CS208 (Semester 1) Week 8 : Predicate Logic: Semantics

Dr. Robert Atkey

Computer & Information Sciences

Predicate Logic: Semantics, Part 1

Models

So far: Syntax and Proof

1. The syntax of predicate logic
   What sequences of symbols are well formed?
2. Proofs for predicate logic
   When are formulas consequences of other formulas?

Missing so far: semantics

1. For Propositional Logic, we defined the *semantics* ("meaning") of a formula P:
   - For every *valuation* v,
     the formula P is assigned a meaning $[\![P]\!]\, v$ which is either T or F.
2. This definition enabled us to give a definition of *entailment*:

$$P_1, \ldots, P_n \models Q$$

which defines consequence without using proofs.

Semantics for Predicate Logic

The plan:

1. Fix a vocabulary
2. Define *models* $\mathcal{M}$
3. Interpret a formula P in a model $\mathcal{M}$

Fixing a Vocabulary

The function symbols we will use, and their *arities* (number of arguments):

Function name(s)	Arity
$\mathrm{socrates}$	0
$\mathrm{dayAfter}$	1
$+, -$	2

We write "func/n" for function symbol func with arity n

Fixing a Vocabulary

The predicates / relation symbols we will use, and their arities:

Predicate name(s)	Arity
$\mathrm{human}, \mathrm{mortal}$	1
$<, \leq, =$	2
$\mathrm{between}$	3

We write "pred/n" for predicate symbol pred with arity n

A simplification

To keep things simple, I’m going to assume that we don’t have any function symbols in our vocabulary.

Example: Orderings

- $\leq/2$ “less than”

Example: Places

- city/1 “is a city”
- within/2 “is within”

Example: Forestry and Birdwatching

- tree/1 “is a tree”
- green/1 “is green”
- bird/1 “is a bird”
- satIn/2 “has sat in”

Models

With a fixed vocabulary, a *model* $\mathcal{M}$ is:

1. A *universe* U, which is a set of individuals:

$$\mathsf{U} = \{1, 2, \mathsf{socrates}, \mathsf{hypatia}, \mathsf{noether}, \mathsf{alexandria}, \mathsf{glasgow}, \ldots\}$$

2. For each predicate pred/n, an n-ary relation on the set U.

Relations

Several ways of understanding what a relation is:

1. For every $\mathfrak{n}$ elements from $\mathfrak{U}$, the interpretation of $\mathrm{pred}/\mathfrak{n}$ assigns the value T or F.

2. The interpretation of $\mathrm{pred}/\mathfrak{n}$ is a (possibly infinite) table of elements of $\mathfrak{U}$ with $\mathfrak{n}$ columns.

3. The interpretation of $\mathrm{pred}/\mathfrak{n}$ is a subset of the $\mathfrak{n}$-fold *cartesian product* $\underbrace{\mathfrak{U} \times \cdots \times \mathfrak{U}}_{\mathfrak{n}\ \text{times}}$.

Example: Places, interpretation 1

$$
\begin{aligned}
\mathcal{U} &= \{\mathrm{aberdeen}, \mathrm{edinburgh}, \mathrm{glasgow}, \mathrm{scotland}, \mathrm{birmingham}, \mathrm{england}\} \\
\mathrm{city} &= \{(\mathrm{aberdeen}), (\mathrm{edinburgh}), (\mathrm{glasgow}), (\mathrm{birmingham})\} \\
\mathrm{within} &= \{(\mathrm{aberdeen}, \mathrm{scotland}), (\mathrm{edinburgh}, \mathrm{scotland}), \\
&\quad (\mathrm{glasgow}, \mathrm{scotland}), (\mathrm{birmingham}, \mathrm{england})\}
\end{aligned}
$$

Example: Places, interpretation 1

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$$U = \{\text{aberdeen}, \text{edinburgh}, \text{glasgow}, \text{scotland}, \text{birmingham}, \text{england}\}$$

As tables:

city
aberdeen
edinburgh
glasgow
birmingham

within	
aberdeen	scotland
edinburgh	scotland
glasgow	scotland
birmingham	england

Example: Places, interpretation 2

$$\begin{aligned}
\mathcal{U} &= \{\text{planet-b}\} \\
\text{city} &= \{\} \\
\text{within} &= \{(\text{planet-b}, \text{planet-b})\}
\end{aligned}$$

Example: Interpreting ordering with natural numbers

1. $\mathcal{U} = \{0, 1, 2, \ldots\} = \mathbb{N}$ (all positive whole numbers)
2. The interpretation of $\leq/2$ is all pairs (x, y) such that $x \leq y$

Example: Interpreting ordering with rational numbers

1. $U = \{0, -1, 1, -\frac{1}{2}, \frac{1}{2}, -2, 2, \dots\} = \mathbb{Q}$
2. The interpretation of $\leq/2$ is all pairs (x, y) such that $x \leq y$

Example: Interpreting ordering with a small set

1. $U = \{a, b, c\}$
2. The interpretation of $\leq/2$ is the set:

$$\{(a, b), (a, c)\}$$

Note! not necessarily what we might think of as $\leq$! Need to add axioms.

