Invited talk

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Title: Solution chains and independent systems of word equations

Abstract: Two fundamental properties of word equations are the following. First the defect theorem which claims that if a set of n words satisfies a nontrivial equation the elements of it can be expressed as products of at most $n-1$ words. Second the compactness property of word equations which states that any system of constant-free equations with a finite number of variable is equivalent to one of its finite subsystems. Of course, the equivalence means here that the systems have exactly the same solutions.

Both of these results state something which can be viewed as a dimension property of word equations. The goal of this presentation is to analyze in more details these phenomena, and in particular to point out that many natural and simply formulated problems are still un- answered.

Central notions of our considerations are independent systems of equations and chains of solution sets of word equations. We say that a system of (word) equations is independent if it is not equivalent to any of its proper subsystems. By a chain of solution sets of word equations we mean a strictly decreasing sequence of solutions of systems of equations where the previous system is always a proper subset of the next one. More intuitively the latter notion aims to capture how many constrains on words we can introduce in such a way that in each step we obtain a proper restriction.

It follows from the above mentioned compactness property that all sets of independent word equations, as well as chains of solutions sets are finite. However, and this is a major open problem of the field, no upper bound, depending e.g. on the number of unknowns, is known, and, in fact, it is not known whether such a bound exists. The best lower bounds are quartic in terms of the number of unknowns, as we shall explain in this lecture.

In the above spirit we can also ask how many constrains (as equations) we can introduce such the they force only a minimal defect effect (guaranteed already by a single equation) or they just avoid the maximal defect effect (allowing still a non-cyclic solution set). We recall known results of this research, but even more interestingly again point out many open problems.

Indeed, it is amazing that even in the case of three unknowns the basic problems are not solved. Namely, no upper bound is known for the maximal size of independent systems of three unknown equations, and accordingly no upper bound is known for maximal chains of such solution sets, either. The best known lower bound for such a maximal chain is 7.

This is joint work with Aleksi Saarela.