On the representation number of a crown graph

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Abstract

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. It is known [9] that any word-representable graph G is k-word-representable for some k, that is, there exists a word w representing G such that each letter occurs exactly k times in w. The minimum such k is called G's representation number.

A crown graph (also known as a cocktail party graph) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. In this paper, we show that for $n \geq 5$, $H_{n,n}$'s representation number is $\lceil n/2 \rceil$. This result not only provides a complete solution to the open Problem 7.4.2 in [8], but also gives a negative answer to the question raised in Problem 7.2.7 in [8] on 3-word-representability of bipartite graphs. As a byproduct, we obtain a new example of a graph class with a high representation number.

Keywords: word-representable graph, crown graph, cocktail party graph, representation number

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1 Introduction

Suppose that w is a word over some alphabet and x and y are two distinct letters in w. We say that x and y *alternate* in w if after deleting in w all letters but the copies of x and y we either obtain a word $xyxy\cdots$ (of even or odd length) or a word $yxyx\cdots$ (of even or odd length).

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. For example, the cycle graph on 4 vertices labeled by 1, 2, 3 and 4 in clockwise direction can be represented by the word 14213243.

There is a long line of research on word-representable graphs, which is summarised in the recently published book [8]. The roots of the theory of word-representable graphs are in the study of the celebrated Perkins semigroup [10, 13] which has played a central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. However, the most interesting aspect of word-representable graphs from an algebraic point of view seems to be the notion of a semi-transitive orientation [6], which generalizes partial orders. It was shown in [6] that a graph is word-representable if and only if it admits a semi-transitive orientation.

More motivation points to study word representable graphs include the fact exposed in [8] that these graphs generalize several important classes of graphs such as *circle graphs* [4], 3-*colourable graphs* [1] and *comparability graphs* [12]. Relevance of word-representable graphs to scheduling problems was explained in [6] and it was based on [5]. Furthermore, the study of word-representable graphs is interesting from an algorithmic point of view as was explained in [8]. For example, the *Maximum Clique problem* is polynomial time solvable on word-representable graphs [8] while this problem is generally NP-complete [3]. Finally, word-representable graphs is an important class among other graph classes considered in the literature that are defined using words. Examples of other such classes of graphs are *polygon-circle graphs* [11] and *word-digraphs* [2].

It was shown in [9] that if a graph G is word-representable then it is k-*word-representable* for some k, that is, G can be represented by a k-*uniform* word w, i.e. a word containing k copies of each letter. In such a context we say that w k-*represents* G. For example, the cycle graph on 4 vertices mentioned above can be 2-represented by the word 14213243.

Thus, when discussing word-representability, one need only consider k-uniform words. A nice property of such words is that any cyclic shift of a k-uniform word represents the same graph [9]. (If a word $w = uv$ for two non-empty words u and v, then the word vu is a cyclic shift of w.) The minimum k for which a word-representable graph G is k-word-representable is called the G's *representation number*.

The following observation follows trivially from the definitions.

Observation 1. *The class of complete graphs coincides with the class of* 1*-word-representable graphs. In particular, the complete graph's representation number is* 1*.*

A less trivial, but still simple fact mentioned in [6] is that the class of 2-word-representable graphs is precisely the class of circle graphs [4], which are defined by intersecting chords. Circle graphs were generalized to polygon-circle graphs [11], where edges are defined by intersecting inscribed k-gons for a fixed k. Note that except for the case of $k = 2$, such graphs are *not* the same as k-word-representable graphs. Indeed, in the case of k-word-representable graphs, $k \geq 3$, we have an edge xy if and only if x and y alternate, while in the case of polygon-circle graphs defined by intersecting inscribed k-gons, xy is an edge if and only if no cyclic shift of the subword induced by x and y, when reading the labels of the polygon corners around the circle in either direction, is $x \cdots xy \cdots y$.

1.1 Representation of crown graphs

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y. A bipartite graph is *complete* if every vertex in X is connected to each vertex in Y. $K_{n,m}$ denotes the complete bipartite graph with the disjoint sets of sizes n and m, respectively. A *crown graph* (also known as a *cocktail party graph*) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. Formally, $V(H_{n,n}) = \{1, \ldots, n, 1', \ldots, n'\}$ and $E(H_{n,n}) = \{ij' \mid i \neq j\}$. First four examples of such graphs are presented in Figure 1.

