The Sub-Additives:

A Proof Theory for Probabilistic Choice extending Linear Logic

Fundamental Structure of Computation and Deduction (FSCD'19)

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Lets talk about Coffee

Processes as Formulae

In process calculi:

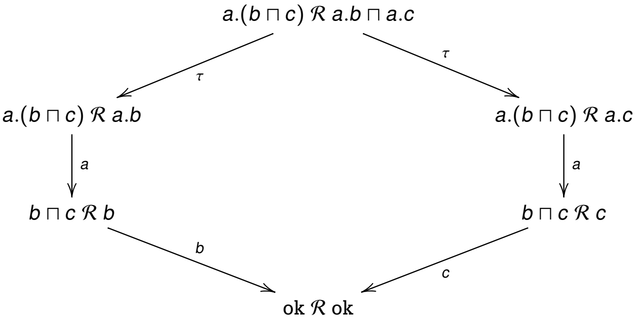
$a.(b \sqcap c)$ simulates $a.b \sqcap a.c$

In MAV:

$\vdash a \triangleleft (b \mathbin{\&} c) \multimap (a \triangleleft b) \mathbin{\&} (a \triangleleft c)$

Processes as Formulae

In process calculi:



In MAV:

$$\dfrac{\dfrac{\vdash \overline{a}\,,\,a \qquad \dfrac{\vdash \overline{b}\,,\,b}{\vdash \overline{b} \oplus \overline{c}\,,\,b}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\, a \triangleleft b} \qquad \dfrac{\vdash \overline{a}\,,\,a \qquad \dfrac{\vdash \overline{c}\,,\,c}{\vdash \overline{b} \oplus \overline{c}\,,\,c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\, a \triangleleft c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\, (a \triangleleft b) \,\&\, (a \triangleleft c)}$$

From Proofs in MAV to Simulations

$$\dfrac{\dfrac{\vdash \overline{a}\,,\, a \qquad \dfrac{\vdash \overline{b}\,,\, b}{\vdash \overline{b} \oplus \overline{c}\,,\, b}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\,,\, a \triangleleft b} \qquad \dfrac{\vdash \overline{a}\,,\, a \qquad \dfrac{\vdash \overline{c}\,,\, c}{\vdash \overline{b} \oplus \overline{c}\,,\, c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\,,\, a \triangleleft c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\,,\, (a \triangleleft b) \mathbin{\&} (a \triangleleft c)}$$

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By cut elimination.

$$\dfrac{\vdash \overline{a}\,,\, a \qquad \dfrac{\vdash \overline{b}\,,\, b}{\vdash \overline{b} \oplus \overline{c}\,,\, b}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\, a \triangleleft b} \qquad\qquad \dfrac{\vdash \overline{a}\,,\, a \qquad \dfrac{\vdash \overline{c}\,,\, c}{\vdash \overline{b} \oplus \overline{c}\,,\, c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\, a \triangleleft c}$$

From Proofs in MAV to Simulations

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By cut elimination.

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Removing left-most interacting atoms.

$$\dfrac{\vdash \overline{b}\,,\,b}{\vdash \overline{b}\oplus\overline{c}\,,\,b} \qquad\qquad \dfrac{\vdash \overline{c}\,,\,c}{\vdash \overline{b}\oplus\overline{c}\,,\,c}$$

From Proofs in MAV to Simulations

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Removing left-most interacting atoms.

$$\dfrac{\vdash \overline{b}\,,\, b}{\vdash \overline{b} \oplus \overline{c}\,,\, b} \qquad\qquad \dfrac{\vdash \overline{c}\,,\, c}{\vdash \overline{b} \oplus \overline{c}\,,\, c}$$

Removing left-most interacting atoms.

$$\vdash \circ\,,\, \circ$$

Processes as Formulae

$$\dfrac{\dfrac{\vdash \overline{a}\,,\,a \qquad \dfrac{\vdash \overline{b}\,,\,b}{\vdash \overline{b} \oplus \overline{c}\,,\,b}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\,a \triangleleft b} \qquad \dfrac{\vdash \overline{a}\,,\,a \qquad \dfrac{\vdash \overline{c}\,,\,c}{\vdash \overline{b} \oplus \overline{c}\,,\,c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\,a \triangleleft c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\,(a \triangleleft b) \mathbin{\&} (a \triangleleft c)}$$

