Riordan graphs I: Structural properties*

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This paper is dedicated to the memory of Professor Jeff Remmel, who recently passed away.

Abstract

In this paper, we use the theory of Riordan matrices to introduce the notion of a Riordan graph. The Riordan graphs are a far-reaching generalization of the well known and well studied Pascal graphs and Toeplitz graphs, and also some other families of graphs. The Riordan graphs are proved to have a number of interesting (fractal) properties, which can be useful in creating computer networks with certain desirable features, or in obtaining useful information when designing algorithms to compute values of graph invariants. The main focus in this paper is the study of structural properties of families of Riordan graphs obtained from infinite Riordan graphs, which includes a fundamental decomposition theorem and certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle. We will study spectral properties of the Riordan graphs in a follow up paper.

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1 Introduction

Pascal's triangle is a classical combinatorial object, and its roots can be traced back to the 2nd century BC. In 1991, Shapiro, Getu, Woan and Woodson [15] have introduced the notion of a *Riordan array*, also known as a *Riordan matrix*, in order to define a class of infinite lower triangular matrices with properties analogous to those of the Pascal triangle (matrices) [11]. Since then, Riordan matrices became an active area of research. See [14] by Merlini and Sprugnoli, and references there in, for examples of results in this direction. Also, see [6] for a recent paper about Lie theory on the Riordan group, the set of invertible Riordan matrices. In particular, Riordan matrices found applications in the context of the computation of combinatorial sums [17].

The notion of Pascal's triangle was also influential in graph theory and its applications to computer networks. Indeed, in 1983, Deo and Quinn [8] have introduced the *Pascal*

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graphs that are constructed using Pascal's triangle modulo 2. These graphs attracted much attention in the literature (see [5] and references there in) and they are optimal graphs for computer networks with certain desirable properties, such as

- the design is to be simple and recursive;
- there must be a universal vertex, i.e. a vertex adjacent to all others;
- there must exist several paths between each pair of vertices.

Another important object of interest to us is the well studied notion of a *Toeplitz graph*, that is based on the notion of a *Toeplitz matrix* (see [12] and references there in). A Toeplitz graph $G = (V, E)$ is a graph with $V = \{1, \ldots, n\}$ and $E = \{ij \mid |i - j| \in \{t_1, \ldots, t_k\}, 1 \le t_1 < \cdots < t_k \le n - 1\}$.

In this paper we introduce the notion of a *Riordan graph*. This notion not only provides a far-reaching generalizaiton of the notions of a Pascal graph and a Toeplitz graph, but also extends the theory of Riordan matrices to the domain of graph theory, similarly to the introduction of the Pascal graphs based on Pascal's triangles. The Riordan graphs are proved to have a number of interesting (fractal) properties, which can be useful in creating computer networks with certain desired features, or in obtaining useful information when designing algorithms to compute values of graph invariants. Indeed, the Pascal graphs are just one instance of a family of Riordan graphs having a universal vertex and a simple recursive structure. Thus, other members of that family could be used instead of the Pascal graphs in designing computer networks, and depending on the context, these could be a better choice than the Pascal graphs.

Our basic idea here is in building the infinite adjacency matrix $\mathcal{A}$ based on an infinite Riordan matrix modulo 2, and considering the leading principal matrices giving (finite) Riordan graphs. See Section 2.2 for the precise definitions. We introduce various families of Riordan graphs based on the choice of the generating functions defining these graphs (via Riordan matrices). For example, the Riordan graphs of the *Appell type* are precisely the class of Toeplitz graphs including the *Fibonacci graphs*, while the Riordan graphs of the *Bell type* include the Pascal graphs, *Catalan graphs* and *Motzkin graphs*.

One of the basic questions one can ask in our context is whether or not a given labelled or unlabelled graph is a Riordan graph (defined by a pair of generating functions). It turns out that all unlabelled graphs on at most four vertices are Riordan graphs (see Figure 1), while non-Riordan unlabelled graphs always exist for larger graphs on any number of vertices. However, the main focus in this paper is the study of *labelled* Riordan graphs, and we give structural properties of certain families of graphs obtained from infinite Riordan graphs.

Throughout the paper, we normally label graphs on n vertices by the elements of $[n] := \{1, \ldots, n\}$. However, we also meet graphs labelled by odd numbers, or even numbers, or consecutive subintervals in $[n]$. For two isomorphic graphs, G and H, we write $G \cong H$. For a graph G, $V(G)$ (resp., $E(G)$) denotes the set of vertices (resp., edges) in G. Also, for a subset of vertices V in a graph G, we let $\langle V \rangle$ denote the graph induced by the vertices in V. Moreover, for a formal power series $f = f(z) = \sum_{n \ge 0} f_n z^n$, $[z^i]f$ denotes the coefficient f_i of z^i in the sum. Finally, we let $\mathbb{N}_0 = \{0, 1, \ldots\}$.

This paper is organized as follows. In Section 2 we review the notion of a Riordan matrix and use it to introduce the notion of an (infinite) Riordan graph in the *labelled* and *unlabelled* cases. However, the main focus in this paper is the *labelled* case, so unless we use the word "unlabelled" explicitely, our Riordan graphs are *labelled.* A number of basic results on Riordan graphs are established in Section 2.2, and this includes the number of Riordan graphs on n vertices (see Proposition 2.3), and the necessary conditions on Riordan graphs (see Theorem 2.5). In Section 2.3 we define the *product* $\otimes_R$ of two Riordan graphs and then give its combinatorial interpretation in terms of directed walks in certain graphs (see Theorem 2.11). We also discuss the *ring sum* $\oplus$ of two graphs in Section 2.3 that can be used to define certain classes of Riordan graphs (see Section 2.4). Various families of Riordan graphs are introduced in Section 2.4. These include, but are not limited to Riordan graphs of the *Appell type*, *Bell type*, *Lagrange type*, *checkerboard type*, *derivative type*, and *hitting time type.*

In Section 3 we give structural results applicable to any Riordan graphs. In particular, in Section 3.1 we show that every Riordan graph is a fractal (see Theorem 3.6). Also, the *reverse relabelling* of proper Riordan graphs is defined and studied in Section 3.2. Further, in Section 3.3 we prove the *Riordan Graph Decompostion* theorem (see Theorem 3.12). In addition, in Section 3.4 we give certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle.

In Section 4.1, we consider *io-decomposable* and *ie-decomposable* proper Riordan graphs, and provide a characterization result for these graphs (see Theorem 4.2). One of the main focuses in this paper is the study of Riordan graphs of the *Bell type* conducted in Section 4.2. In particular, we provide two characterization results for io-decomposable Riordan graphs of the Bell type (see Lemma 4.4 and Theorem 4.6) and use the results to show that the *Pascal graphs* and *Catalan graphs* are io-decomposable, while the *Motzkin graphs* are not io-decomposable. Also, in Section 4.2 we study the following properties of io-decomposable Riordan graphs of the Bell type: number of edges, number of universal vertices, clique number, chromatic number, diameter, and others. In Section 4.3 we provide two characterization results (Lemma 4.25 and Theorem 4.26) for ie-decomposable Riordan graphs of the *derivative type.* Finally, in Section 5 we provide concluding remarks and state directions for further research.

We study spectral properties of the Riordan graphs in the follow up paper [4].

2 From Riordan matrices to Riordan graphs

After reviewing the notion of a *Riordan matrix* in Section 2.1, we will introduce the notion of a *Riordan graph* in Section 2.2. Then, in Section 2.4 we introduce various families of Riordan graphs.

2.1 Riordan matrices

Let $\kappa[[z]]$ be the ring of formal power series in the variable z over an integral domain κ. If there exists a pair of generating functions $(g,f)\in\kappa[[z]]\times\kappa[[z]]$, $f(0)=0$ such that for $j\geq 0$,

$$g\cdot f^j=\sum_{i\geq 0}\ell_{i,j}z^i,$$

then the matrix $L=[\ell_{ij}]_{i,j\geq 0}$ is called a *Riordan matrix* (or, a *Riordan array*) over κ generated by g and f. Usually, we write $L=(g,f)$. Since $f(0)=0$, every Riordan matrix

(g, f) is infinite and a lower triangular matrix. If a Riordan matrix is invertible, it is called *proper*. Note that (g, f) is invertible if and only if $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$.

If we multiply (g, f) by a column vector $(c_0, c_1, \ldots)^T$ with the generating function Φ over an integral domain κ with characteristic zero, then the resulting column vector has the generating function $g\Phi(f)$. This property is known as the *fundamental theorem of Riordan matrices* (*FTRM*). This leads to the multiplication of Riordan matrices, which can be described in terms of generating functions as

$$(g, f) * (h, \ell) = (g \cdot h(f), \ell(f)). \tag{1}$$

The set of all proper Riordan matrices under the above *Riordan multiplication* forms a group called the *Riordan group*. The identity of the group is $(1, z)$, the usual identity matrix and $(g, f)^{-1} = (1/g(\overline{f}), \overline{f})$ where $\overline{f}$ is the *compositional inverse* of f, i.e. $\overline{f}(f(z)) = f(\overline{f}(z)) = z$.

Throughout this paper, we write $a \equiv b$ for $a \equiv b \pmod 2$.

For a Riordan matrix (g, f) over $\mathbb{Z}$, the $(0, 1)$-matrix $L = [\ell_{ij}]_{i,j\geq 0}$ defined by

$$\ell_{ij} \equiv [z^i]gf^j,$$

is called a *binary Riordan matrix*, and it is denoted by $\mathcal{B}(g, f)$. The *leading principal matrix of order* n in $\mathcal{B}(g, f)$ (resp., (g, f)) is denoted by $\mathcal{B}(g, f)_n$ (resp., $(g, f)_n$). If $\kappa = \mathbb{Z}$ then the fundamental theorem gives

$$\mathcal{B}(g, f)\Phi \equiv g\Phi(f). \tag{2}$$

It is known [13] that an infinite lower triangular matrix $L = [\ell_{i,j}]_{i,j\geq 0}$ with $\ell_{0,0} \neq 0$ is a proper Riordan matrix if and only if there is a unique sequence $(a_0, a_1, \ldots)$ with $a_0 \neq 0$ such that, for $i \geq j \geq 0$,

$$\ell_{i+1,j+1} = a_0\ell_{i,j} + a_1\ell_{i,j+1} + \cdots + a_{i-j}\ell_{i,i}.$$

This sequence is called the *A-sequence* of the Riordan array. Also, if $L = (g, f)$ then

$$f = zA(f), \quad \text{or equivalently} \quad A = z/\bar{f} \tag{3}$$

where A is the generating function of the A-sequence of (g, f). In particular, if L is a binary Riordan matrix $\mathcal{B}(g, f)$ with $f'(0) = 1$ then the sequence is called the *binary A-sequence* $(1, a_1, a_2, \ldots)$ where $a_k \in \{0, 1\}$.

2.2 Riordan graphs

The following definition gives the notion of a Riordan graph in both *labelled* and *unlabelled* cases. We note that throughout this paper the graphs are assumed to be *labelled* unless otherwise specified.

Definition 2.1 A simple *labelled* graph G with n vertices is a *Riordan graph* of order n if the adjacency matrix of G is an $n \times n$ symmetric $(0, 1)$-matrix given by

$$\mathcal{A}(G) = \mathcal{B}(zg, f)_n + \mathcal{B}(zg, f)_n^T$$

for some Riordan matrix (g, f) over $\mathbb{Z}$. We denote such G by $G_n(g, f)$, or simply by G_n when the matrix (g, f) is understood from the context, or it is not important. A simple *unlabelled* graph is a *Riordan graph* if at least one of its labelled copies is a Riordan graph.

We note that the choice of the functions g and f in Definition 2.1 may not be unique. If $G = G_n(g, f)$ is a Riordan graph and $\mathcal{A}(G) = [r_{ij}]_{1\le i,j\le n}$ then for $i > j \ge 1$,

$$r_{i,j} \equiv [z^{i-1}]zgf^{j-1} \equiv [z^{i-2}]gf^{j-1}. \tag{4}$$

Thus the $n \times n$ adjacency matrix $\mathcal{A}(G)$ satisfies that

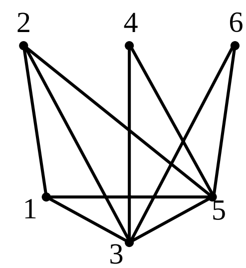
- its main diagonal entries are all 0, and
- its lower triangular part below the main diagonal is the $(n-1) \times (n-1)$ binary Riordan matrix $\mathcal{B}(g, f)_{n-1}$.

For example, the *Catalan graph* $CG_6 = G_6(C, zC)$ where

$$C = \frac{1 - \sqrt{1-4z}}{2z} = \sum_{n\ge 0} \frac{1}{n+1}\binom{2n}{n} z^n = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

is given by

$$\mathcal{A}(CG_6) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$



Definition 2.2 A Riordan graph $G_n(g, f)$ is *proper* if the binary Riordan matrix $\mathcal{B}(g, f)_{n-1}$ is proper.

If a Riordan graph $G_n(g, f)$ is proper then the Riordan matrix (g, f) is also proper because $g(0) \equiv f'(0) \equiv 1$. The converse to this statement is not true. For instance, $(1, 2z + z^2)$ is a proper Riordan matrix but $G_n(1, 2z + z^2)$ is not a proper Riordan graph.

For any Riordan graph $G_n(g, f)$, we can think of the sequence of induced subgraphs

$$G_1 = \langle\{1\}\rangle\,,\ G_2 = \langle\{1, 2\}\rangle\,, \ldots, G_{n-1} = \langle\{1, 2, \ldots, n-1\}\rangle\,,$$

each defined by the same pair of functions, showing the recursive nature of Riordan graphs. From applications point of view, this property implies that when a new node is added to a network, the entire network does not have to be reconfigured.

Proposition 2.3 *The number of Riordan graphs of order $n \ge 1$ is*

$$\frac{4^{n-1} + 2}{3}.$$

Proof. Let $G_n = G_n(g, f)$ be a labelled Riordan graph and i be the smallest index such that $g_i = [z^i]g \equiv 1$.