Important Points

Every model $\mathcal{M}$ has

1. a universe; and
2. a relation for each predicate symbol pred/n,

but the domain can be empty, or the predicate symbols' interpretations may be empty!

The model needn't match our intuition about the symbols!

- Will assume formulas that will filter the possible models.

Relationship to Valuations

If all our predicate symbols have arity 0 (take no arguments), then a model consists of:

1. A universe $\mathcal{U}$; and
2. An assignment of T or F to each predicate symbol $\mathrm{pred}/0$.

Apart from the universe, this is the same as a *valuation* in Propositional Logic (Week 01).

Summary

We interpret Predicate Logic formulas in a *model* $\mathcal{M}$.

- A universe $\mathbb{U}$ – the set of all “things”.
- A relation between elements of $\mathbb{U}$ for every predicate.

Useful intuition: models are (possibly infinite) databases.

Predicate Logic: Semantics, Part 2

Interpreting Formulas

Meaning of free variables

Assume a vocabulary $\mathcal{V}$ and model $\mathcal{M}$ are fixed.

Consider the formula:

$$\mathrm{city}(x) \land \mathrm{within}(x, y)$$

we can't give it a truth value until we know what x and y mean.

Cities Model

$U = \{\text{aberdeen}, \text{edinburgh}, \text{glasgow}, \text{scotland}, \text{birmingham}, \text{england}\}$

As tables:

city
aberdeen
edinburgh
glasgow
birmingham

within	
aberdeen	scotland
edinburgh	scotland
glasgow	scotland
birmingham	england

Meaning of free variables

With the cities model, if we set:

$$x = \mathsf{glasgow} \qquad\qquad y = \mathsf{scotland}$$

then $\mathrm{city}(x) \land \mathrm{within}(x, y)$ should be assigned the truth value T.

Meaning of free variables

With the cities model, if we set:

$$x = \mathsf{glasgow} \qquad\qquad y = \mathsf{scotland}$$

then $\mathrm{city}(x) \wedge \mathrm{within}(x, y)$ should be assigned the truth value T.

If we set:

$$x = \mathsf{scotland} \qquad\qquad y = \mathsf{edinburgh}$$

then $\mathrm{city}(x) \wedge \mathrm{within}(x, y)$ should be assigned the truth value F.

Interpreting Formulas

If we fix:

1. a vocabulary $\mathcal{V}$;
2. a model $\mathcal{M}$ of that vocabulary;
3. an assignment v of elements of U to free variables of P.

then we can give a truth value $[\![P]\!](\mathcal{M}, v)$ to P.

Interpreting Formulas

Relations:

$$
\begin{aligned}
[\![R(x_1, \ldots, x_n)]\!](\mathcal{M}, v) &= \mathsf{T} \quad \text{if} \quad (v(x_1), \ldots, v(x_n)) \in R \\
&= \mathsf{F} \quad \text{otherwise} \\
[\![x = y]\!](\mathcal{M}, v) &= \mathsf{T} \quad \text{if} \quad v(x) = v(y) \\
&= \mathsf{F} \quad \text{otherwise}
\end{aligned}
$$

where R is one of the relations in $\mathcal{M}$.

Interpreting Formulas (Example)

With the cities model $\mathcal{M}$:

$$[\![\mathrm{within}(x, y)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}, y \mapsto \mathsf{scotland}]) = \mathsf{T}$$

$$[\![\mathrm{within}(x, y)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}, y \mapsto \mathsf{england}]) = \mathsf{F}$$

Interpreting Formulas

Quantifiers:

$$
\begin{aligned}
[\![\forall x.P]\!](\mathcal{M}, v) &= \mathsf{T} \quad \text{if for all } a \in U,\ [\![P]\!](\mathcal{M}, v[x \mapsto a]) = \mathsf{T} \\
&= \mathsf{F} \quad \text{otherwise} \\
[\![\exists x.P]\!](\mathcal{M}, v) &= \mathsf{T} \quad \text{if exists } a \in U,\ \text{with } [\![P]\!](\mathcal{M}, v[x \mapsto a]) = \mathsf{T} \\
&= \mathsf{F} \quad \text{otherwise}
\end{aligned}
$$

Notation $v[x \mapsto a]$ means the assignment that maps x to a and any other variable to whatever v mapped it to.