Crown graphs are of special importance in the theory of word-representable graphs. More precisely, they appear in the construction of graphs requiring long words representing them [6]. Note that these graphs also appear in the theory of partially ordered sets as those

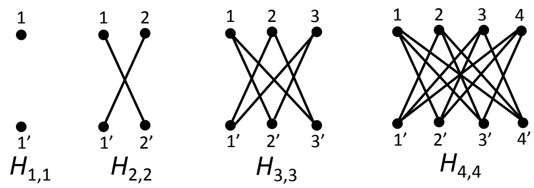

Figure 1: The crown graph $H_{n,n}$ for $n = 1, 2, 3, 4$

n	representation of $H_{n,n}$ by concatenation of n permutations
1	$11'1'1$
2	$12'21'21'12'$
3	$123'32'1'132'23'1'231'13'2'$
4	$1234'43'2'1'1243'34'2'1'1342'24'3'1'2341'14'3'2'$

Table 1: Representing $H_{n,n}$ as a concatenation of n permutations

defining partial orders that require many linear orders to be represented.

Each crown graph, being a bipartite graph, is a *comparability graph* (that is, a transitively orientable graph), and thus it can be represented by a concatenation of permutations [10]. Moreover, it follows from [6], and also is discussed in Section 7.4 in [8], that $H_{n,n}$ can be represented as a concatenation of n permutations but it cannot be represented as a concatenation of a fewer permutations. These results on crown graphs were obtained by exploiting the idea of representation of a poset as an intersection of several linear orders. Thus, the representation number of $H_{n,n}$ is at most n. See Table 1 (appearing in [8]) for the words representing the graphs in Figure 1 as concatenation of permutations.

It was noticed in [7] that, for example, $H_{3,3}$ can be represented using two copies of each letter as $3'32'1'132'23'1'231'1$ (as opposed to three copies used in Table 1 to represent it) if we drop the requirement to represent crown graphs as concatenation of permutations. On the other hand, $H_{4,4}$ is the three-dimensional cube, which is the *prism graph* Pr_4, so that $H_{4,4}$ is 3-word-representable by Proposition 15 in [9], while four copies of each letter are used in Table 1 to represent this graph. Note that $H_{4,4}$ is not 2-word-representable by Theorem 18 in [7].

These observations led to Problem 7.4.2 on page 172 in [8] essentially asking to find the representation number of a crown graph $H_{n,n}$. A relevant Problem 7.2.7 on page 169 in [8] asks whether each bipartite graph is 3-word-representable. When we started to investigate these problems, we established that both $H_{5,5}$ and $H_{6,6}$ are 3-word-representable, which not only suggested that the representation number of a crown graph could be the constant 3, but also that any bipartite graph could be 3-word-representable since crown graphs seem to be the most difficult among them to be represented.

In this paper we completely solve the former problem (Problem 7.4.2) and provide the negative answer to the question in the latest problem (Problem 7.2.7) by showing that if $n \geq 5$ then the crown graph $H_{n,n}$, being a bipartite graph, is $\lceil n/2 \rceil$-representable (see Theorem 5). Thus, crown graphs are another example of a graph class with high representation number. Note that non-bipartite graphs obtained from crown graphs by adding an all-adjacent vertex (i.e. apex) require roughly twice as long words representing them (see Section 4.2.1 in [8]).

1.2 Organization of the paper and some definitions

The paper is organized as follows. In Section 2, we find a lower bound for the representation number of $H_{n,n}$, while in Section 3 we provide a construction of words representing $H_{n,n}$ that match our lower bound. Finally, in Section 4 we provide some concluding remarks including directions for further research.

We conclude the introduction with a number of technical definitions to be used in the paper.

A *factor* of a word is a number of consecutive letters in the word. For example, the set of all factors of the word 1132 of length at most 2 is $\{1, 2, 3, 11, 13, 32\}$. A *subword* of a word is a subsequence of letters in the word. For instance, 56, 5212 and 361 are examples of subwords in 3526162. The subword of a word w *induced* by a set A is obtained by removing all elements in w not belonging to A. For example, if $A = \{2, 4, 5\}$ then the subword of 223141565 induced by A is 22455.