- If $i \ge n-1$ then G_n is the null graph N_n.
- If $0 \le i \le n-2$ then we can assume that $g = z^i + g_{i+1}z^{i+1} + \cdots + g_{n-2}z^{n-2}$ and $f = f_1 z + f_2 z^2 + \cdots + f_{n-i-2}z^{n-i-2}$ where $g_{i+1}, \ldots, g_{n-2}, f_1, \ldots, f_{n-i-2} \in \{0, 1\}$.

Thus the number of possibilities to create G_n is

$$1+\sum_{i=0}^{n-2} 2^{2(n-i-2)} = \frac{4^{n-1}+2}{3}$$

where the 1 corresponds to the null graph. ■

Definition 2.4 Any Riordan matrix (g, f) over $\mathbb{Z}$ naturally defines the *infinite* graph

$$G := G(g, f) = \lim_{n\to\infty} G_n(g, f),$$

which we call the *infinite Riordan graph* corresponding to the Riordan matrix (g, f).

We note that even if an *unlabelled* graph is Riordan, its random labelling is likely to result in a non-Riordan graph. The following theorem gives necessary conditions for a graph to be Riordan. These conditions are formulated in terms of the subdiagonal elements in the adjacency matrix of a Riordan graph.

Theorem 2.5 (Necessary conditions for Riordan graphs) *Let $G_n = G_n(g, f)$ be a Riordan graph of order n. Then one of the following holds:*

(i) $i(i+1) \notin E(G_n)$ *for* $1 \le i \le n-1$.

(ii) $12 \in E(G_n)$ *and* $i(i+1) \notin E(G_n)$ *for* $2 \le i \le n-1$.

(iii) $i(i+1) \in E(G_n)$ *for* $1 \le i \le n-1$, *i.e.* G_n *has the Hamiltonian path* $1 \to 2 \to \cdots \to n$.

Proof. Let $\mathcal{A}(G_n) = [r_{i,j}]_{1\le i,j\le n}$. From the definition of the Riordan matrix (g, f), we have $r_{i+1,i} \equiv g_0 f_1^{i-1}$ for $1 \le i \le n-1$ where $g_0 = [z^0]g$ and $f_1 = [z^1]f$. Going through the four possibilities of choosing g_0 and f_1 in $\{0, 1\}$, we obtain the required result. ■

Figure 1 justifies that all *unlabelled* graphs on at most four vertices are Riordan (proper labelling and the corresponding Riordan matrices (g, f) are provided in the figure). On the other hand, the following proposition shows that not all unlabelled graphs on n vertices are Riordan for $n \ge 5$.

Proposition 2.6 *The unlabelled graph $H_{n+1} \cong K_n \cup K_1$ obtained from a complete graph K_n by adding an isolated vertex is not Riordan for $n \ge 4$.*

Proof. Suppose that there exist g and f such that a labelled copy of H_{n+1} is the Riordan graph $G_n(g, f)$. We consider two cases depending on whether the isolated vertex in H_{n+1} is labelled by 1 or not.

Let the isolated vertex be labelled by 1. Since there are no edges $1i \in E(H_{n+1})$ for $i = 2, \ldots, n+1$, we have $g = 0$ so that $G_n(g, f)$ is the null graph N_n. This is a contradiction.

Let $i \ne 1$ be the label of the isolated vertex and $\mathcal{A}(H_{n+1}) = [r_{i,j}]_{1\le i,j\le n+1}$. Since

$$(r_{2,1}, \ldots, r_{n+1,n}) = \begin{cases} (\underbrace{1,\ldots,1}_{(i-2)\text{times}}, 0, 0, \underbrace{1,\ldots,1}_{(n-i)\text{ times}}) & \text{if } i \ne n \\ (\underbrace{1,\ldots,1}_{(n-1)\text{ times}}, 0) & \text{if } i = n \end{cases}$$

by Theorem 2.5 this is also a contradiction. Hence the proof follows. ■

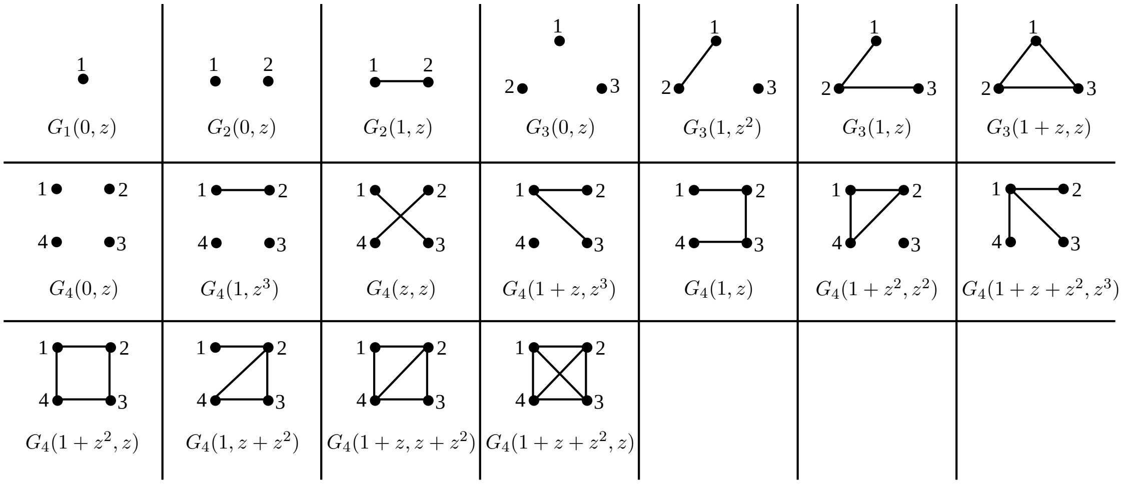


Figure 1: All unlabelled graphs on at most four vertices are Riordan graphs

2.3 Operations on Riordan graphs

There are many graph operations studied in the literature. However, in general we cannot guarantee that a particular operation applied to Riordan graphs results in a Riordan graph. An example of an operation that can be used in our context is the *ring sum* of two graphs defined as follows.

Definition 2.7 *Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define the* ring sum $G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$. *Thus, an edge is in $G_1 \oplus G_2$ if and only if it is an edge in G_1, or and edge in G_2, but not both.*

The ring sum is well defined on Riordan graphs $G_n(g, f)$ with a fixed f and a fixed vertex set (e.g. $[n]$) due to the fact that $(g, f) + (h, f) = (g + h, f)$, so

$$G_n(g, f) \oplus G_n(h, f) = G_n(g + h, f).$$

Next, we define a new graph operation, the *product* $\otimes_R$ of two Riordan graphs, and then we give its combinatorial interpretation in terms of directed walks in certain graphs.

Definition 2.8 The *product* of two Riordan graphs $G_n(g, f)$ and $G_n(h, \ell)$ is the graph

$$G_n(g, f) \otimes_R G_n(h, \ell) := G_n(g \cdot h(f), \ell(f)). \tag{5}$$

The set of all proper Riordan graphs forms a group under the binary operation $\otimes_R$ given by (5), which follows from (1). The identity of the group is the path graph $G_n(1, z)$. The adjacency matrix $\mathcal{A}(G_n)$, where $G_n = G_n(g, f) \otimes_R G_n(h, \ell)$ is given by

$$\mathcal{A}(G_n) = \mathcal{B}(zgh(f), \ell(f))_n + \mathcal{B}(zgh(f), \ell(f))_n^T.$$

Definition 2.9 Let $G_n = G_n(g, f)$ and $H_n = G_n(h, \ell)$. We define the *RGB-graph* $\mathcal{D}_n = \mathcal{D}(G_n, H_n)$ to be a digraph on the set of vertices $[n]$ with colored edges as follows:

- For *each* edge ij in G_n, $i > j$, add the edge $i \to j$ to $\mathcal{D}_n$ and color it in *Red*.

- For *each* edge ij in H_n, $i > j$, add the edge $i \to j$ to $\mathcal{D}_n$ and color it in *Blue.*
- For each i, $1 \le i \le n-1$, add the edge $i \to i+1$ to $\mathcal{D}_n$ and color it in *Green.*

The adjacency matrix of the RGB-graph $\mathcal{D}_n$ is defined as the $n \times n$ (0,1)-matrix whose (i,j)-entry is 1 if and only if $i \to j$.

See Example 2.13 below for an instance of a graph $\mathcal{D}_6$.

Definition 2.10 An *RGB-walk* in $\mathcal{D}_n = \mathcal{D}(G_n, H_n)$ is a directed walk of length 3 in $\mathcal{D}_n$ such that the first edge in it is Red, the second edge is Green, and the third edge is Blue.

Theorem 2.11 *Let $G_n = G_n(g,f)$ and $H_n = G_n(h,\ell)$. Then two vertices i and j in $G_n \otimes_R H_n$, for $i > j$, are adjacent if and only if the number of RGB-walks in $\mathcal{D}(G_n, H_n)$ from i to j is odd.*

Proof. Let $R_n = [r_{i,j}]_{1\le i,j\le n} \equiv \mathcal{B}(zg,f)_n$, $M_n = [m_{i,j}]_{1\le i,j\le n} \equiv \mathcal{B}(z,z)_n^T$ and $B_n = [b_{i,j}]_{1\le i,j\le n} \equiv \mathcal{B}(zh,\ell)_n$ be $(0,1)$-matrices. Thus, R_n (resp., M_n; B_n) is the adjacency matrix of the directed subgraph of $\mathcal{D}(G_n, H_n)$ formed by the red (resp., green; blue) edges. All entries in M_n are 0 except $m_{k,k+1} = 1$ for $1 \le k \le n-1$. Thus if $D_n = [d_{i,j}]_{1\le i,j\le n} = R_n M_n B_n$ then

$$d_{i,j} = \begin{cases} \sum_{k=j}^{i-1} r_{i,k} m_{k,k+1} b_{k+1,j} & \text{if } i > j \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It implies that if $i > j$ then $d_{i,j}$ counts the number of RGB-walks in the digraph $\mathcal{D}(G_n, H_n)$ from i to j . Now let $\mathcal{A} = [a_{i,j}]_{1\le i,j\le n}$ be the adjacency matrix of $G_n \otimes_R H_n$. Since

$$\mathcal{B}(gh(f), \ell(f))_{n-1} \equiv \mathcal{B}(g,f)_{n-1}\mathcal{B}(h,\ell)_{n-1}$$

we have

$$a_{ij} \equiv \sum_{k=j}^{i-1} r_{i,k} b_{k+1,j} \quad \text{if } i > j. \tag{7}$$

Since $m_{k,k+1} = 1$ for $1 \le k \le n-1$ it follows from (6) and (7) that if $i > j$ then $d_{i,j} \equiv a_{i,j}$. It means that i and j for $i > j$ are adjacent, i.e. $a_{i,j} = 1$ if and only if $d_{i,j}$ is odd. Hence the proof follows. ∎

Remark 2.12 Let $G_n = G_n(g,f)$ and $H_n = G_n(g,f)^{-1}$. Since $G_n \otimes_R H_n = G_n(1,z)$ is the path graph, by Theorem 2.11 the number of RGB-walks from i to j is even (resp., odd) when $i - j \ge 2$ (resp., $i - j = 1$).

Example 2.13 Consider the graph $\mathcal{D}_6 = \mathcal{D}(G_6, H_6)$ in Figure 2 where $G_6 = G_6\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ and $H_6 = G_6\left(\frac{1}{1-z^2}, z^2\right)$. Since $R_6 = \mathcal{B}\left(\frac{z}{1-z}, \frac{z}{1-z}\right)_6$, $M_6 = \mathcal{B}(z,z)_6^T$ and $B_6 = \mathcal{B}\left(\frac{z}{1-z^2}, z^2\right)_6$, from (6) we obtain

$$D_6 = [d_{i,j}]_{1\le i,j\le 6} = R_6 M_6 B_6 = \begin{pmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 2&1&0&0&0&0 \\ 2&1&0&0&0&0 \\ 2&1&1&0&0&0 \end{pmatrix}.$$

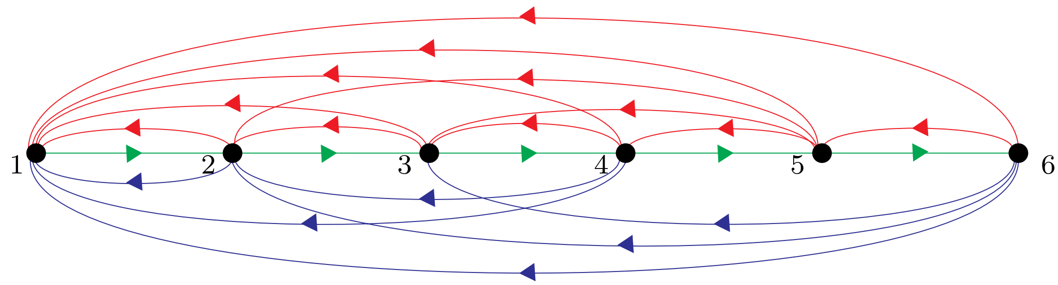


Figure 2: The RGB-graph $\mathcal{D}_6 = \mathcal{D}(G_6, H_6)$

By counting RGB-walks in Figure 2, we see that if $i > j$ then $d_{i,j}$ counts the number of RGB-walks from the vertex i to the vertex j. For instance, $d_{5,1} = 2$ because there are two RGB-walks, $5{\rightarrow}1{\rightarrow}2{\rightarrow}1$ and $5{\rightarrow}3{\rightarrow}4{\rightarrow}1$, from the vertex 5 to the vertex 1.

Since $G_n \otimes_R H_n = G_n\left(1+z, \frac{z^2}{1-z^2}\right)$, the adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

which should be the case by Theorem 2.11.

