CS208 (Semester 1) Week 10 : Metatheory of Predicate Logic

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Metatheory of Predicate Logic, Part 1

Metatheory

1. Is our proof system sound and complete?
2. Are our axiomatisations complete enough?
3. Can we automate mathematics?

Soundness

The proof system we have seen so far is *sound*:

$$\Gamma \vdash Q \quad \Rightarrow \quad \Gamma \models Q$$

"Every provable judgement is valid."
Can be checked by checking that every rule preserves validity.

Completeness

If we add a rule for excluded middle ($P \vee \neg P$ for any formula P), then it is *complete*:

$$\Gamma \vDash Q \quad \Rightarrow \quad \Gamma \vdash Q$$

"Every valid judgement is provable."
This is not a simple fact. "Gödel's Completeness Theorem"

Automating Mathematics?

If our proof system is sound and complete, then we should be able to automatically prove things by searching for proofs?

This is one of the oldest branches of AI.

Automating Mathematics?

We have seen so far that there are many axiomatisations for describing certain bits of mathematics:

1. Monoid axioms: addition with a zero.
2. Peano's axioms: arithmetic
3. Zermelo-Frankel axioms: set theory

Are these axiomatisations suitable for finding proofs?

Syntactic Completeness

An axiomatisation Ax is *syntactically complete* if for all formulas P, we can prove one of:

$$\mathrm{Ax} \vdash \mathrm{P}$$

or

$$\mathrm{Ax} \vdash \neg\mathrm{P}$$

if we can prove both, then the theory is *inconsistent*.

Effectively Generatable

An axiomatisation Ax is *effectively generatable* if we can write a computer program that generates all the valid axioms.

There may be infinitely many axioms, but each one will eventually be generated.

Automation

If an axiomatisation Ax is syntactically complete and effectively generatable, then we can (in principle) write a program to search for a proof of some P:

1. Search for a proof $\mathrm{Ax} \vdash P$
   try proofs of size 1, then proofs of size 2, then proofs of size 3…
2. Interleaved with this: search for a proof of $\mathrm{Ax} \vdash \neg P$

Since one of them is provable, we will eventually terminate.

Is every interesting axiomatisation syntactically complete?

Peano's axioms (PA)

1. $\forall x.\ \neg(0 = S(x))$
2. $\forall x.\forall y.\ S(x) = S(y) \to x = y$
3. $\forall x.\ \mathrm{add}(0, x) = x$
4. $\forall x.\forall y.\ \mathrm{add}(S(x), y) = S(\mathrm{add}(x, y))$
5. $\forall x.\ \mathrm{mul}(0, x) = 0$
6. $\forall x.\forall y.\ \mathrm{mul}(S(x), y) = \mathrm{add}(y, \mathrm{mul}(x, y))$

$+$ induction

is effectively generatable.

Robinson's axioms ($\mathcal{Q}$)

1. $\forall x.\, \neg(0 = S(x))$
2. $\forall x.\forall y.\, S(x) = S(y) \rightarrow x = y$
3. $\forall x.\, \mathrm{add}(0, x) = x$
4. $\forall x.\forall y.\, \mathrm{add}(S(x), y) = S(\mathrm{add}(x, y))$
5. $\forall x.\, \mathrm{mul}(0, x) = 0$
6. $\forall x.\forall y.\, \mathrm{mul}(S(x), y) = \mathrm{add}(y, \mathrm{mul}(x, y))$
7. $\forall x.(x = 0) \vee (\exists y. x = S(y))$ **(instead of induction)**

Gödel's 1st Incompleteness Theorem

For any *effectively generatable* consistent set of axioms Ax that imply those of Robinson arithmetic, there exists a formula P such that it is **not** possible to prove either of

$$\mathrm{Ax} \vdash \mathrm{P}$$

or

$$\mathrm{Ax} \vdash \neg\mathrm{P}$$

Consequences

PA is not syntactically complete, so our attempt to use it to automate mathematics fails.

In fact, provability in PA is undecidable, so all attempts are doomed.

Consequences

1. PA is incomplete, so there is a formula P such that neither of:

$$\mathrm{PA} \vdash \mathrm{P} \qquad \text{and} \qquad \mathrm{PA} \vdash \neg \mathrm{P}$$

   are provable.
2. Inspection of Gödel's proof shows that the formula P it generates is actually true in "the" natural numbers.
3. So we could use the axioms $\mathrm{PA} + \mathrm{P}$, but then goto 1.

Consequences

So:

1. PA does not cover everything that is “true” about arithmetic
2. Every attempt to fix it is doomed

Consequences

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Some people have said that Gödel's Incompleteness theorem shows that there are fundamental limitations to what computers can reason about.

The reasoning (roughly) goes:

1. Computers can only use effectively generatable axioms
2. This means that there are truths they cannot prove
3. Humans can perceive "real" truth to see these truths
4. Therefore, Humans are better than computers, and AI is impossible.

Consequences

Two problems with this:

1. Humans only know that the formula generated by Gödel's Incompleteness Theorem is true by some *larger* axiomatisation we are (maybe implicitly) using. Computers can use this axiomatisation.
2. The theorem depends on the theory being consistent. How do we *know* this? Definitely not obvious for Zermelo-Frankel set theory.

(In)Completeness?

Gödel proved:

1. Completeness "Everything that is true is provable"
2. Incompleteness "There exist true things that are not provable"

Surely a contradiction?

(In)Completeness?

There is no contradiction.

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Completeness Theorem says that if something is true in *every* model of the axioms, it is provable.

(In)Completeness?

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Completeness Theorem says that if something is true in *every* model of the axioms, it is provable.

Incompleteness Theorem only gives something that is true for *"the"* natural numbers. It might be false in other models.

Automating Mathematics?

If we can't completely automate arithmetic, then what can we do?

1. Do proof search with a timeout
2. Restrict to weaker systems to gain decidability, e.g.:
   - Pure equality
   - Linear Arithmetic: only addition, no multiplication

Automated proof for fragments of logic is a large and ongoing topic of research, with applications in software engineering, computer security, optimisation, ...

Summary

1. Our proof system is sound
2. If we add excluded middle, it is complete
3. Gödel's Incompleteness theorem:
    - If some axioms can prove basic facts about arithmetic, then there are statements that it can neither prove nor disprove.
4. Not every theory is decidable, but some useful ones are.