For a vertex v in a graph G denote by $N(v)$ the neighbourhood of v, i.e. the set of vertices adjacent to v. Clearly, if a graph is bipartite then the neighbourhood of each vertex induces an independent set, that is, no pair of vertices in the neighbourhood is connected by an edge.

2 A lower bound for the representation number of $H_{n,n}$

For a (sub)word w, let $l(w)$ and $r(w)$ be its first and last letters, respectively.

Let w be a word that k-represents a graph $G = (V, E)$. A subset $A \subseteq V$ is *splittable* if there is a cyclic shift of the word w such that the subword induced by the set A has the form $P_1 \cdots P_k$ where each P_i is a permutation of A. For a splittable set A, a *canonical shift* of w, with respect to A, is a cyclic shift of w that puts $l(P_1)$ at the beginning of the word. Note that any permutation P_i can play the role of P_1, since word-representation is invariant under cyclic shift; however, if P_1 is fixed, then there is a unique canonical shift. The following proposition gives an example of a splittable set.

Proposition 1. *Let G be a word-representable graph and let $v \in V(G)$. Then, $N(v)$ is splittable.*

Proof. Consider a cyclic shift of a word w k-representing G that puts v at the beginning of the word. Then between any two occurrences of v (and after the last one) each letter from $N(v)$ occurs exactly once, i.e. the subword induced by $N(v)$ is a concatenation of permutations. Hence, $N(v)$ is splittable. □

For a letter x, denote by x_i its i-th occurrence in a word w (from left to right). We write $x_i < y_j$ if the i-th occurrence of x is to the left of the j-th occurrence of y in w. Clearly, if A is splittable, then in the canonical shift for every $a, b \in A$ and for all i, j such that $1 \leq i < j \leq k$, we have $a_i < b_j$.

Lemma 2. *Let w be a word k-representing G and let $A \subseteq V(G)$ be a splittable set. Furthermore, let $a, b \in A$, $x \notin A$ and $ax, bx \in E(G)$. If in a canonical shift of w we have $a_1 < x_1 < b_1$ then $ab \in E(G)$.*

Proof. Let $a_1 < x_1 < b_1$. Since A is splittable, $b_i < a_{i+1}$ for each i. Since both a and b are adjacent to x, we have $a_i < x_i < b_i$ for every $i = 1, \ldots, k$. Therefore, a and b alternate in w and must be adjacent in G. □

Lemma 3. *If $n \geq 5$ then in any word w k-representing $H_{n,n}$ the set $A = \{1, \ldots, n\}$ is splittable.*

Proof. By Proposition 1, the set $B := N(1') = \{2, \ldots, n\}$ is splittable, i.e. there is a cyclic shift of w in which the letters of B form the subword $P'_1 \cdots P'_k$, where P'_i is a permutation of B. Let a canonical shift of w with respect to B be $w' = P_1 I_1 \cdots P_k I_k$, where for $i = 1, \ldots, k$, the factor P_i begins at $l(P'_i)$ and ends at $r(P'_i)$, and I_is are (possibly empty) factors lying between $r(P'_i)$ and $l(P'_{i+1})$. We begin by proving the following claim.

Claim 1. For every $t \geq 1$ and $i \geq 1$ such that $i+t-1 \leq k$, the factor $U = P_i I_i \cdots I_{i+t-2} P_{i+t-1}$ of w' contains at most t copies of the letter 1.

Proof of Claim 1. Indeed, suppose not. Using a cyclic shift of w if necessary, without loss of generality we can assume that in a problematic case $i = 1$. First consider the case $t = 1$. That is, we assume that P_1 contains at least two 1s. Let $a = l(P_1), b = r(P_1)$ and $x \in V \setminus \{1, a, b, 1', 2', \ldots, n'\}$. Recall that a, b belong to the splittable set B. Then the letter x' occurs exactly once between any two consecutive occurrences of 1, in particular, between the first two occurrences. Hence we have $a_1 < x'_1 < b_1$. Since both a and b are adjacent to x', it follows from Lemma 2 that $ab \in E$, contradiction.