2.4 Families of Riordan graphs

Below, we introduce a number of classes of Riordan graphs and give examples of graphs in these classes. The names of the classes come from the widely used names of the Riordan matrices defining the respective Riordan graphs; such matrices are obtained by imposing various restrictions on the pairs of functions (g, f). Additionally, in Section 4 we introduce *o-decomposable*, *e-decomposable*, *io-decomposable* and *ie-decomposable* Riordan graphs. Also, more classes of Riordan graphs can be introduced using the operations $\oplus$ and $\otimes_R$ defined in Section 2.3, and we discuss these at the end of this subsection.

Note that the most general definition of the *null graphs* N_n (also known as the *empty graphs*) in our terms is $G_n(0, f)$ for any f where $f(0) = 0$. Also note that the empty graphs, the star graphs $G_n(\frac{1}{1-z}, 0)$, and the *complete k-ary trees* for $k \geq 2$ defined by $G_n(1 + z + \cdots + z^{k-1}, z^k)$ are examples of non-proper Riordan graphs; other examples of non-proper Riordan graphs can be obtained from (v) in Theorem 3.14, and even more such examples are discussed at the end of this subsection. However, most of Riordan graphs considered in this paper are proper.

Riordan graphs of the *Appell type*. This class of graphs is defined by an *Appell matrix* (g, z), and thus it is precisely the class of Toeplitz graphs. Examples of graphs in this class are

- the null graphs N_n defined by $G_n(0, z)$;
- the path graphs P_n defined by $G_n(1, z)$;
- the complete graphs K_n defined by $G_n\left(\frac{1}{1-z}, z\right)$;
- the complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ defined by $G_n(\frac{1}{1-z^2}, z)$; and
- the *Fibonacci graph* FG_n defined by $G_n\left(\frac{1}{1-z-z^2}, z\right)$.

Riordan graphs of the *Bell type.* This class of graphs is defined by a *Bell matrix* (g, zg). Examples of graphs in this class are

- the null graphs N_n defined by $G_n(0, 0)$;
- the path graphs P_n defined by $G_n(1, z)$;
- the Pascal graphs PG_n defined by $G_n(\frac{1}{1-z}, \frac{z}{1-z})$;
- the *Catalan graphs* CG_n defined by $G_n\left(\frac{1-\sqrt{1-4z}}{2z}, \frac{1-\sqrt{1-4z}}{2}\right)$; and
- the *Motzkin graphs* MG_n defined by $G_n\left(\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}, \frac{1-z-\sqrt{1-2z-3z^2}}{2z}\right)$.

Riordan graphs of the *Lagrange type.* This class of graphs is defined by a *Lagrange matrix** $(1, f)$, and it is trivially related to Riordan graphs of the Bell type. Indeed, letting $f = zg$, we see that removing the vertex 1 in $G_n(1, zg)$ gives the graph $G_{n-1}(g, zg)$ of the Bell type. Conversely, given a graph $G_{n-1}(g, zg)$, we can always relabel each vertex i by $i + 1$, and add a new vertex labelled by 1 and connected to the vertex 2, to obtain the graph $G_n(1, zg)$ of the Lagrange type.

Riordan graphs of the *checkerboard type.* This class of graphs is defined by a *checkerboard matrix* (g, f) such that g is an even function and f is an odd function. Examples of graphs in this class are

- the path graphs P_n defined by $G_n(1, z)$; and
- the complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ defined by $G_n(\frac{1}{1-z^2}, z)$.

Riordan graphs of the *derivative type.* This class of graphs is defined by a *derivative matrix* (f', f). Examples of graphs in this class are

- the null graphs N_n defined by $G_n(0, 0)$; and
- the path graphs P_n defined by $G_n(1, z)$.

Riordan graphs of the *hitting time type.* This class of graphs is defined by a *hitting time matrix* $(zf'/f, f)$, $f'(0) = 1$, and it is trivially related to Riordan graphs of the derivative type. Indeed, the lower triangular part below the main diagonal of the adjacency matrix $\mathcal{A}(G)$ of any graph $G = G_n(zf'/f, f)$ is

$$\mathcal{B}_{n-1}(zf'/f, f) = \left[\left\{zf'/f\right\}, \left\{zf'\right\}, \left\{zf'f\right\}, \ldots, \left\{zf'f^{n-3}\right\}\right]_{(n-1)\times(n-1)},$$

*Riordan matrices of the form $(1, f)$ are also known as *associated matrices* in the literature.

where the coefficients are taking modulo 2. Removing the first row and the first column of $\mathcal{A}(G)$, corresponding to removing the vertex 1, gives the graph $G_{n-1}(f', f)$ of the derivative type. Conversely, given a Riordan graph $G_{n-1}(f', f)$ of the derivative type, one can relabel each vertex i by $i+1$, and add a new vertex, labelled by 1, that is connected to the vertices defined by the coefficients of the function zf'/f to obtain $G_n(zf'/f, f)$.

Thus Riordan graphs of the Bell type (Section 4.2) and the derivative type (Section 4.3) are of the main interest in Section 4.

As is mentioned above, more classes of Riordan graphs can be introduced using the operations $\oplus$ and $\otimes_R$ defined in Section 2.3. Indeed, to illustrate this idea, note that the ring sum $\oplus$ of a Riordan graph $G_n(g, zg)$ of the Bell type and the Riordan graph $G_n((zg)', zg)$ of the derivative type is well defined. Such a sum results in a new class of graphs defined by the Riordan matrices of the form (zg', zg). Indeed, since $2g \equiv 0 \pmod 2$, we have

$$G_n(g, zg) \oplus G_n((zg)', zg) = G_n(g + g + zg', zg) = G_n(zg', zg).$$

Note that $G_n(zg', zg)$ is not proper, as is the ring sum of any two proper Riordan graphs.

We end the subsection by noticing that the class of Riordan graphs of the Appell type (i.e. Toeplitz graphs) are closed under the operations $\oplus$ and $\otimes_R$. It is not difficult to show that this class of graphs on n vertices forms a *commutative ring* with the identity element $G_n(1, z)$ and the zero element $G_n(0, z) = N_n$.

2.5 The complement of a Riordan graph

For any Riordan graph $G_n = G_n(g, z)$ of the Appell type, the ring sum $G_n(g, z) \oplus G_n\left(\frac{1}{1-z}, z\right)$ gives the *complement* of G_n, i.e. the graph in which edges of G_n become non-edges, and vice versa.

In general, it is not true that the complement of a Riordan (labelled or unlabelled) graph is Riordan. Indeed, the complement of the star graph $G_n\left(\frac{1}{1-z}, 0\right)$, $n \geq 5$, is a labelled copy of the graph H_n in Proposition 2.6, which is non-Riordan. Thus, Riordan graphs can become non-Riordan, and thus versa, under taking the complement.

Note that the complement of the Riordan graph $G_4(1 + z^2, z^2)$ in Figure 1 is the Riordan graph $G_4(1 + z + z^2, z^3)$ showing that the operation of the complement preserves the property of being Riordan for some graphs of non-Appell type.

3 Structural properties of Riordan graphs

We begin with basic properties of a Riordan graph $G_n(g, f)$, which can be directly determined in terms of binary column generating functions denoted $\{gf^k\}$ of the binary Riordan array $\mathcal{B}(g, f)$. Let

$$\mathcal{B}(g, f)_{n-1} = \left[\{g\}, \{gf\}, \{gf^2\}, \ldots, \{gf^{n-2}\}\right]_{(n-1)\times(n-1)}$$

and let $\{gf^k\}_n(1) := \sum_{j=k}^{n}[z^j]\{gf^k\}$ be the substitution of $z = 1$ in the Taylor expansion in z up to degree n of $\{gf^k\}$ modulo 2.

In this paper, $d_G(i)$ denotes the *degree* of a vertex i in a graph G. If G is understood from the context, we simply write $d(i)$.

Theorem 3.1 (Basic Properties) *Let $G_n(g,f) = ([n], E)$ be a Riordan graph. Then*

(i) *For $i > j$, $ij \in E$ if and only if $[z^{i-2}]\left\{gf^{j-1}\right\} = 1$.*

(ii) $d(1) = \{g\}_{n-2}(1)$.

(iii) $d(n) = \sum_{j=0}^{n-2}[z^{n-2}]\left\{gf^j\right\}$.

(iv) *For $k \notin \{1, n\}$, $d(k) = \left\{gf^{k-1}\right\}_{n-2}(1) + \sum_{j=0}^{k-2}[z^{k-2}]\left\{gf^j\right\}$.*

(v) $|E| = \sum_{j=0}^{n-2}\left\{gf^j\right\}_{n-2}(1)$.

(vi) *If $G_n(g,f)$ is proper then it has the Hamiltonian path $1 \to 2 \to \cdots \to n$. If additionally $[z^{n-2}]\{g\} = 1$ then $G_n(g,f)$ has the Hamiltonian cycle $1 \to 2 \to \cdots \to n \to 1$.*

Proof. The (i)–(iii) are straightforward. The (iv) follows from the fact that the degree of a vertex k is the summation of all entries located in both the $(k-1)$th column and the $(k-2)$th row of $\mathcal{B}(g,f)_{n-1}$. The (v) follows from the fact that the number of edges in $G_n(g,f)$ is equal to the number of 1s in $\mathcal{B}(g,f)_{n-1}$. The (vi) follows from the fact that if $G_n(g,f)$ is proper then all entries of the subdiagonal in its adjacency matrix are 1s, i.e. i is adjacent to $i+1$ for $i = 1, \ldots, n-1$. ■

In what follows, the *matching number* $\beta(G)$ is the size of a *maximal matching* in a graph G.

Theorem 3.2 *Let G_n be a proper Riordan graph. Then $\beta(G_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. By (vi) in Theorem 3.1, G_n has the Hamiltonian path $1 \to 2 \to \cdots \to n$. Therefore, for even and odd n, respectively, maximal matchings are $\{12,\ 34,\ \ldots,\ (n-1)n\}$ and $\{12,\ 34,\ \ldots,\ (n-2)(n-1)\}$. This completes the proof. ■

3.1 Fractal properties of Riordan graphs

A *fractal* is an object exhibiting similar patterns at increasingly small scales. Thus, fractals use the idea of a detailed pattern that repeats itself.

In this section, we show that every Riordan graph $G_n(g,f)$ with $f'(0) = 1$ has fractal properties by using the notion of the A-sequence of a Riordan matrix.

Definition 3.3 Let G be a graph. A pair of vertices $\{k,t\}$ in G is a *cognate pair* with a pair of vertices $\{i,j\}$ in G if

- $|i-j| = |k-t|$ and
- i is adjacent to $j \Leftrightarrow k$ is adjacent to t.

The set of all cognate pairs of $\{i,j\}$ is denoted by $\mathrm{cog}(i,j)$.

Definition 3.4 The A-sequence of the binary Riordan matrix $\mathcal{B}(g,f)$ defining a Riordan graph $G_n(g,f)$ with $f'(0) = 1$ is called the *binary A-sequence of the graph.*

The following theorem gives a relationship between cognate pairs and the A-sequence of a Riordan graph.

Theorem 3.5 *For $\ell \geq 0$, let $A = (a_k)_{k\geq 0} = (1, \underbrace{0, \ldots, 0}_{\ell \text{ times}}, 1, a_{\ell+2}, \ldots)$ be the binary A-sequence for a Riordan graph $G_n(g,f)$ where $f \neq z$ and $f'(0) = 1$. Then*

$$\text{cog}(i,j) = \{\{i + m2^s, j + m2^s\} \mid i + m2^s,\ j + m2^s \in [n]\}$$

where $\left\lfloor \frac{|i-j|-1}{m2^s} \right\rfloor \leq \ell$ for integers m and $s \geq 0$.

Proof. Let $\mathcal{A}(G_n) = [r_{i,j}]_{1\leq i,j\leq n}$ be the adjacency matrix of $G_n(g,f)$. Without loss of generality, we may assume that $i > j \geq 1$. By Lemma 3.11, (3) and (4), we obtain

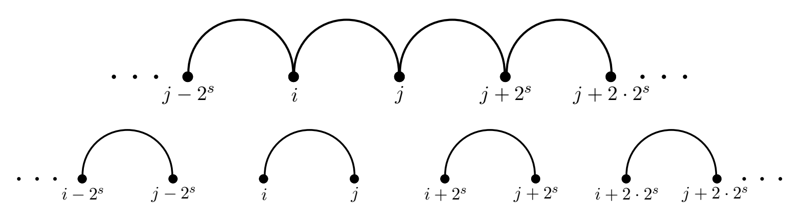
$$\begin{aligned} r_{i+m2^s,j+m2^s} &\equiv \left[z^{i+m2^s-2}\right] gf^{j+m2^s-1} = \left[z^{i+m2^s-2}\right] gf^{j-1}(zA(f))^{m2^s} \\ &\equiv \left[z^{i-2}\right] gf^{j-1}A(f^{m2^s}) = \left[z^{i-2}\right] \sum_{k\geq 0} a_k g f^{j-1+km2^s} \\ &\equiv \sum_{k=0}^{\alpha} a_k r_{i,j+km2^s}. \end{aligned} \tag{8}$$

where $\alpha = \max\{k \in \mathbb{N}_0 \mid 0 \leq km2^s \leq i-j-1\} = \lfloor (i-j-1)/m2^s \rfloor$, $m \neq 0$. Since $a_k = 0$ for $1 \leq k \leq \ell$ it follows that $\alpha \leq \ell$ if and only if

$$r_{i+m2^s,j+m2^s} \equiv r_{i,j}. \tag{9}$$

Thus we obtain the desired result. ■

In particular, if i is adjacent to j then the pairs cognate with $\{i,j\}$ are those connected by edges in the following figures:



for $|i - j| = 2^s$

for $|i - j| < 2^s$

The following theorem shows that every Riordan graph $G_n(g,f)$ with $f'(0) = 1$ is a fractal.