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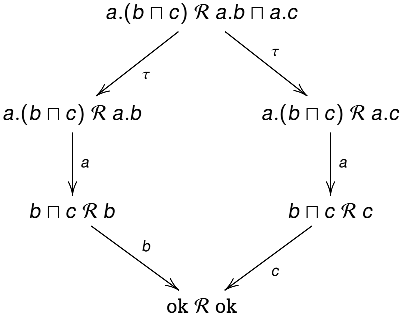
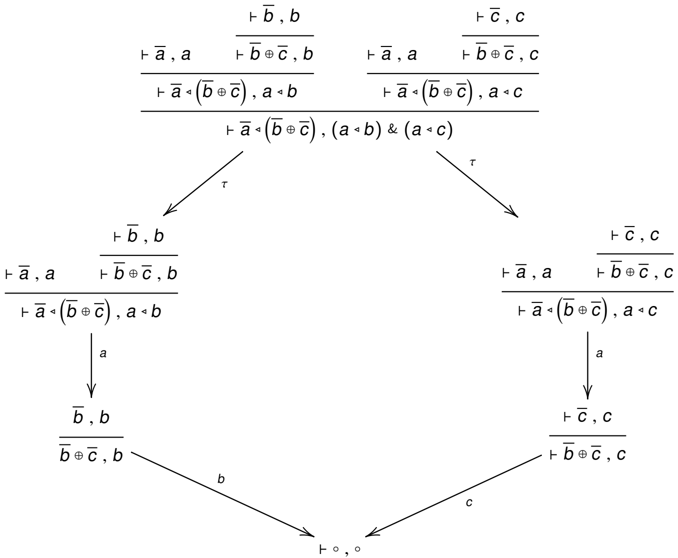
$$\dfrac{\overline{b}\,,\,b}{\overline{b} \oplus \overline{c}\,,\,b} \qquad \dfrac{\vdash \overline{c}\,,\,c}{\vdash \overline{b} \oplus \overline{c}\,,\,c}$$

$$\vdash \circ\,,\,\circ$$



Processes as Formulae



Theorem (soundness)

If $\vdash [\![P]\!] \multimap [\![Q]\!]$ *then* P *simulates* Q.

$a.b \sqcap a.c$ does not simulate $a.(b \sqcap c)$

Consider probabilistic test:

$$\overline{a}.\big(\overline{b}.\omega +_{1/2} \overline{c}.\omega\big)$$

$a.b \sqcap a.c$ does not simulate $a.(b \sqcap c)$

Consider probabilistic test:

$$\overline{a}.\left(\overline{b}.\omega +_{1/2} \overline{c}.\omega\right)$$

passes 50% v.s. passes 100%

Consequently:

$$\nvdash (a \triangleleft b) \mathbin{\&} (a \triangleleft c) \multimap a \triangleleft (b \mathbin{\&} c)$$

In process calculi:

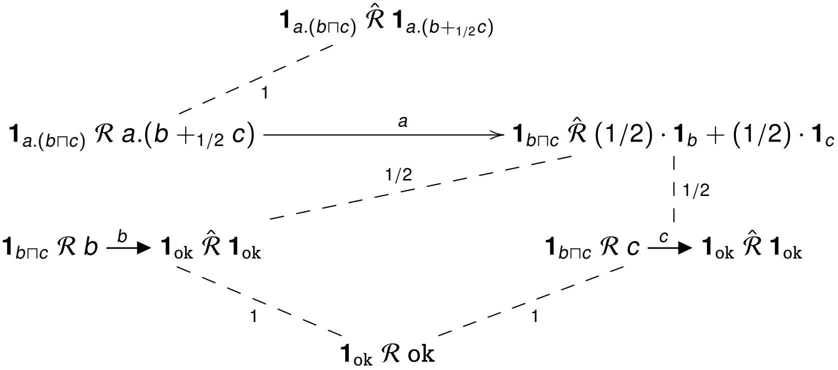
$$a.(b \sqcap c) \qquad \textit{simulates} \qquad a.(b +_{1/2} c)$$