Interpreting Formulas (Example)

$$[\![\forall x.\mathrm{city}(x)]\!](\mathcal{M}, []) = \mathsf{F}$$

because all of the following would need to be T:

$$\begin{aligned}
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{aberdeen}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{glasgow}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{birmingham}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{scotland}]) &= \mathsf{F} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{england}]) &= \mathsf{F}
\end{aligned}$$

Interpreting Formulas (Example)

$$[\![\exists x.\mathrm{city}(x)]\!](\mathcal{M}, []) = \mathsf{T}$$

because only one of the following needs to be T:

$$\begin{aligned}
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{aberdeen}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{glasgow}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{birmingham}]) &= \mathsf{T} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{scotland}]) &= \mathsf{F} \\
[\![\mathrm{city}(x)]\!](\mathcal{M}, [x \mapsto \mathsf{england}]) &= \mathsf{F}
\end{aligned}$$

Interpreting Formulas

Propositional Connectives:

$$
\begin{aligned}
[\![P \wedge Q]\!](\mathcal{M}, v) &= [\![P]\!](\mathcal{M}, v) \wedge [\![Q]\!](\mathcal{M}, v) \\
[\![P \vee Q]\!](\mathcal{M}, v) &= [\![P]\!](\mathcal{M}, v) \vee [\![Q]\!](\mathcal{M}, v) \\
[\![P \to Q]\!](\mathcal{M}, v) &= [\![P]\!](\mathcal{M}, v) \to [\![Q]\!](\mathcal{M}, v) \\
[\![\neg P]\!](\mathcal{M}, v) &= \neg [\![P]\!](\mathcal{M}, v)
\end{aligned}
$$

Interpreting Formulas (Example)

$$[\![\mathrm{city}(x) \wedge \mathrm{within}(x, y)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}, y \mapsto \mathsf{scotland}]) = \mathsf{T}$$

and

$$[\![\mathrm{city}(x) \wedge \mathrm{within}(x, y)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}, y \mapsto \mathsf{birmingham}]) = \mathsf{F}$$

and

$$[\![\mathrm{city}(x) \vee \mathrm{within}(x, y)]\!](\mathcal{M}, [x \mapsto \mathsf{edinburgh}, y \mapsto \mathsf{birmingham}]) = \mathsf{T}$$

Some notation

We write

$$\mathcal{M} \models \mathsf{P}$$

when

$$[\![\mathsf{P}]\!](\mathcal{M}, \square) = \mathsf{T}$$

meaning that $\mathcal{M}$ is a model of P.

Examples

If $\mathcal{M}$ is the cities model, then

$$\mathcal{M} \models \exists x.\mathrm{city}(x)$$

and

$$\mathcal{M} \not\models \forall x.\mathrm{city}(x)$$

and

$$\mathcal{M} \models \forall x.\mathrm{city}(x) \to (\exists y.\mathrm{within}(x, y))$$

Entailment

Relative to a model $\mathcal{M}$:

$$\mathcal{M}; P_1, \ldots, P_n \models Q$$

exactly when:

$$\text{if all } [\![P_i]\!](\mathcal{M}, \Box) = \mathsf{T}, \text{ then } [\![Q]\!](\mathcal{M}, \Box) = \mathsf{T}.$$

If all the assumptions are true, then the conclusion must be true

Entailment

$$P_1, \ldots, P_n \models Q$$

exactly when *for all* $\mathcal{M}$, we have $\mathcal{M}; P_1, \ldots, P_n \models Q$.

Checking this is infeasible (at least naively): there are infinitely many models, and the models themselves may be infinite.

Which is one reason to use proof.

Axiomatisations

Some collections of formulas are called *axiomatisations.*

For example,

$$
\begin{aligned}
&\forall x. x \leq x \\
&\forall x. \forall y. (x \leq y \land y \leq x) \to x = y \\
&\forall x. \forall y. \forall z. (x \leq y \land y \leq z) \to x \leq z
\end{aligned}
$$

axiomatises what it means to be a *partial order.*

Axiomatisations

If we write Γ_{partial} for those formulas $\bigwedge$ed together, then collection of models $\mathcal{M}$ such that:

$$\mathcal{M} \models \Gamma_{\text{partial}}$$

is the collection of partial orders.

And the collection of formulas Q such that

$$\Gamma_{\text{partial}} \models Q$$

is the collection of facts that are true about all partial orders.

Summary

We have defined what it means for a Predicate Logic P formula to be true in some model $\mathcal{M}$.

- Just as with Propositional Logic, this is done by breaking down the formula into its constituent parts
- Care must be taken to ensure that all free variables have an interpretation.
- When a formula is true in some model, we write:

$$\mathcal{M} \models P$$

- Gives us a basis to talk about axiomatisations.

Predicate Logic: Semantics, Part 3

Model Checking Formulas