Theorem 3.6 *Let $A = (a_k)_{k\geq 0} = (1, \underbrace{0, \ldots, 0}_{\ell \text{ times}}, 1, a_{\ell+2}, a_{\ell+3}, \ldots)$, $\ell \geq 0$, be the binary A-sequence of a Riordan graph $G_n = G_n(g,f)$ with $f'(0) = 1$. For each $s \geq 0$ and $k \in \{0, \ldots, \ell\}$, G_n has the following fractal properties:*

(i) $\langle \{1, \ldots, (k+1)2^s + 1\} \rangle \cong \langle \{\alpha(k+1)2^s + 1, \ldots, (\alpha+1)(k+1)2^s + 1\} \rangle$

(ii) $\langle \{1, \ldots, (k+1)2^s\} \rangle \cong \langle \{\alpha(k+1)2^s + 1, \ldots, (\alpha+1)(k+1)2^s\} \rangle$

where $\alpha \geq 1$.

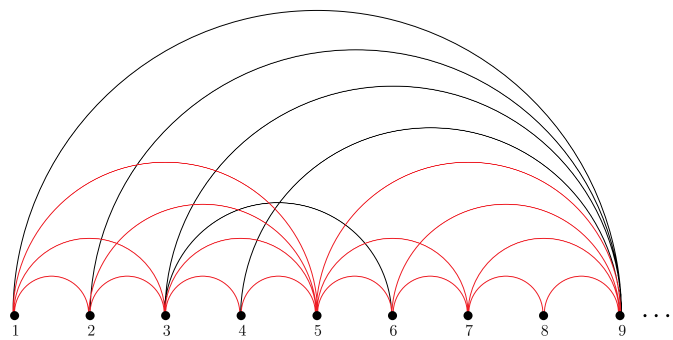


Figure 3: The Catalan graph $G_9(C, zC)$

Proof. Let $i, j \in \{1, \ldots, (k+1)2^s+1\} \in V(G_n)$ and $k = 0, \ldots, \ell$. Consider $m = \alpha(k+1)$ in Theorem 3.5. Since

$$\left\lfloor \frac{|i-j|-1}{\alpha(k+1)2^s} \right\rfloor \leq \left\lfloor \frac{(k+1)2^s-1}{(k+1)2^s} \right\rfloor = 0 \leq \ell,$$

it follows from Theorem 3.5 that

$$\{i + \alpha(k+1)2^s, j + \alpha(k+1)2^s\} \in \text{cog}(i, j). \tag{10}$$

Thus i is adjacent to j if and only if $i + \alpha(k+1)2^s$ is adjacent to $j + \alpha(k+1)2^s$. Hence we obtain (i). Similarly we obtain (ii). Hence the proof follows. ■

Example 3.7 Consider the Catalan graph $CG_n = G_n(C, zC)$ where $C = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function of the *Catalan numbers*. Since the binary A-sequence of CG_n is $(1, 1, \ldots)$, i.e. $\ell = 0$ so that $k = 0$, it follows from Theorem 3.6 that

$$CG_{2^s+1} \cong \langle \{1, \ldots, 2^s+1\} \rangle \cong \langle \{\alpha 2^s + 1, \ldots, (\alpha+1)2^s+1\} \rangle .$$

If $n = 9$ then we obtain $\langle \{1, 2, 3, 4, 5\} \rangle \cong \langle \{5, 6, 7, 8, 9\} \rangle$ for $s = 2$ and $\alpha = 1$. Figure 3 shows a way to draw CG_9 in a "fractal form". See Figure 4 for examples of adjacency matrices of Riordan graphs presented as fractals.

3.2 Reverse relabelling of Riordan graphs

Relabelling a Riordan graph does not necessarily result in a Riordan graph. However, if for a proper Riordan graph $G_n(g, f)$, the relabelling is done by *reversing* the vertices in $[n]$, that is, by replacing a label i by $n + 1 - i$ for each $i \in [n]$, then the resulting graph will always be a proper Riordan graph, as will be shown in this subsection.

Proposition 3.8 ([3]) *Let $(g, f)_n$ be an $n \times n$ leading principal matrix of a proper Riordan matrix (g, f). Then the flip-transpose $L_n^F = E(g, f)_n^T E$ is the proper Riordan matrix given by*

$$L_n^F = (g(\bar{f}) \cdot \bar{f}' \cdot (z/\bar{f})^n, \bar{f})_n$$

where E is the $n \times n$ backward identity matrix, i.e.

$$E = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Proposition 3.8 implies the following theorem.

Theorem 3.9 *The reverse relabelling of a Riordan graph $G_n(g,f)$ with $f'(0)=1$ is the Riordan graph*

$$G_n(g(\bar{f})\cdot\bar{f}'\cdot(z/\bar{f})^{n-1},\bar{f}).$$

Example 3.10 Consider the Catalan graph $CG_{2^j-1}=G_{2^j-1}(C,zC)$. Since the compositional inverse of $f=zC$ is $\bar{f}=z+z^2$ and $C(\bar{f})=\frac{1}{1-z}$, by Theorem 3.9 we obtain

$$\begin{aligned} g(\bar{f})\cdot\bar{f}'\cdot(z/\bar{f})^{2^j-2} &= \frac{1}{1-z}\cdot(1-2z)\cdot\left(\frac{1}{1-z}\right)^{2^j-2} \equiv \left(\frac{1}{1-z}\right)^{2^j-1} \\ &= \sum_{k\geq 0}\binom{2^j+k-2}{k}z^k. \end{aligned}$$

By Lucas theorem [9] asserting that if the base p (a prime) expansion of n is $n=n_0+n_1p+n_2p^2+\cdots$ then

$$\binom{n}{k}\equiv_p\prod_i\binom{n_i}{k_i},$$

we obtain

$$\binom{2^j+k-2}{k}\equiv 1 \text{ for } k=0,1 \quad\text{and}\quad \binom{2^j+k-2}{k}\equiv 0 \text{ for } 2\leq k\leq 2^j-1.$$

Hence the reverse relabelling of the Catalan graph CG_{2^j-1} is $G_{2^j-1}(1-z,z-z^2)$. We note that the Riordan matrix $(1-z,z-z^2)$ is the inverse matrix of (C,zC). Figure 4 illustrates, as fractals, the adjacency matrices of the two graphs.

3.3 Decomposition of Riordan graphs

Let $G=(V,E)$ be any simple graph of order n. There exists a permutation matrix P such that

$$P\mathcal{A}(G)P^T = \begin{pmatrix} \mathcal{A}(\langle V_1\rangle) & B \\ B^T & \mathcal{A}(\langle V_2\rangle) \end{pmatrix} \tag{11}$$

where V_1 and V_2 are nonempty disjoint subsets of V such that $V_1\bigcup V_2=V$. Theorem 3.12 gives a description for the adjacency matrix in the case of a Riordan graph where V_1 and V_2 are assumed to be the sets of odd and even labelled vertices, respectively.

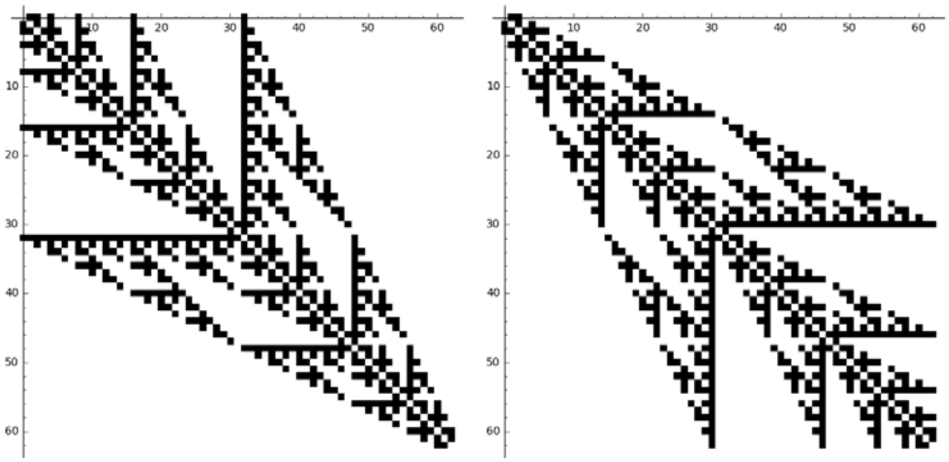


Figure 4: The fractal nature of $\mathcal{A}(CG_{63})$ (to the left) and $\mathcal{A}\left(G_{63}(1-z, z-z^2)\right)$ (to the right)

Lemma 3.11 *Let $g, f \in \mathbb{Z}[[z]]$. Then*

$$g^2(f) \equiv g(f^2).$$

Proof. Letting $g = g(z) := \sum_{n\geq 0} g_n z^n$, we obtain

$$\begin{aligned} g^2(f) &= \left(\sum_{n\geq 0} g_n f^n\right)^2 = \sum_{n\geq 0}\left(\sum_{i=0}^{n} g_i g_{n-i}\right) f^n \\ &= g_0^2 + 2g_0g_1 f + (2g_0f_1 + g_1^2)f^2 + (2g_0g_3 + 2g_1g_2)f^3 + \cdots \\ &\equiv \sum_{n\geq 0} g_n^2 f^{2n} \equiv \sum_{n\geq 0} g_n f^{2n} = g(f^2). \end{aligned}$$

■

Theorem 3.12 (Riordan Graph Decomposition) *Let $G_n = G_n(g, f)$ be a Riordan graph with $f'(0) = 1$. Then the following holds.*

(i) *The adjacency matrix $\mathcal{A}(G_n)$ satisfies*

$$\mathcal{A}(G_n) = P^T \begin{pmatrix} X & B \\ B^T & Y \end{pmatrix} P \qquad (12)$$

where $P = \left[e_1 \mid e_3 \mid \cdots \mid e_{2\lceil n/2\rceil -1} \mid e_2 \mid e_4 \mid \cdots \mid e_{2\lfloor n/2\rfloor}\right]^T$ is the $n\times n$ permutation matrix and e_i is the elementary column vector with the ith entry being 1 and the others entries being 0.

(ii) *The matrix X is the adjacency matrix of the induced subgraph of $G_n(g, f)$ by $V_o = \{2i-1 \mid 1 \leq i \leq \lceil n/2\rceil\}$. In particular, the induced subgraph $\langle V_o\rangle$ is isomorphic to a Riordan graph of order $\lceil n/2\rceil$ given by $G_{\lceil n/2\rceil}(g'(\sqrt{z}), f(z))$.*

(iii) *The matrix* Y *is the adjacency matrix of the induced subgraph of* $G_n(g,f)$ *by* $V_e = \{2i \mid 1 \le i \le \lfloor n/2 \rfloor\}$. *In particular, the induced subgraph* $\langle V_e \rangle$ *is isomorphic to a Riordan graph of order* $\lfloor n/2 \rfloor$ *given by* $G_{\lfloor n/2 \rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f(z)\right)$.

(iv) *The matrix* B *representing the edges between* V_o *and* V_e *can be expressed as the sum of binary Riordan matrices as follows:*

$$B = \mathcal{B}(z \cdot (gf)'(\sqrt{z}), f(z))_{\lceil n/2 \rceil \times \lfloor n/2 \rfloor} + \mathcal{B}((zg)'(\sqrt{z}), f(z))^T_{\lfloor n/2 \rfloor \times \lceil n/2 \rceil}.$$

Proof. (i) It is easy to see that $P\mathcal{A}(G_n)P^T$ is equal to the block matrix in (12), that is, $\mathcal{A}(G_n)$ is permutationally equivalent to the block matrix.

(ii)–(iii) Taking into account the form of P, clearly X is the adjacency matrix of the induced subgraph $\langle V_o \rangle$ of order $\lceil n/2 \rceil$ of the Riordan graph $G_n = G_n(g,f)$. Let $\mathcal{A}(G_n) = [r_{i,j}]_{1 \le i,j \le n}$. Since $f = zA(f)$ where $A(z) \in \mathbb{Z}[[z]]$ is the generating function of the A-sequence $a_0, a_1, \ldots$ for the Riordan matrix (g,f), it follows from Lemma 3.11 that for $i > j \ge 3$

$$\begin{aligned} r_{i,j} &\equiv \left[z^{i-2}\right] gf^{j-1} = \left[z^{i-2}\right] gf^{j-3}(zA(f))^2 \equiv \left[z^{i-4}\right] gf^{j-3}A\left(f^2\right) \\ &= \left[z^{i-4}\right]\left(a_0 gf^{j-3} + a_1 gf^{j-1} + a_2 gf^{j+1} + \cdots\right) \\ &= \sum_{k=0}^{\lfloor (i-j-1)/2 \rfloor} a_k r_{i-2,j-2+2k}. \end{aligned} \tag{13}$$

Since $g'(z) \equiv r_{3,1} + r_{5,1}z^2 + r_{7,1}z^4 + \cdots$ and

$$X = \begin{bmatrix} 0 & r_{3,1} & r_{5,1} & r_{7,1} & \cdots & r_{2\lceil n/2 \rceil -1,1} \\ r_{3,1} & 0 & r_{5,3} & r_{7,3} & \cdots & r_{2\lceil n/2 \rceil -1,3} \\ r_{5,1} & r_{5,3} & 0 & r_{7,5} & \cdots & r_{2\lceil n/2 \rceil -1,5} \\ r_{7,1} & r_{7,3} & r_{7,5} & 0 & \cdots & r_{2\lceil n/2 \rceil -1,7} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{2\lceil n/2 \rceil -1,1} & r_{2\lceil n/2 \rceil -1,3} & r_{2\lceil n/2 \rceil -1,5} & r_{2\lceil n/2 \rceil -1,7} & \cdots & 0 \end{bmatrix},$$

we have $X = \mathcal{A}\left(G_{\lceil n/2 \rceil}(g'(\sqrt{z}), f)\right)$ from (13). Similarly, Y is the adjacency matrix of the induced subgraph $\langle V_e \rangle$ of order $\lfloor n/2 \rfloor$ of G_n. In addition,

$$Y = \mathcal{A}\left(G_{\lfloor n/2 \rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f\right)\right).$$