In ΔMAV:

$$\vdash a \triangleleft (b \,\&\, c) \multimap a \triangleleft (b \oplus_{1/2} c)$$

Probabilistic Processes

In process calculi:



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Probabilistic Processes

In process calculi:

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In ΔMAV:

$$\dfrac{\vdash \overline{a}\,,\,a \qquad \dfrac{\vdots}{\vdash \overline{b} \oplus \overline{c}\,,\, b \oplus_{1/2} c}}{\vdash \overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right),\, a \triangleleft (b \oplus_{1/2} c)}$$

Probabilistic Processes

In process calculi:

$$a.(b \sqcap c) \qquad \textit{simulates} \qquad a.(b +_{1/2} c)$$

In ΔMAV:

$$\circ \triangleleft P \equiv P$$

$$\dfrac{\mathcal{C}\{\ \circ\ \}}{\mathcal{C}\{\ \bar{a} \parr a\ \}}$$

$$\dfrac{\mathcal{C}\{\ (P \parr R) \triangleleft (Q \parr S)\ \}}{\mathcal{C}\{\ (P \triangleleft Q) \parr (R \triangleleft S)\ \}}$$

$$\dfrac{\dfrac{\vdots}{(\bar{b} \oplus \bar{c}) \parr (b \oplus_{1/2} c)}}{\dfrac{(\bar{a} \parr a) \triangleleft \left((\bar{b} \oplus \bar{c}) \parr (b \oplus_{1/2} c)\right)}{\left(\bar{a} \triangleleft (\bar{b} \oplus \bar{c})\right) \parr (a \triangleleft (b \oplus_{1/2} c))}}$$

Probabilistic Processes

In process calculi:

$$a.(b \sqcap c) \qquad \textit{simulates} \qquad a.(b +_{1/2} c)$$

In ΔMAV:

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\left(\left(\overline{b} \oplus \overline{c}\right) \mathbin{\color{red}{\&_{1/2}}} \left(\overline{b} \oplus \overline{c}\right)\right) \parr \left(b \oplus_{1/2} c\right)}}{\left(\overline{b} \oplus \overline{c}\right) \parr \left(b \oplus_{1/2} c\right)}}{\left(\overline{a} \parr a\right) \triangleleft \left(\left(\overline{b} \oplus \overline{c}\right) \parr \left(b \oplus_{1/2} c\right)\right)}}{\left(\overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\right) \parr \left(a \triangleleft \left(b \oplus_{1/2} c\right)\right)}$$

$$P \equiv P \mathbin{\color{red}{\&_p}} P$$

Probabilistic Processes

In process calculi:

$$a.(b \sqcap c) \qquad \textit{simulates} \qquad a.(b +_{1/2} c)$$

In ΔMAV:

$$\vdots$$

$$\left(\left(\overline{b} \oplus \overline{c}\right) \parr b\right) \mathbin{\&_{1/2}} \left(\left(\overline{b} \oplus \overline{c}\right) \parr c\right)$$

$$\left(\left(\overline{b} \oplus \overline{c}\right) \mathbin{\&_{1/2}} \left(\overline{b} \oplus \overline{c}\right)\right) \parr (b \oplus_{1/2} c)$$

$$\left(\overline{b} \oplus \overline{c}\right) \parr (b \oplus_{1/2} c)$$

$$(\overline{a} \parr a) \triangleleft \left(\left(\overline{b} \oplus \overline{c}\right) \parr (b \oplus_{1/2} c)\right)$$

$$\left(\overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\right) \parr (a \triangleleft (b \oplus_{1/2} c))$$

$$\frac{\mathcal{C}\left\{\, (P \parr R) \mathbin{\&_p} (Q \parr S) \,\right\}}{\mathcal{C}\left\{\, \left(P \mathbin{\&_p} Q\right) \parr \left(R \oplus_p S\right) \,\right\}}$$

