

Summary

- To prove $\exists x.P(x)$ we must give a witness t and prove $P(t)$.
- To use $\exists x.P(X)$ we get to assume there is some y and $P(y)$.

Predicate Logic : Natural Deduction, Part 5

Using the interactive prover