(iv) Let

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ r_{3,2} & 0 & 0 & \cdots \\ r_{5,2} & r_{5,4} & 0 & \cdots \\ r_{7,2} & r_{7,4} & r_{7,6} & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} r_{2,1} & 0 & 0 & 0 & \cdots \\ r_{4,1} & r_{4,3} & 0 & 0 & \cdots \\ r_{6,1} & r_{6,3} & r_{6,5} & 0 & \cdots \\ r_{8,1} & r_{8,3} & r_{8,5} & r_{8,7} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $(gf)'(z) \equiv r_{3,2} + r_{5,2}z^2 + r_{7,2}z^4 + \cdots$ and $(zg)'(z) \equiv r_{2,1} + r_{4,1}z^2 + r_{6,1}z^4 + \cdots$, it follows from (13) that

$$B_1 = \mathcal{B}\left(z \cdot (gf)'(\sqrt{z}), f(z)\right) \quad \text{and} \quad B_2 = \mathcal{B}\left((zg)'(\sqrt{z}), f(z)\right). \tag{14}$$

Using (14) and

$$B = \begin{bmatrix} r_{1,2} & r_{1,4} & r_{1,6} & r_{1,8} & \cdots & r_{1,2\lfloor n/2\rfloor} \\ r_{3,2} & r_{3,4} & r_{3,6} & r_{3,8} & \cdots & r_{3,2\lfloor n/2\rfloor} \\ r_{5,2} & r_{5,4} & r_{5,6} & r_{5,8} & \cdots & r_{5,2\lfloor n/2\rfloor} \\ r_{7,2} & r_{7,4} & r_{7,6} & r_{7,8} & \cdots & r_{7,2\lfloor n/2\rfloor} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{2\lceil n/2\rceil-1,2} & r_{2\lceil n/2\rceil-1,4} & r_{2\lceil n/2\rceil-1,6} & r_{2\lceil n/2\rceil-1,8} & \cdots & r_{2\lceil n/2\rceil-1,2\lfloor n/2\rfloor} \end{bmatrix}$$
$$= (B_1 + B_2^T)_{\lceil n/2\rceil\times\lfloor n/2\rfloor},$$

where $(B_1 + B_2^T)_{\lceil n/2\rceil\times\lfloor n/2\rfloor}$ is the $\lceil n/2\rceil \times \lfloor n/2\rfloor$ leading principal matrix of $B_1 + B_2^T$, we obtain the desired result. ■

Definition 3.13 Let *o-decomposable Riordan graphs*, standing for *odd decomposable Riordan graphs*, be the class of graphs defined by requiring in (12) $Y = O$, where O is the zero matrix of Y's size. Also, let *e-decomposable Riordan graphs*, standing for *even decomposable Riordan graphs*, be the class of graphs defined by requiring in (12) $X = O$.

The parts (ii) and (iii) in the following theorem justify our choice for the names of o-decomposable and e-decomposable Riordan graphs.

Theorem 3.14 *Let $g \not\equiv 0$ and V_o and V_e be, respectively, the odd and even labelled vertex sets of a Riordan graph $G_n = G_n(g, f)$ with $f'(0) = 1$. Then*

(i) *For even n, $\langle V_o\rangle \cong \langle V_e\rangle$ if and only if $[z^{2m-1}]g \equiv [z^{2m}]gf$ for all $m \geq 1$.*

(ii) *The induced subgraph $\langle V_o\rangle$ is a null graph, i.e. G_n is e-decomposable if and only if $[z^{2m-1}]g \equiv 0$ for all $m \geq 1$, that is, g is an even function modulo 2.*

(iii) *The induced subgraph $\langle V_e\rangle$ is a null graph, i.e. G_n is o-decomposable if and only if $[z^{2m}]gf \equiv 0$ for all $m \geq 1$, that is, gf is an odd function modulo 2.*

(iv) *G_n is a bipartite graph with parts V_o and V_e if and only if $[z^{2m+1}]g \equiv [z^{2m}]f \equiv 0$ for all $m \geq 0$, that is, G_n is of the checkerboard type.*

(v) *There is no edge between a vertex $i \in V_o$ and a vertex $j \in V_e$ if and only if $[z^{2m}]g \equiv [z^{2m}]f \equiv 0$ for all $m \geq 0$, i.e. $G \cong \langle V_o\rangle \bigcup \langle V_e\rangle$.*

Proof. (i) Let n be even. From Theorem 3.12, $\langle V_o\rangle \cong \langle V_e\rangle$ if and only if the matrices X and Y in the block matrix in (12) are given by

$$\mathcal{A}\left(G_{n/2}\left(g'(\sqrt{z}), f\right)\right) = X \equiv Y = \mathcal{A}\left(G_{\lfloor n/2\rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f\right)\right) \quad \Leftrightarrow \quad g' \equiv \left(\frac{gf}{z}\right)'. \tag{15}$$

Since $[z^{2m-1}]h' = 2m[z^{2m}]h \equiv 0$ and $[z^{2m-2}]h' \equiv [z^{2m-1}]h$ for all $h \in \mathbb{Z}[[z]]$, the second equation in (15) is equivalent to

$$[z^{2m-1}]g \equiv [z^{2m-1}]\frac{gf}{z} \quad \Leftrightarrow \quad [z^{2m-1}]g \equiv [z^{2m}]gf \tag{16}$$

which proves (i).

(ii) From Theorem 3.12, the induced subgraph $\langle V_o \rangle$ is a null graph if and only if the matrix X in (12) is a zero matrix, i.e.

$$\mathcal{A}\left(G_{\lceil n/2 \rceil}(g'(\sqrt{z}), f(z))\right) = O \quad \Leftrightarrow \quad g' \equiv 0 \quad \Leftrightarrow \quad [z^{2m-1}]g \equiv 0 \text{ for all } m \geq 1. \qquad (17)$$

Hence the proof follows.

(iii) From Theorem 3.12, the induced subgraph $\langle V_e \rangle$ is a null graph if and only if the matrix Y in (12) is a zero matrix, i.e.

$$\mathcal{A}\left(G_{\lfloor n/2 \rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f(z)\right)\right) = O \;\; \Leftrightarrow \;\; \left(\frac{gf}{z}\right)' \equiv 0 \;\; \Leftrightarrow \;\; [z^{2m}]gf \equiv 0 \text{ for all } m \geq 1.$$

Hence the proof follows.

(iv) From Theorem 3.12, G_n is a bipartite graph with parts V_o and V_e if and only if the matrices X and Y in (12) are zero matrices, i.e.

$$\begin{aligned} & \mathcal{A}\left(G_{\lceil n/2 \rceil}(g'(\sqrt{z}), f(z))\right) = O \;\; \textit{and} \;\; \mathcal{A}\left(G_{\lfloor n/2 \rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f(z)\right)\right) = O \\ \Leftrightarrow \quad & g' \equiv 0 \;\text{ and }\; (gf/z)' \equiv 0 \quad \Leftrightarrow \quad g' \equiv 0 \;\; \textit{and} \;\; (f/z)' \equiv 0 \\ \Leftrightarrow \quad & [z^{2m+1}]g \equiv [z^{2m}]f \equiv 0 \text{ for all } m \geq 0. \end{aligned}$$

Hence the proof follows.

(v) Form Theorem 3.12, there is no edge between a vertex $i \in V_o$ and a vertex $j \in V_e$ if and only if the matrix B in (12) is a zero matrix, i.e.

$$\begin{aligned} & \mathcal{B}(z \cdot (gf)'(\sqrt{z}), f(z))_{\lceil n/2 \rceil \times \lfloor n/2 \rfloor} + \mathcal{B}((zg)'(\sqrt{z}), f(z))^T_{\lfloor n/2 \rfloor \times \lceil n/2 \rceil} = O \\ \Leftrightarrow \quad & (gf)'(\sqrt{z}) \equiv 0 \;\text{ and }\; (zg)'(\sqrt{z}) \equiv 0 \quad \Leftrightarrow \quad (zg)' \equiv 0 \;\; \textit{and} \;\; (f/z)' \equiv 0 \\ \Leftrightarrow \quad & [z^{2m}]g \equiv [z^{2m}]f \equiv 0 \text{ for all } m \geq 0. \end{aligned}$$

Hence the proof follows. ■

3.4 Eulerian and Hamiltonian Riordan graphs

In this section we give certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle. As the result, we obtain large classes of Riordan graphs which are Eulerian or Hamiltonian.

Theorem 3.15 *Let $G_n = G_n(g, f)$ be a Riordan graph with $f'(0) = 1$ and $n \geq 2$ vertices. Then the degree $d(i)$ of a vertex i is odd (resp., even) if and only if*

$$\phi_{i-1} + \varphi_{n-i} \equiv 1 \;\; \textit{(resp., 0)}$$

where for $j = 0, \ldots, n-1$

$$\phi_j \equiv [z^j]\frac{zg}{1-f} \quad \textit{and} \quad \varphi_j \equiv [z^j]\frac{zg(\bar{f}) \cdot (\bar{f})' \cdot (z/\bar{f})^{n-1}}{1-\bar{f}}.$$

Proof. From Proposition 3.8 let $\mathcal{B}(L_n)^F = E\mathcal{B}(zg,f)_n^T E$. Then we obtain $\mathcal{B}(L_n)^F = \mathcal{B}(zg(\bar{f})\cdot\bar{f}'\cdot(z/\bar{f})^{n-1},\bar{f})_n$. Now let $\mathcal{A}(G_n)$ be the $n\times n$ adjacency matrix of the graph $G_n(g,f)$ and let $\mathbf{d}=(d(1),\ldots,d(n))^T$ denote the vertex degree sequence. Applying the fundamental theorem given by (2) to $\mathcal{B}(zg,f)_n$ and $\mathcal{B}(L_n)^F$, respectively, we obtain

$$\begin{aligned}\mathbf{d} &= \mathcal{A}(G_n)\mathbf{1} = \left(\mathcal{B}(zg,f)_n + \mathcal{B}(zg,f)_n^T\right)\mathbf{1} = \left(\mathcal{B}(zg,f)_n + E\mathcal{B}(L_n)^F E\right)\mathbf{1} \\ &= \mathcal{B}(zg,f)_n\mathbf{1} + E\mathcal{B}(L_n)^F\mathbf{1} \\ &\equiv \begin{pmatrix}\phi_0\\ \phi_1\\ \vdots\\ \phi_{n-1}\end{pmatrix} + \begin{pmatrix}\varphi_{n-1}\\ \varphi_{n-2}\\ \vdots\\ \varphi_0\end{pmatrix} \equiv \begin{pmatrix}\phi_0+\varphi_{n-1}\\ \phi_1+\varphi_{n-2}\\ \vdots\\ \phi_{n-1}+\varphi_0\end{pmatrix}\end{aligned}$$

where $\mathbf{1}=(1,1,\ldots,1)^T\in\mathbb{Z}^n$ whose generating function is $\frac{1}{1-z}$. Thus the vertex degree $d(i)$ is odd (resp., even) if and only if for $i=1,\ldots,n$

$$\phi_{i-1}+\varphi_{n-i}\equiv 1 \ \ \text{(resp., 0)}$$

as required. ■

Since a simple graph G has an Eulerian trail if and only if G has no odd vertex or exactly two odd vertices, by Theorem 3.15 we obtain the following corollary.

Corollary 3.16 *A Riordan graph $G_n = G_n(g,f)$ with $f'(0)=1$ and $n\geq 2$ vertices has an Eulerian trail if and only if ϕ_{i-1} and φ_{n-i} satisfy either*

$$\phi_{i-1}+\varphi_{n-i}\equiv 0 \quad \textit{for all } i=1,\ldots,n, \tag{18}$$

or for $i_1\neq i_2$

$$\phi_{i-1}+\varphi_{n-i}\equiv\begin{cases}1 & \textit{if } i=i_1,i_2\in\{1,\ldots,n\}\\ 0 & \textit{if } i\in\{1,\ldots,n\}\setminus\{i_1,i_2\}\end{cases}.$$

In particular, if (18) *is satisfied then G_n is Eulerian.*

Example 3.17 The Pascal graph $PG_n = G_n\left(\frac{1}{1-z},\frac{z}{1-z}\right)$ for $n=2^k+1$ $(k\geq 2)$ has an Eulerian trail. Indeed, since $\bar{f}=\frac{z}{1+z}$, simple computations show that

$$\begin{aligned}\phi_{i-1} &\equiv [z^{i-1}]\frac{z}{1-2z}\equiv[z^{i-1}]z,\\ \varphi_{n-i} &\equiv [z^{n-i}]z(1+z)^{n-1} = \binom{n-1}{i}.\end{aligned}$$

By Lucas theorem, if $n=2^k+1$ and $i=2^k$ then $\binom{n-1}{i}\equiv 1$, and 0 otherwise. Thus we obtain

$$\phi_{i-1}+\varphi_{n-i}\equiv\begin{cases}1 & \text{if } i=2 \text{ or } i=2^k,\\ 0 & \text{if } i\in\{1,\ldots,2^k+1\}\setminus\{2,2^k\}.\end{cases}$$

By Corollary 3.16, PG_{2^k+1} has an Eulerian trail. Similarly, if $n=2^k$ for $k\geq 3$ then we can show that PG_n does not have an Eulerian trail.

A *palindromic* number sequence $(a_k)_{k=0}^n$ is a sequence that is the same when written forwards or backwards, i.e. $a_k = a_{n-k}$ for $k = 0, \dots, n$. For instance, $\left(\binom{n}{k}\right)_{k=0}^n$ is a palindromic sequence with the generating function $(1+z)^n$.