$$P \equiv P \mathbin{\&_p} P$$

Probabilistic Processes

In process calculi:

$$a.(b \sqcap c) \qquad \textit{simulates} \qquad a.(b +_{1/2} c)$$

In ΔMAV:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\circ \mathbin{\&_{1/2}} \circ}}{\left(\overline{b} \parr b\right) \mathbin{\&_{1/2}} \left(\overline{c} \parr c\right)}}{\left(\left(\overline{b} \oplus \overline{c}\right) \parr b\right) \mathbin{\&_{1/2}} \left(\left(\overline{b} \oplus \overline{c}\right) \parr c\right)}}{\left(\left(\overline{b} \oplus \overline{c}\right) \mathbin{\&_{1/2}} \left(\overline{b} \oplus \overline{c}\right)\right) \parr \left(b \oplus_{1/2} c\right)}}{\left(\overline{b} \oplus \overline{c}\right) \parr \left(b \oplus_{1/2} c\right)}}{\left(\overline{a} \parr a\right) \triangleleft \left(\left(\overline{b} \oplus \overline{c}\right) \parr \left(b \oplus_{1/2} c\right)\right)}}{\left(\overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\right) \parr \left(a \triangleleft \left(b \oplus_{1/2} c\right)\right)}$$

$$\frac{\mathcal{C}\{\, \circ \,\}}{\mathcal{C}\{\, \overline{a} \parr a \,\}}$$

$$\frac{\mathcal{C}\{\, P \,\}}{\mathcal{C}\{\, P \oplus Q \,\}}$$

$$\frac{\mathcal{C}\{\, Q \,\}}{\mathcal{C}\{\, P \oplus Q \,\}}$$

$$\frac{\mathcal{C}\left\{\, (P \parr R) \mathbin{\&_p} (Q \parr S) \,\right\}}{\mathcal{C}\left\{\, \left(P \mathbin{\&_p} Q\right) \parr \left(R \oplus_p S\right) \,\right\}}$$

$$P \equiv P \mathbin{\&_p} P$$

Probabilistic Processes

In process calculi:

$$a.(b \sqcap c) \qquad \textit{simulates} \qquad a.(b +_{1/2} c)$$

In ΔMAV:

$$
\begin{array}{c}
\circ \\ \hline
\circ \mathbin{\&_{1/2}} \circ \\ \hline
\left(\overline{b} \parr b\right) \mathbin{\&_{1/2}} \left(\overline{c} \parr c\right) \\ \hline
\left(\left(\overline{b} \oplus \overline{c}\right) \parr b\right) \mathbin{\&_{1/2}} \left(\left(\overline{b} \oplus \overline{c}\right) \parr c\right) \\ \hline
\left(\left(\overline{b} \oplus \overline{c}\right) \mathbin{\&_{1/2}} \left(\overline{b} \oplus \overline{c}\right)\right) \parr \left(b \oplus_{1/2} c\right) \\ \hline
\left(\overline{b} \oplus \overline{c}\right) \parr \left(b \oplus_{1/2} c\right) \\ \hline
\left(\overline{a} \parr a\right) \triangleleft \left(\left(\overline{b} \oplus \overline{c}\right) \parr \left(b \oplus_{1/2} c\right)\right) \\ \hline
\left(\overline{a} \triangleleft \left(\overline{b} \oplus \overline{c}\right)\right) \parr \left(a \triangleleft \left(b \oplus_{1/2} c\right)\right)
\end{array}
$$

$$\frac{\mathcal{C}\{\ \circ\ \}}{\mathcal{C}\{\ \overline{a} \parr a\ \}}$$

$$\frac{\mathcal{C}\{\ P\ \}}{\mathcal{C}\{\ P \oplus Q\ \}}$$

$$\frac{\mathcal{C}\{\ Q\ \}}{\mathcal{C}\{\ P \oplus Q\ \}}$$

$$\frac{\mathcal{C}\left\{\ (P \parr R) \mathbin{\&_p} (Q \parr S)\ \right\}}{\mathcal{C}\left\{\ \left(P \mathbin{\&_p} Q\right) \parr \left(R \oplus_p S\right)\ \right\}}$$