Now let $t \geq 2$. Let $a = l(P_1), b = r(P_1), c = r(P_t)$ and $x \in V \setminus \{1, a, b, c, 1', 2', \ldots, n'\}$ (recall that $n \geq 5$), and suppose that there are at least $t+1$ occurrences of 1 between a and c. Note that $a \neq b$, but it is possible that $a = c$ or $a = b$. Since $1x' \in E$, there must be at least t occurrences of x' between a and c. By Lemma 2, no x' can appear between a_1 and b_1. However, c appears exactly once between a_1 and b_1 (possibly coinciding with one of them) because P_1 contains the permutation P'_1 over B as a subword. Moreover, there are exactly t occurrences of c in U. Therefore, the subword of U induced by c and x' starts and ends with c and contains at least t copies of x'. Clearly, such subword cannot be alternating, which contradicts $cx' \in E$. Claim 1 is proved. □

It follows from Claim 1 that each $P_i I_i$ contains at most two 1s, since $P_i I_i P_{i+1}$ contains at most two 1s, for $1 \leq i \leq k-1$. If each of $P_i I_i, 1 \leq i \leq k-1$ contains exactly one 1 then add 1 to each P'_i to obtain the concatenation of permutations for the set A showing that it is splittable. Otherwise, for some $1 \leq i \leq k-1$ the factor $P_i I_i$ must contain exactly two 1s. Without loss of generality, we can assume that $i = 1$ (otherwise, we can apply a cyclic shift and rename the permutations). By Claim 1 applied to P_1 and $P_1I_1P_2$, at least one of 1s must be in I_1 and P_2 contains no 1s. So, add the first occurrence of 1 to P'_1 and the second one to the beginning of P'_2. If I_2 contains no 1s we apply the same arguments to the word obtained from w by removing the factor $P_1I_1P_2I_2$. Otherwise, again by Claim 1, applied to $P_1I_1P_2I_2P_3$, I_2 has one 1, P_3 has no 1 and we add this 1 to the beginning of P'_3 and continue in the same way showing that w contains as a subword a concatenation of permutations over A. Hence, A is splittable. □

Lemma 4. *Let $n \geq 5$, w be k-representing $H_{n,n}$ and let $P'_1 \cdots P'_k$ be a subword of (a cyclic shift of) w that is a concatenation of permutations over $A = \{1, \ldots, n\}$. Then, for every $a \in A$ there is $j \in \{1, \ldots, k\}$ such that $a = l(P'_j)$ or $a = r(P'_j)$.*

Proof. First, observe that such a cyclic shift of w exists since A is splittable by Lemma 3. Suppose on the contrary that the letter 1 is never the first or the last letter of any permutation P'_j. Consider a canonical shift of w for the set A and define the subwords P_i and I_i for permutations P'_i in the same way as in the proof of Lemma 3. Since $l(P_1) \neq 1$ and $r(P_1) \neq 1$ by Lemma 2, no $1'$ can appear between $l(P_1)$ and $r(P_1)$. This is true for any P_i since we can apply a cyclic shift and rename P_i and P_1. Moreover, no I_i can have two or more $1'$s, or no $1'$s at all, because otherwise $1'$ would not be adjacent to the vertices in $\{2, \ldots, n\}$.

But since each P_j for $j = 1, \ldots, k$ contains one 1, the letters 1 and $1'$ alternate in w, i.e. the vertices 1 and $1'$ must be adjacent, contradicting the definition of $H_{n,n}$. □

Theorem 5. *For $n \geq 1$, the representation number of $H_{n,n}$ is at least $\lceil n/2 \rceil$.*

Proof. We consider three cases.

- If $n = 1, 2$, then the statement is trivial since each graph requires at least one copy of each letter to be represented.
- If $n = 3, 4$, then the statement is true by Observation 1 since none of $H_{n,n}$'s is a complete graph.
- If $n \geq 5$, since, by Lemma 3, the set $A = \{1, \ldots, n\}$ is splittable, and, by Lemma 4, each of its n letters must be the first or the last letter of some permutation P'_j for $j = 1, \ldots, k$, we have that $2k \geq n$. Since k is an integer, we obtain the bound $k \geq \lceil n/2 \rceil$.