Corollary 3.18 *Let $G_n = G_n(g, z)$ be a Riordan graph of the Appell type with $f'(0) = 1$ and $n \geq 3$ vertices. If g is the generating function for a palindromic sequence $(g_k)_{k=0}^{n-2}$ with $g(1) \equiv 0$ then G_n is Eulerian.*

Proof. Let $g = \sum_{k=0}^{n-2} g_k z^k$ and $f = z$. Using Theorem 3.15 it can be easily shown that

$$\phi_j = \varphi_j \equiv [z^j]\frac{zg}{1-z} = \begin{cases} 0 & \text{if } j = 0 \\ \sum_{k=0}^{j-1} g_k & \text{if } j = 1, \dots, n-1. \end{cases}$$

Assume that $g_k \equiv g_{n-2-k}$ for $k = 0, \dots, n-2$ and $g(1) \equiv 0$. Then we obtain $\phi_0 = \phi_{n-1}$ and for $j = 1, \dots, n-2$,

$$\phi_j \equiv \sum_{k=0}^{j-1} g_k \equiv \sum_{k=0}^{j-1} g_{n-2-k} \equiv \sum_{k=j}^{n-2} g_{n-2-k} \equiv \phi_{n-1-j}.$$

Thus the sequence $(\phi_k)_{k=0}^{n-1}$ is palindromic. Since $\phi_k = \varphi_k$ it follows from Corollary 3.16 that G_n is Eulerian. ■

Hamiltonian properties of Toeplitz graphs, which are Riordan graphs of the Appell type, have been studied in [7]. Next we give more general results. Recall that every proper Riordan graph $G_n = G_n(g, f)$ has the Hamiltonian path $1 \to 2 \to \cdots \to n$. In particular, if $g = \frac{1}{1-z}$ then G_n is Hamiltonian, i.e. G_n has a Hamiltonian cycle.

Theorem 3.19 *Let $G_n = G_n(g, f)$ be a proper Riordan graph. If one of the following holds then G_n is Hamiltonian.*

(i) *There exists $i \in \{2, \dots, n-1\}$ such that $[z^{i-1}]g \equiv 1$ and $[z^{n-2}](gf^{i-1}) \equiv 1$.*

(ii) *$[z]g \equiv 1$ and $[z^2]f \equiv 0$.*

Proof. (i) Since G_n is proper, we have the path $1 \to 2 \to \cdots \to i$. Since $[z^{n-2}](gf^{i-1}) \equiv 1$, we have the edge $in \in E(G_n)$. Again, since G_n is proper, we have the path $n \to n-1 \to \cdots \to i+1$. Finally, $(i+1)1 \in E(G_n)$ since $[z^{i-1}]g \equiv 1$. Thus we have the following Hamiltonian cycle in G_n:

$$1 \to 2 \to \cdots \to i \to n \to n-1 \to \cdots \to i+1 \to 1$$

as required.

(ii) Since G_n is proper, by Theorem 3.12 and the assumption $[z]g \equiv 1$ we obtain that $\mathcal{A}(\langle V_o \rangle)$ is proper, i.e. we have the path $1 \to 3 \to \cdots \to 2\left\lceil \frac{n}{2} \right\rceil - 1$. Our assumption that $[z^2]f \equiv 0$ implies $[z^2](gf) \equiv 1$, and by Theorem 3.12 we obtain that $\mathcal{A}(\langle V_e \rangle)$ is proper, i.e. we have the path $2 \to 4 \to \cdots \to 2\left\lfloor \frac{n}{2} \right\rfloor$. Finally, we obtain the following Hamiltonian cycle in G_n

$$1 \to 3 \to \cdots \to 2\left\lceil \frac{n}{2} \right\rceil - 1 \to 2\left\lfloor \frac{n}{2} \right\rfloor \to \cdots \to 4 \to 2 \to 1$$

as required. ■

Example 3.20 Recall that the Catalan graph CG_n is the Riordan graph $G_n(C, zC)$ where $C = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\geq 0} C_n z^n$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. CG_n is Hamiltonian for $n = 2^k + 1 \geq 3$. Indeed, from [9] we have $C_n = [z^n]C \equiv 1$ if and only if $n = 2^k - 1$ for $k \geq 1$. Further, consider the case of $i = n - 1$ in (i) of Theorem 3.19. If $n = 2^k + 1$ then we obtain

$$[z^{n-2}]C = [z^{2^k-1}]C \equiv 1 \text{ and } [z^{n-2}](C(zC)^{n-2}) = [z^0]C^{n-1} \equiv 1.$$

Thus the Catalan graph CG_{2^k+1} is Hamiltonian.

The following result is obtained by Theorem 3.1 in [9] and Proposition 1 in [1].

Lemma 3.21 *The Motzkin number M_n is even if and only if either $n \in S_1$ or $n \in S_2$ where $S_1 = \{4^i(2j-1) - 2 \mid i, j \geq 1\}$ and $S_2 = \{4^i(2j-1) - 1 \mid i, j \geq 1\}$.*

Theorem 3.22 *If $n \neq 4^i(2j-1)$ for $i, j \geq 1$ then the Motzkin graph $MG_n = G_n(M, zM)$ is Hamiltonian for $n \geq 3$.*

Proof. Consider the generating function M of the Motzkin numbers M_n:

$$M = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2} = \sum_{n\geq 0} M_n z^n = 1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 51z^6 + \cdots .$$

Note that $S_1 \cap S_2 = \emptyset$ in Lemma 3.21. Next consider the case of $i = n - 1$ in (i) of Theorem 3.19. By Lemma 3.21 we obtain

$$[z^{n-2}]M = M_{n-2} \equiv 1 \text{ if and only if } n \notin S_3 \cup S_4$$

where $S_3 := \{4^i(2j-1) \mid i, j \geq 1\}$ and $S_4 := \{4^i(2j-1) + 1 \mid i, j \geq 1\}$. In addition, $[z^{n-2}](M(zM)^{n-2}) = [z^0]M^{n-1} \equiv 1$. Thus by Theorem 3.19, if $n \notin S_3 \cup S_4$ then MG_n is Hamiltonian.

It remains to prove that if $n \in S_4$ then MG_n is Hamiltonian. To show this, consider the case of $i = 2$ in (i) of Theorem 3.19. Since

$$[z^{n-2}]zM^2(z) \equiv [z^{n-2}]zM(z^2) = [z^{n-2}]\sum_{k\geq 0} M_k z^{2k+1}$$

it follows from $n - 2 = 2k + 1$ that if $n \in S_4$ then

$$k \in \{4^{i-1}(4j-2) - 1 \mid i, j \geq 1\} = \{1, 5, 7, 9, 23, \ldots\}. \tag{19}$$

Since $k \notin S_1 \cup S_2$ for any k in (19), we have $M_k \equiv 1$ by Lemma 3.21 so that $[z^{n-2}]zM^2(z) \equiv 1$. In addition, $[z]M = M_1 \equiv 1$. Thus by Theorem 3.19, if $n \in S_4$ then MG_n is Hamiltonian as required. ■

A complete split graph $CS_{\omega, n-\omega}$, $\omega \leq n$, is a graph on n vertices consisting of a clique K_ω on ω vertices and a stable set (i.e. independent set) on the remaining $n - \omega$ vertices, such that any vertex in the clique is adjacent to each vertex in the stable set. The bipartite graph $G(\omega, n-\omega)$ obtained from $CS_{\omega, n-\omega}$ by deleting all edges in K_ω is referred to as the bipartite graph corresponding to $CS_{\omega, n-\omega}$.

Lemma 3.23 ([2]) *In a split graph $CS_{\omega,n-\omega}$, if $\omega < n-\omega$ then $CS_{\omega,n-\omega}$ is not Hamiltonian. If $\omega = n-\omega$ then $CS_{\omega,n-\omega}$ is Hamiltonian if and only if the corresponding bipartite graph $G(\omega,\, n-\omega)$ is Hamiltonian.*

Theorem 3.24 *If $G_n = G_n(g,f)$ is an improper Riordan graph with $[z]f = f_1 \equiv 0$ then G_n is not Hamiltonian.*

Proof. If f satisfies $[z]f = f_1 \equiv 0$ then for any g the Riordan graph $G_n(g,f)$ is a subgraph of $G_n(\frac{1}{1-z}, z^2)$. Thus it is sufficient to prove the claim for $G_n(\frac{1}{1-z}, z^2)$.

Let $\mathcal{A}(G_n) = [a_{i,j}]_{1\le i,j\le n}$ be the adjacency matrix of G_n, and let $C_j(z)$ be the j-th column generating function of $\mathcal{B}(zg,f)$ where $g = \frac{1}{1-z}$ and $f = z^2$. We may assume that $i > j$. Then $C_{\lfloor n/2\rfloor+1}(z) = g\cdot z^{2\lfloor n/2\rfloor}$, i.e. $a_{i,j} = 0$ if $j \ge \lfloor n/2\rfloor + 1$. Thus the subgraph of G_n induced by $\langle\{\lfloor n/2\rfloor+1,\ldots,n\}\rangle$ is a null graph. Clearly, G_n is a subgraph of the complete split graph $CS_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. If n is odd then by Lemma 3.23 $CS_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is not Hamiltonian. It follows that G_n is not Hamiltonian. Otherwise, n is even. Again, by Lemma 3.23, $CS_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is Hamiltonian if the corresponding bipartite graph $G(\lfloor n/2\rfloor,\lceil n/2\rceil)$ is Hamiltonian. But in this case $C_{\lfloor n/2\rfloor}(z) = g\,z^{n-2}$. Now if $g_0 \equiv 1$, then $a_{n/2,(n-2)} \equiv 1$ and otherwise if $g_0 \equiv 0$ then $a_{n/2,(n-2)} \equiv 0$. Thus the vertex $n/2$ has maximum degree 1 in the graph $G(\lfloor n/2\rfloor,\lceil n/2\rceil)$, i.e. $G(\lfloor n/2\rfloor,\lceil n/2\rceil)$ is not Hamiltonian. Again, by Lemma 3.23, $CS_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ has no Hamiltonian cycle so that G_n is not Hamiltonian. ■

Proposition 3.25 *Let $G_n = G_n(g,f)$ be an e-decomposable/o-decomposable Riordan graph and n be even. Then G_n is Hamiltonian if and only if the bipartite graph with partitions V_o and V_e is Hamiltonian. Moreover, if G_n is an e-decomposable Riordan graph and n is odd then G_n is not Hamiltonian.*

Proof. Let G_n be an e-decomposable/o-decomposable Riordan graph and n be even. In this case, G_n contains an independent set on $n/2$ vertices and so G_n is a subgraph of the complete split graph $CS_{n/2,n/2}$. By Lemma 3.23, $CS_{n/2,n/2}$ is Hamiltonian if and only if the corresponding bipartite graph $G(n/2,\, n/2)$ is Hamiltonian.

If G_n is an e-decomposable Riordan graph and n is odd, then G_n is a subgraph of the complete split graph $CS_{(n-1)/2,\,(n+1)/2}$. Again, by Lemma 3.23 we obtain the desired result. ■

We end this section by observing that every Riordan graph $G_n = G_n(g,f)$ of the checkerboard type of odd order n is not Hamiltonian since G_n is bipartite of odd order by (iv) in Theorem 3.14.

4 Four families of Riordan graphs

In this section, we consider io-decomposable and ie-decomposable Riordan graphs, and also Riordan graphs of the Bell type and of the derivative type.

4.1 io-decomposable and ie-decomposable Riordan graphs

Definition 4.1 Let $G_n = G_n(g,f)$ be a proper Riordan graph with the odd and even vertex sets V_o and V_e, respectively.

- If $\langle V_o\rangle \cong G_{\lceil n/2\rceil}(g,f)$ and $\langle V_e\rangle$ is a null graph then G_n is *io-decomposable*.
- If $\langle V_o\rangle$ is a null graph and $\langle V_e\rangle \cong G_{\lfloor n/2\rfloor}(g,f)$ then G_n is *ie-decomposable*.

"io" and "ie" stand for "isomorphically odd" and "isomorphically even", respectively.

Theorem 4.2 *Let $G_n = G_n(g,f)$ be a proper Riordan graph.*

(i) *G_n is io-decomposable if and only if*

$$g' \equiv g^2 \quad \text{and} \quad gf \equiv z\cdot (f/z)'.$$

(ii) *G_n is ie-decomposable if and only if*

$$g' \equiv 0 \quad \text{and} \quad z^2 g \equiv zf' + f.$$

Proof. (i) Since $g(z^2)\equiv g^2(z)$ by Lemma 3.11, by the definitions and Theorem 3.12, G_n is io-decomposable if and only if the matrix X in (12) is given by

$$\mathcal{A}\left(G_{\lceil n/2\rceil}(g'(\sqrt{z}), f(z))\right) = \mathcal{A}\left(G_{\lceil n/2\rceil}(g(z), f(z))\right) \quad \Leftrightarrow \quad g'(z) = g(z^2) \equiv g^2(z)$$

and

$$\begin{aligned} G_{\lfloor n/2\rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f(z)\right) \cong N_{\lfloor n/2\rfloor} \quad &\Leftrightarrow \quad 0 \equiv \left(\frac{gf}{z}\right)' \equiv g^2\frac{f}{z} + g\cdot\left(\frac{f}{z}\right)' \\ &\Leftrightarrow \quad gf \equiv z\cdot (f/z)'. \end{aligned}$$

(ii) Since $g(z^2)\equiv g^2(z)$ by Lemma 3.11, by the definitions and Theorem 3.12, G_n is ie-decomposable if and only if the matrix X in (12) is given by

$$G_{\lceil n/2\rceil}(g'(\sqrt{z}), f(z)) \cong N_{\lceil n/2\rceil} \quad \Leftrightarrow \quad g'(z)\equiv 0$$

and

$$\begin{aligned} &\mathcal{A}\left(G_{\lfloor n/2\rfloor}\left(\left(\frac{gf}{z}\right)'(\sqrt{z}), f(z)\right)\right) = \mathcal{A}\left(G_{\lfloor n/2\rfloor}(g(z), f(z))\right) \\ \Leftrightarrow \quad & g^2 \equiv \left(\frac{gf}{z}\right)' \equiv g'\frac{f}{z} + g\cdot\left(\frac{f}{z}\right)' \equiv g\cdot\left(\frac{f}{z}\right)' \\ \Leftrightarrow \quad & z^2 g \equiv zf' + f. \end{aligned}$$

■

4.2 Riordan graphs of the Bell type

Consider a Riordan graph $G_n(g, zg)$ of Bell type with odd and even vertex sets V_o and V_e, respectively. By Theorems 3.12 and 4.3, its adjacency matrix is permutationally equivalent to

$$\begin{pmatrix} X & B \\ B^T & O \end{pmatrix}$$

where X is the adjacency matrix of $\langle V_o\rangle = G_{\lceil n/2\rceil}(g'(\sqrt{z}), zg(z))$ and

$$B = \mathcal{B}(zg, zg)_{\lceil n/2\rceil,\lfloor n/2\rfloor} + \mathcal{B}((zg)'(\sqrt{z}), zg))^T_{\lfloor n/2\rfloor,\lceil n/2\rceil}. \tag{20}$$

Theorem 4.3 *Any Riordan graph $G_n = G_n(g, zg)$ of the Bell type is o-decomposable.*

Proof. Since for $m \geq 1$

$$[z^{2m}]z \cdot g^2(z) = [z^{2m-1}]g^2(z) \equiv [z^{2m-1}]g(z^2) \equiv 0,$$

by (iii) of Theorem 3.14 G_n is o-decomposable. ■

Now, we consider io-decomposable Riordan graphs of the Bell type. We first derive conditions on the A-sequences of such graphs.