$$P \equiv P \mathbin{\&_p} P$$

The Sub-Additives

$$P \oplus_p Q$$

$$P \mathbin{\&} Q$$

The Sub-Additives

$$P \oplus Q$$

$$P \oplus_p Q$$

$$P \&_p Q$$

$$P \& Q$$

Distributivity Properties

In process calculi:

$$(a \parallel c) +_{1/2} (b \parallel d) \qquad \text{is unrelated to} \qquad (a +_{1/2} b) \parallel (c +_{1/2} d)$$

In ΔMAV:

$$(a \otimes c) \oplus_{1/2} (b \otimes d) \qquad \text{is unrelated to} \qquad (a \oplus_{1/2} b) \otimes (c \oplus_{1/2} d)$$

Distributivity Properties

In process calculi:

$$(a \sqcap b) \oplus_{1/2} (a \sqcap c) \qquad \text{simulates} \qquad a \sqcap (b \oplus_{1/2} c)$$

In ΔMAV:

$$\vdash (a \mathbin{\&} b) \oplus_{1/2} (a \mathbin{\&} c) \multimap a \mathbin{\&} (b \oplus_{1/2} c)$$

Proof:

$$\circ$$

tidy

$$\circ \mathbin{\&} \circ$$

idempotency

$$\left(\circ \mathbin{\&_p} \circ\right) \mathbin{\&} \left(\circ \mathbin{\&_p} \circ\right)$$

by atomic interaction

$$\left((\overline{a} \parr a) \mathbin{\&_p} (\overline{a} \parr a)\right) \mathbin{\&} \left((\overline{b} \parr b) \mathbin{\&_p} (\overline{c} \parr c)\right)$$

by *confine*

$$\left((\overline{a} \mathbin{\&_p} \overline{a}) \parr (a \oplus_{1/2} a)\right) \mathbin{\&} \left((\overline{c} \mathbin{\&_p} \overline{c}) \parr (c \oplus_{1/2} c)\right)$$

idempotency

$$\left((\overline{a} \mathbin{\&_p} \overline{a}) \parr a\right) \mathbin{\&} \left((\overline{b} \mathbin{\&_p} \overline{c}) \parr (b \oplus_{1/2} c)\right)$$

by *choose*

$$\left(\left((\overline{a} \oplus \overline{b}) \mathbin{\&_p} (\overline{a} \oplus \overline{c})\right) \parr a\right) \mathbin{\&} \left(\left((\overline{a} \oplus \overline{b}) \mathbin{\&_p} (\overline{a} \oplus \overline{c})\right) \parr (b \oplus_{1/2} c)\right)$$

by *external*

$$\left((\overline{a} \oplus \overline{b}) \mathbin{\&_p} (\overline{a} \oplus \overline{c})\right) \parr \left(a \mathbin{\&} (b \oplus_{1/2} c)\right)$$

Conclusions

Sub-additives arise from the probabilistic content of linear logic.

Further remarks (in paper):

- Preserved in all contexts.
- Permits action refinement.

$$\nvdash a \triangleleft a \multimap a \otimes a \qquad \text{(no auto-concurrency)}$$

- **Medial rules** necessary for cut elimination.

$$\frac{(P \sqcap R) \sqcup (Q \sqcap S)}{(P \sqcup Q) \sqcap (R \sqcup S)}\ \text{medial} \qquad \text{where } (\sqcap, \sqcup) \in \{\ (\parr, \oplus_q), (\&_p, \oplus_q), (\&, \oplus_q), (\&, \&_p), (\parr, \triangleleft), (\&, \triangleleft), (\&_p, \triangleleft), (\triangleleft, \oplus_q)\ \}$$

- Cut elimination demands **decomposition**, based on the topology of proofs.

Questions (for audience):

- Do sub-additives arise in other semantics?