□

3 An upper bound for the representation number of $H_{n,n}$

In this section, we provide a construction that shows that the bound in Theorem 5 is tight for all n except $n = 1, 2, 4$. We need the following auxiliary lemma.

Lemma 6. *If $n = 2k \geq 6$ then for every partition of the set $A = \{1, \ldots, 2k\}$ into k pairs $(a_1, b_1), \ldots, (a_k, b_k)$ there exist permutations $P(a_1, b_1), \ldots, P(a_k, b_k)$ such that:*

1. *$l(P(a_i, b_i)) = a_i, r(P(a_i, b_i)) = b_i$ for each $i = 1, \ldots, k$, and*
2. *for every $x, y \in A$ there exist i, j such that $x < y$ in $P(a_i, b_i)$ and $y < x$ in $P(a_j, b_j)$.*

Proof. Let P be an arbitrary permutation over the set $A \backslash \{a_1, a_2, b_1, b_2\}$, $Rev(P)$ be obtained from P by writing it in the reverse order, P' be an arbitrary permutation over the set $A \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and, for each $i = 4, \ldots, k$, let P_i be an arbitrary permutation over the set $A \setminus \{a_i, b_i\}$. Define the sought permutations as follows: $P(a_1, b_1) = a_1 b_2 P a_2 b_1$, $P(a_2, b_2) = a_2 b_1 Rev(P) a_1 b_2$, $P(a_3, b_3) = a_3 b_2 a_1 P' b_1 a_2 b_3$ and $P(a_i, b_i) = a_i P_i b_i$ for each $i = 4, \ldots, k$. It is straightforward to verify that both requirements of the lemma hold for these permutations. □

Theorem 7. *If $n \geq 5$ then the crown graph $H_{n,n}$ is $\lceil n/2 \rceil$-representable.*

Proof. It is sufficient to prove the theorem only for $n = 2k$, $k \geq 3$, because the case of $n = 2k - 1$ is obtained from the case of $n = 2k$ by removing all occurrences of the letters $2k$ and $(2k)'$ from the respective word. First, consider the following k-uniform word, where the permutations $P(x, y)$s are defined in Lemma 6 and $P(x', y')$s are obtained from these by adding primes.

$$w' = P(1, 2)P(2', 3')P(3, 4)P(4', 5') \cdots P(n - 1, n)P(n', 1').$$

It follows from item 2 in Lemma 6 that w' represents the complete bipartite graph $K_{n,n}$. Shift w' cyclicly one position to the left to obtain the word w'' where for every even i there is exactly one occurrence of the factor ii' and for every odd i there is exactly one occurrence of the factor $i'i$. Let w be the word obtained from w'' by switching i and i' in each of these factors. This operation makes the subword induced by i and i' non-alternating (thus removing the edges ii' in $K_{n,n}$) but does not affect any other alternations in the word. Therefore, w k-represents $H_{n,n}$, as desired. □

Note that for $n < 4$ the graph $H_{n,n}$ is 2-word-representable, which is given by the words $w_1 = 11'1'1$, $w_2 = 12'21'21'12'$ and $w_3 = 12'3'123'1'231'2'3$, respectively (see pages 172 and 173 in [8]). As for $n = 4$, note that $H_{4,4}$ is the three-dimensional cube, which is the prism graph Pr_4. Thus, $H_{4,4}$ is 3-word-representable by Proposition 15 in [9] and it is not 2-word-representable by Theorem 18 in [7]. An example of 3-representation of $H_{4,4}$ given on page 90 in [8] is

$$414'343'231'12'24'1'3'44'2'33'11'22'.$$

4 Concluding remarks

In this paper, we found the representation number of any crown graph $H_{n,n}$ solving two open problems in [8]. $H_{n,n}$ has the largest known representation number among bipartite graphs on $2n$ vertices. We propose the following conjecture.

Conjecture 1. *Each bipartite graph on n vertices has representation number at most $n/4$.*

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