Lemma 4.4 *A Riordan graph $G_n(g, zg)$ is io-decomposable if and only if*

$$g^2 \equiv g', \ \textit{i.e.}\ [z^j]g \equiv [z^{2j+1}]g.$$

Proof. Since $\langle V_e \rangle$ is a null graph, by Theorem 4.2 $G_n(g, zg)$ is io-decomposable if and only if $g' \equiv g^2$. ■

Remark 4.5 *Let $[z^j]g = g_j$ with $g_0 = 1$. By Lemma 4.4, a Riordan graph $G_n(g, zg)$ is io-decomposable if and only if $g_{2m} \equiv g_{(2m+1)2^k-1}$ for each $m \geq 0$ and $k \geq 1$, i.e. the generating function of g is*

$$g = \sum_{n\geq 0} g_{2n} \left(\sum_{k\geq 0} z^{(2n+1)2^k-1} \right).$$

Since g depends only on its even coefficients, the number of io-decomposable Riordan graphs $RG_n(g, zg)$ is given by $2^{\lfloor n/2 \rfloor -1}$.

We note that a binary Riordan matrix $[r_{i,j}]_{i,j\geq 0}$ is of the Bell type given by $\mathcal{B}(g, zg)$ if and only if, for $i \geq j \geq 0$,

$$\begin{aligned} r_{i+1,0} &= a_1 r_{i,0} + a_2 r_{i,1} + \cdots + a_{i+1} r_{i,i}, \\ r_{i+1,j+1} &= r_{i,j} + a_1 r_{i,j+1} + \cdots + a_{i-j} r_{i,i} \end{aligned}$$

where $(1, a_1, \ldots)$ is the binary A-sequence of $\mathcal{B}(g, zg)$.

Theorem 4.6 *A Riordan graph $G_n(g, zg)$ is io-decomposable if and only if its binary A-sequence is*

$$(1, 1, a_2, a_2, a_4, a_4, \ldots)$$

where $a_{2j} \in \{0, 1\}$ for all $j \geq 1$, i.e. its A-sequence generating function is

$$A(z) \equiv (1 + z) + (1 + z) \sum_{j\geq 1} a_{2j} z^{2j}.$$

Proof. Let $G_n(g, zg)$ be io-decomposable. Since there is a unique generating function $A = \sum_{i\geq 0} a_i z^i$ such that $g = A(zg)$, by applying derivative to both sides, we obtain

$$g' \equiv (g + zg') \cdot A'(zg).$$

Since $g^2 \equiv g'$ by Lemma 4.4, it implies

$$g \equiv (1+zg) \cdot A'(zg), \text{ i.e. } A(zg) \equiv (1+zg) \cdot A'(zg). \tag{21}$$

Since $A'(z) \equiv \sum_{i \geq 0} a_{2i+1} z^{2i}$, the equation (21) is equivalent to

$$\begin{aligned} \sum_{j \geq 0} a_j (zg)^j &\equiv (1+zg) \left(\sum_{i \geq 0} a_{2i+1} (zg)^{2i} \right) \\ &= a_1 + a_1 zg + a_3 (zg)^2 + a_3 (zg)^3 + a_5 (zg)^4 + a_5 (zg)^5 + \cdots . \end{aligned}$$

Thus $a_{2i} \equiv a_{2i+1}$ for $i \geq 0$ as desired. ■

Corollary 4.7 *Let $A(z)$ be a generating function of the binary A-sequence for $\mathcal{B}(g, zg)$ where $g(0) = 1$. For any $s \in \mathbb{N}_0$ and $t \in \mathbb{Z}$, we have the following.*

(i) *If $A(z) \equiv \sum_{j=0}^{2s+1} z^n$ or $A(z) \equiv (1+z)^{2t-1}$ then $G_n(g, zg)$ is io-decomposable.*

(ii) *If $A(z) \equiv \sum_{j=0}^{2s} z^n$ or $A(z) \equiv (1+z)^{2t}$ then $G_n(g, zg)$ is not io-decomposable.*

Proof. (i) If $A(z) \equiv \sum_{j=0}^{2s+1} z^n$ then clearly $G_n(g, zg)$ is io-decomposable by Theorem 4.6. Now let $A(z) = (1+z)^{2t-1}$. Since (g, zg) is of the Bell type, we obtain

$$g \equiv A(zg) = (1+zg)^{2t-1}.$$

Applying derivative to both sides and using $2t - 1 \equiv 1$, we obtain

$$g' \equiv (g + zg')(1+zg)^{2t-2} = \frac{g + zg'}{1+zg} (1+zg)^{2t-1} \equiv \frac{g+zg'}{1+zg} g.$$

It implies that

$$g'(1+zg) \equiv (g+zg')g, \text{ i.e. } g' \equiv g^2.$$

By Lemma 4.4, $G_n(g, zg)$ is io-decomposable.

(ii) If $A \equiv \sum_{j=0}^{2s} z^n$ then clearly $G_n(g, zg)$ is not io-decomposable by Theorem 4.6. Now let $A = (1+z)^{2t}$. Using $g = A(zg) \equiv (1+zg)^{2t}$ we obtain $g' \equiv 0$. Since $g^2(z) \equiv g(z^2) \not\equiv 0$ we have $g' \not\equiv g^2$. Thus $G_n(g, zg)$ is not io-decomposable. ■

Example 4.8 Note that the Pascal graph PG_n, the Catalan graph CG_n and the Motzkin graph MG_n have the generating functions $1+z$, $(1+z)^{-1}$ and $1+z+z^2$ for the A-sequences, respectively. By Corollary 4.7 we see that PG_n and CG_n are io-decomposable, but MG_n is not io-decomposable.

Lemma 4.9 *If $G_n = G_n(g, zg)$ is io-decomposable then*

$$m(G_n) = 2m(G_{\lceil n/2 \rceil}) + m(H_{\lfloor n/2 \rfloor + 1})$$

where $H_{\lfloor n/2 \rfloor + 1} \cong G_{\lfloor n/2 \rfloor + 1} \left((zg)'(\sqrt{z}), zg \right)$.

Proof. Let $\sigma(A_n)$ denote the number of 1s in the adjacency matrix $A_n = \mathcal{A}(G_n)$. Since G_n is io-decomposable, by (20) we obtain

$$\sigma(A_n) = \sigma(A_{\lceil n/2 \rceil}) + 2\sigma(B_1) + 2\sigma(B_2) \tag{22}$$

where $B_1 = \mathcal{B}((zg)'(\sqrt{z}), zg)_{\lfloor n/2 \rfloor \times \lceil n/2 \rceil}$ and $B_2 = \mathcal{B}(zg, zg)_{\lceil n/2 \rceil \times \lfloor n/2 \rfloor}$. We can see that

$$2\sigma(B_1) = \sigma\left(\mathcal{A}\left(H_{\lfloor n/2 \rfloor + 1}\right)\right) \text{ and } 2\sigma(B_2) = \sigma(A_{\lceil n/2 \rceil}).$$

Applying this to (22), we obtain

$$\sigma(A_n) = 2\sigma(A_{\lceil n/2 \rceil}) + \sigma\left(\mathcal{A}\left(H_{\lfloor n/2 \rfloor + 1}\right)\right)$$

which implies the desired result. ■

Since $d(n) = m(G_n) - m(G_{n-1})$, by Lemma 4.9 we obtain the following lemma.

Lemma 4.10 *If a Riordan graph $G_n = G_n(g, zg)$ is io-decomposable then*

(i) $d_{G_n}(n) = 2\left\{m\left(G_{\frac{n+1}{2}}\right) - m\left(G_{\frac{n-1}{2}}\right)\right\} = 2\, d_{G_{\frac{n+1}{2}}}\left(\frac{n+1}{2}\right)$ *if n is odd*

(ii) $m\left(H_{\frac{n}{2}+1}\right) - m\left(H_{\frac{n}{2}}\right) = d_{H_{\frac{n}{2}+1}}\left(\frac{n}{2}+1\right)$ *if n is even*

where $m(G_0) = 0$ and $H_j \cong G_j((zg)'(\sqrt{z}), zg)$.

Example 4.11 (a) Consider the Catalan graph. Since by using the functional equation $1 + zC^2 = C^2$ we obtain

$$(zC)' = \frac{d}{dz}\left(z + (zC)^2\right) \equiv 1,$$

it follows from Lemmas 4.9 and 4.10 that we obtain the following:

$$m(CG_n) = 2\, m\left(CG_{\lceil n/2 \rceil}\right) + m\left(CG_{\lfloor n/2 \rfloor}\right) + 1.$$

Equivalently, we obtain

$$m(CG_n) = \begin{cases} 2\, m\left(CG_{\frac{n+1}{2}}\right) + m\left(CG_{\frac{n-1}{2}}\right) + 1 & \text{if } n \text{ is odd} \\ 3\, m\left(CG_{\frac{n}{2}}\right) + 1 & \text{if } n \text{ is even.} \end{cases}$$

This recurrence relation gives

$$m\left(CG_{2^k}\right) = \frac{3^k - 1}{2} \quad \text{and} \quad m\left(CG_{2^k+1}\right) = \frac{3^k - 1}{2} + 2^k.$$

(b) Let $PG_n = G_n\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ be the Pascal graph. Since $(zg)' = \left(\frac{z}{1-z}\right)' \equiv \frac{1}{1-z^2}$, by Lemma 4.9 we obtain

$$m(PG_n) = 2m(PG_{\lceil n/2 \rceil}) + m(PG_{\lfloor n/2 \rfloor + 1}).$$

This recurrence relation gives

$$m(PG_{2^k+1}) = 3^k \quad \text{and} \quad m(PG_{2^k}) = 3^k - 2^k$$

which also known as in [16].

Definition 4.12 A vertex in a graph G is *universal* (or *apex*, or *dominating vertex*) if it is adjacent to all other vertices in G.

It is known [8] that the Pascal graphs PG_n, which are io-decomposable by Example 4.8, have two or three universal vertices for $n \geq 2$. The following theorem gives a result in this direction for any io-decomposable Riordan graphs of the Bell type with $2^i + 1$ or $2^i + 2$ vertices.

Theorem 4.13 *Let $G_n = G_n(g, zg)$ be io-decomposable. If $n = 2^i + 1$ for $i \geq 0$, then G_n and G_{n+1} have at least one universal vertex, namely the vertex $2^i + 1$.*

Proof. Let $n = 2^i + 1$. It is enough to show that $d_{G_n}(n) = 2^i$. We prove this by induction on $i \geq 0$. Let $i = 0$. Since $g(0) = 1$, $d_{G_2}(2) = 1$. Thus it holds for $i = 0$. Let $i \geq 1$. Then we obtain

$$\begin{aligned} d_{G_n}(n) &= 2\{m(G_{2^{i-1}+1}) - m(G_{2^{i-1}})\} \qquad \text{(by Lemma 4.10)} \\ &= 2\{2m(G_{2^{i-2}+1}) + m(H_{2^{i-2}+1}) - 2m(G_{2^{i-2}}) - m(H_{2^{i-2}+1})\} \quad \text{(by Lemma 4.9)} \\ &= 2^2\{m(G_{2^{i-2}+1}) - m(G_{2^{i-2}})\} \\ &= 2\, d_{G_{2^{i-1}+1}}(2^{i-1}+1) \qquad \text{(by Lemma 4.10)} \\ &= 2^i \qquad \text{(by the induction hypothesis).} \end{aligned}$$

Thus if $n = 2^i + 1$ then the vertex n is a universal vertex of G_n. In addition, two vertices n and $n+1$ are adjacent in G_{n+1} since G_{n+1} is proper. Thus the vertex n is also a universal vertex of G_{n+1} if $n = 2^i + 1$. ■

Theorem 4.14 *An io-decomposable Riordan graph $G_n(g, zg)$ is $(\lceil \log_2 n \rceil + 1)$-partite.*

Proof. We proceed by induction on $n \geq 2$. Let $n = 2$. Since $G_2(g, zg)$ is clearly bi-partite, the theorem holds for $n = 2$. Let $n \geq 3$. Since $\langle V_o \rangle \cong G_{\lceil n/2 \rceil}(g, zg)$ is io-decomposable, and $\langle V_e \rangle$ is a null graph, by the induction hypothesis, $G_n(g, zg)$ is the $(\lceil \log_2 \lceil n/2 \rceil \rceil + 2)$-partite graph.

Now it is enough to show that $\lceil \log_2 \lceil n/2 \rceil \rceil = \lceil \log_2 n \rceil - 1$. For all $k \geq 0$, when $2^k < n \leq 2^{k+1}$ we have

$$\lceil \log_2 \lceil n/2 \rceil \rceil = k = \lceil \log_2 n \rceil - 1.$$

Hence we obtain the desired result. ■

Remark 4.15 We note that if $G_n(g, zg)$ is io-decomposable then it is $(\lceil \log_2 n \rceil + 1)$-partite with partitions $V_1, V_2, \ldots, V_{\lceil \log_2 n \rceil + 1}$ such that

$$V_j = \left\{ 2^{j-1} + 1 + i2^j \mid 0 \leq i \leq \left\lfloor \frac{n - 1 - 2^{j-1}}{2^j} \right\rfloor \right\} \quad \text{if } 1 \leq j \leq \lceil \log_2 n \rceil$$

and $V_{\lceil \log_2 n \rceil + 1} = \{1\}$.

Definition 4.16 A *clique* is a subset of vertices of a graph G such that its induced subgraph is a complete graph. The *clique number* of G is the number of vertices in a maximum clique in G, and it is denoted by $\omega(G)$.

Theorem 4.17 *For $n \geq 1$, if $G_n = G_n(g, zg)$ is io-decomposable then*

$$\omega(G_n) = \lceil \log_2 n \rceil + 1.$$

Proof. It follows from Theorem 4.14 that

$$\omega(G_n) \leq \lceil \log_2 n \rceil + 1. \tag{23}$$

Let $\tilde{V} = \{1\} \cup \{2^i + 1 \mid 0 \leq i \leq \lceil \log_2 n \rceil - 1\} \subseteq V(G_n)$. By Theorem 4.13, for every i, $0 \leq i \leq \lceil \log_2 n \rceil$, the vertex $2^i + 1 \in \tilde{V}$ is adjacent to all vertices in $\{1\} \cup \{2^j + 1 \mid 0 \leq j \leq i - 1\}$. Thus the induced subgraph $\left\langle \tilde{V} \right\rangle$ is

$$\left\langle \tilde{V} \right\rangle = K_{\lceil \log_2 n \rceil + 1}. \tag{24}$$

By (23) and (24), we obtain the desired result. ■

Since the complete graph K_5 is not planar, from Theorem 4.17 we immediately obtain the following corollary.

Corollary 4.18 *An io-decomposable graph $G_n(g, zg)$ is not planar for all $n \geq 9$.*

It is known [8] that the Pascal graph PG_n is planar for $n \leq 7$ but it is not for $n \geq 8$. Also we may check that the Catalan graph CG_n is planar for $n \leq 8$ but it is not for $n \geq 9$.

Definition 4.19 The *chromatic number* of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, and it is denoted by $\chi(G)$.

Theorem 4.20 *For $n \geq 1$, if a Riordan graph $G_n(g, zg)$ is io-decomposable then*

$$\chi(G_n) = \lceil \log_2 n \rceil + 1.$$

Proof. Since a k-partite graph is k-colorable and $\chi(G_n) \geq \omega(G_n)$, we obtain the desired result by Theorems 4.14 and 4.17. ■

Definition 4.21 The *distance* between two vertices u, v in a graph G is the number of edges in a shortest path between u and v, and it is denoted by $d(u, v)$. The *diameter* of G is the maximum distance between all pairs of vertices, and it is denoted by $\mathrm{diam}(G)$.

It is obvious that if G has a universal vertex then $\mathrm{diam}(G) = 1$ or 2.

Theorem 4.22 *If a Riordan graph $G_n = G_n(g, zg)$ is io-decomposable then*

$$\mathrm{diam}(G_n) \leq \lfloor \log_2 n \rfloor. \tag{25}$$

In particular, if $n = 2^k + 2$ or $2^{k+1} + 1$, for $k \geq 1$, then

$$\mathrm{diam}(G_n) = 2.$$

Proof. We proceed by induction on $n \geq 2$. Let $n = 2$. Since clearly $\text{diam}(G_2) = 1 \leq \lfloor \log_2 2 \rfloor$, the statement is true for $n = 2$. Suppose $n \geq 3$. Let $V_1 = \{i \mid 1 \leq i \leq 2^b + 1\}$ and $V_2 = \{i \mid 2^b + 1 \leq i \leq n\}$ where b is an integer such that $2^b < n \leq 2^{b+1}$. By Theorem 4.13, $2^b + 1$ is a universal vertex in the induced subgraph $\langle V_1 \rangle$. Thus, $\text{diam}(\langle V_1 \rangle) \leq 2$. From Corollary 3.6 and Lemma 4.6, we obtain

$$\langle V_2 \rangle \cong G_{n-2^b}.$$

By the induction hypothesis, we have

$$\text{diam}(\langle V_2 \rangle) \leq \lfloor \log_2 (n - 2^b) \rfloor \leq \lfloor \log_2 n \rfloor - 1. \tag{26}$$

Let $u \in V_1 \backslash \{2^b + 1\}$ and $v \in V_2 \backslash \{2^b + 1\}$. Now it is enough to show that $d(u, v) \leq \lfloor \log_2 n \rfloor$. Since $2^b + 1$ is an universal vertex of the induced subgraph $\langle V_1 \rangle$, it follows from (26) that

$$d(u, v) \leq d(u, 2^b + 1) + d(2^b + 1, v) \leq \lfloor \log_2 n \rfloor$$

which proves (25).

Let $n = 2^k + 2$ or $2^{k+1} + 1$ for $k \geq 1$. By Theorem 4.13, every io-decomposable Riordan graph $G_n = G_n(g, zg)$ has at least one universal vertex. Thus $\text{diam}(G_n)$ is 1 or 2. Now it is enough to show that G_n is not a complete graph for $n \geq 4$. Let $[r_{i,j}]_{1 \leq i,j \leq n}$ be the adjacency matrix of G_n. Since $r_{4,2} \equiv [z^2] z g^2(z) \equiv [z] g(z^2) = 0$, G_n cannot be K_n for $n \geq 4$. Hence we obtain the desired result. ■

Corollary 4.23 *Let $n \geq 6$ and a Riordan graph $G_n = G_n(g, zg)$ be io-decomposable. If $2^k + 1 < n < 2^{k+1}$ then*

$$\text{diam}(G_n) \leq \lfloor \log_2 (n - 2^k) \rfloor + 1. \tag{27}$$

Proof. Let $V_1 = \{i \mid 1 \leq i \leq 2^k + 1\}$ and $V_2 = \{i \mid 2^k + 1 \leq i \leq n\}$. By Corollary 3.6 and Lemma 4.6, we obtain

$$\langle V_2 \rangle \cong G_{n-2^b}.$$

Thus, by (25)

$$\text{diam}(G_{n-2^b}) \leq \lfloor \log_2 (n - 2^k) \rfloor. \tag{28}$$

Let $u \in V_1 \backslash \{2^k + 1\}$ and $v \in V_2 \backslash \{2^k + 1\}$. Since the vertex $2^k + 1$ is a universal vertex in the induced subgraph $\langle V_1 \rangle$, it follows from (28) that

$$d(u, v) \leq d(u, 2^k + 1) + d(2^k + 1, v) \leq \lfloor \log_2 (n - 2^k) \rfloor + 1,$$

which completes the proof. ■

4.3 Riordan graphs of the derivative type

Consider a Riordan graph $G_n(f', f)$ of the derivative type. By Theorems 3.12 and 4.24, it is permutationally equivalent to

$$\begin{pmatrix} O & B \\ B^T & Y \end{pmatrix}$$

where Y is the adjacency matrix of $\langle V_e \rangle = G_{\lfloor n/2 \rfloor}\left((f'f/z)'(\sqrt{z}), f(z)\right)$ and

$$B = \mathcal{B}(zf'(z), f(z))_{\lceil n/2 \rceil \times \lfloor n/2 \rfloor} + \mathcal{B}(f'(\sqrt{z}), f(z))^T_{\lfloor n/2 \rfloor \times \lceil n/2 \rceil}. \tag{29}$$

Theorem 4.24 *Any Riordan graph $G_n = G_n(f', f)$ of the derivative type is e-decomposable.*

Proof. Let $f = \sum_{i \geq 1} f_i z^i$. Since for $m \geq 1$

$$[z^{2m-1}]f' = [z^{2m-1}]\left(\sum_{i \geq 1} i f_i z^{i-1}\right) = 2m f_{2m} \equiv 0,$$

by (ii) in Theorem 3.14 G_n is o-decomposable. ■

Now, we turn our attention to ie-decomposable Riordan graphs of the derivative type.

Lemma 4.25 *A Riordan graph $G_n = G_n(f', f)$ is ie-decomposable if and only if*

$$(z + z^2)f' \equiv f, \text{ i.e. } [z^{2m-1}]f \equiv [z^{2m}]f \text{ for all } m \geq 1.$$

Proof. Since $\langle V_o \rangle$ in G_n is a null graph and $f'' \equiv 0$ for all $f \in \mathbb{Z}[[z]]$, by Theorem 4.2 G_n is ie-decomposable if and only if $(z + z^2)f' \equiv f$. ■

Theorem 4.26 *A Riordan graph $G_n = G_n(f', f)$ is ie-decomposable if and only if its binary A-sequence is of the form*

$$(1, 1, a_2, 0, a_4, 0, a_6, 0, \ldots)$$

where a_{2i} is 0 or 1 for $i \geq 1$, i.e. $A'(z) \equiv 1$.

Proof. Since $f = zA(f)$ it follows from Lemma 4.25 that G_n is ie-decomposable if and only if $(1 + z)f' \equiv f/z = A(f)$. Applying derivative to both sides of $f = zA(f)$ we have

$$f' = A(f) + zf'A'(f) \equiv (1 + z)f' + zf'A'(f).$$

After simplification we obtain $A'(f) \equiv 1$, i.e. $A'(z) \equiv 1$. Since G_n is proper, it follows that G_n is ie-decomposable if and only if $A(z) \equiv 1 + z + \sum_{i \geq 1} a_{2i} z^{2i}$ where $a_{2i} \in \{0, 1\}$. ■

Example 4.27 *Consider $G_n = G_n\left(\frac{1}{1+z^2}, \frac{z}{1+z}\right)$. Then $A(z) = 1 + z$. Since $A'(z) = 1$, it follows from Theorem 4.26 that G_n is ie-decomposable. For instance, if $n = 9$ then*

$$P \left(\begin{array}{cccc|ccccc} 0&1&0&1&0&1&0&1&0 \\ 1&0&1&1&0&0&1&1&0 \\ 0&1&0&1&0&0&0&1&0 \\ 1&1&1&0&1&1&1&1&0 \\ \hline 0&0&0&1&0&1&0&1&0 \\ 1&0&0&1&1&0&1&1&0 \\ 0&1&0&1&0&1&0&1&0 \\ 1&1&1&1&1&1&1&0&1 \\ 0&0&0&0&0&0&0&1&0 \end{array}\right) P^T = \left(\begin{array}{ccccc|cccc} 0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&1&1&0&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1 \\ \hline 1&1&0&1&0&0&1&0&1 \\ 1&1&1&1&0&1&0&1&1 \\ 1&0&1&1&0&0&1&0&1 \\ 1&1&1&1&1&1&1&1&0 \end{array}\right)$$

where $P = [e_1 \mid e_3 \mid \cdots \mid e_9 \mid e_2 \mid e_4 \mid \cdots \mid e_8]^T$.

Theorem 4.28 *Let* $G_n = G_n(f', f)$ *be ie-decomposable. Then* G_n *is* $(\lfloor \log_2 n \rfloor + 1)$*-partite for* $n \geq 1$.

Proof. We proceed by induction on n. Since G_1 is a single vertex and G_2 is bi-partite, the theorem holds for $n = 1, 2$. Assume $n \geq 3$. Since $\langle V_e \rangle \cong G_{\lfloor n/2 \rfloor}$ is ie-decomposable and $\langle V_o \rangle$ is a null graph, by the induction hypothesis G_n is $(\lfloor \log_2 \lfloor n/2 \rfloor \rfloor + 2)$-partite.

Now it is enough to show that $\lfloor \log_2 \lfloor n/2 \rfloor \rfloor = \lfloor \log_2 n \rfloor - 1$. If $2^k \leq n < 2^{k+1}$ for all $k \geq 1$ then

$$\lfloor \log_2 \lfloor n/2 \rfloor \rfloor = k - 1 = \lfloor \log_2 n \rfloor - 1.$$

Thus we obtain the desired result. ■

5 Concluding remarks and open problems

In this paper, we use the notion of a Riordan matrix to introduce the notion of a Riordan graph, and based on it, to introduce the notion of an *unlabelled* Riordan graph. The studies conducted by us are aimed at *structural properties* of (various classes of) Riordan graphs; *spectral properties* of Riordan graphs are studied by us in the follow up paper [4].

Even though our paper establishes a number of fundamental structural results, many more such results are yet to be discovered. In particular, we would like to extend our results on graph properties for Riordan graphs of the Bell type to other families of Riordan graphs. Other specific problems we would like to be solved are as follows.

Problem 1 Characterize *unlabelled* Riordan graphs.

Problem 2 Enumerate *unlabelled* Riordan graphs.

Problem 3 Characterize Riordan graphs whose complements are Riordan in labelled and unlabelled cases. See Section 2.5 for relevant observations.

Problem 4 What is the complexity of recognizing labelled/unlabelled Riordan graphs?

Problem 5 Characterize Riordan graphs in terms of forbidden subgraphs, or otherwise.

Problem 6 Find graph invariants not considered in this paper for io-decomposable Riordan graphs of the Bell type, e.g. the *independence number*, *Wiener index*, *average path length*, and so on.

Let G_n be an io-decomposable Riordan graph of the Bell type. Then one can check that $\mathrm{diam}(G_1) = 0$ and $\mathrm{diam}(G_n) = 1$ for $n = 2, 3$. The following conjecture shows significance of the Pascal graphs PG_n and the Catalan graphs CG_n.

Conjecture 1 Let G_n be an io-decomposable Riordan graph of the Bell type. Then

$$2 = \mathrm{diam}(PG_n) \leq \mathrm{diam}(G_n) \leq \mathrm{diam}(CG_n)$$

for $n \geq 4$. Moreover, PG_n is the only graph in the class of io-decomposable graphs of the Bell type whose diameter is 2 for *all* $n \geq 4$.

Conjecture 2 We have that $diam(CG_{2^k}) = k$ and there are no io-decomposable Riordan graphs $G_{2^k} \ncong CG_{2^k}$ of the Bell type satisfying $diam(G_{2^k}) = k$ for all $k \geq 1$.

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