Enumerating $(\mathbf{2}+\mathbf{2})$-free posets by the number of minimal elements and other statistics

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Abstract

An unlabeled poset is said to be $(\mathbf{2}+\mathbf{2})$-free if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. Let p_n denote the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n. In a recent paper, Bousquet-Mélou et al. [1] found, using so called ascent sequences, the generating function for the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n: $P(t)=\sum_{n\geq 0} p_n t^n = \sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-t)^i\right)$. We extend this result in two ways. First, we find the generating function for $(\mathbf{2}+\mathbf{2})$-free posets when four statistics are taken into account, one of which is the number of minimal elements in a poset. Second, we show that if $p_{n,k}$ equals the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements, then $P(t,z)=\sum_{n,k\geq 0} p_{n,k} t^n z^k = 1+\sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i)$. The second result cannot be derived from the first one by a substitution. On the other hand, $P(t)$ can easily be obtained from $P(t,z)$ thus providing an alternative proof for the enumeration result in [1]. Moreover, we conjecture a simpler form of writing $P(t,z)$. Our enumeration results are extended to certain restricted permutations and to regular linearized chord diagrams through bijections in [1, 2]. Finally, we define a subset of ascent sequences counted by the Catalan numbers and we discuss its relations with $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets.

1 Introduction

An unlabeled poset is said to be $(\mathbf{2}+\mathbf{2})$-*free* if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. We let $\mathcal{P}$ (resp. $\mathcal{P}_n$) denote the set of $(\mathbf{2}+\mathbf{2})$-free posets (resp. on n elements). Fishburn [7] showed that a poset is $(\mathbf{2}+\mathbf{2})$-free precisely when it is isomorphic to an *interval order*. Another important characterization of $(\mathbf{2}+\mathbf{2})$-free posets, set [5, 6, 10], is that a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if the collection of strict principal down-sets can be linearly ordered by inclusion. Here for any poset $\mathbf{P}=(P,<_p)$ and $x\in P$, the strict principal down set of x, $D(x)$, in $\mathbf{P}$ is the set of all $y\in P$ such that

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$y <_p x$. The *trivial down-set* is the empty set. Thus if $\mathbf{P}$ is a $(\mathbf{2}+\mathbf{2})$-free poset, we can write $D(\mathbf{P}) = \{D(x) : x \in P\}$ as

$$D(\mathbf{P}) = \{D_0, D_1, \ldots, D_k\}$$

where $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_k$. In such a situation, we say that $x \in P$ has level i if $D(x) = D_i$.

Let p_n be the number of of $(\mathbf{2}+\mathbf{2})$-free posets on n elements. El-Zahar [4] and Khamis [9] used a recursive description of $(\mathbf{2}+\mathbf{2})$-free posets to derive a pair of functional equations that define the series $P(t)$. However, they did not solve these equations. Haxell, McDonald and Thomasson [8] provided an algorithm, based on a complicated recurrence relation, to produce the first few values of p_n. Bousquet-Mélou et al. [1] showed that the generating function for the number p_n of $(\mathbf{2}+\mathbf{2})$-free posets on n elements is

$$P(t) = \sum_{n\geq 0} p_n\, t^n = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-t)^i\right). \tag{1}$$

Note that the term corresponding to $n = 0$ in the last sum is 1.

Zagier [14] proved that (1) is also the generating function which counts certain involutions introduced by Stoimenow [12]. Bousquet-Mélou et al. [1] gave a bijections between $(\mathbf{2}+\mathbf{2})$-free posets and such involutions, between $(\mathbf{2}+\mathbf{2})$-free posets and a certain restricted class of permutations, and between $(\mathbf{2}+\mathbf{2})$-free posets and *ascent sequences*. A sequence $(x_1, \ldots, x_n) \in \mathbb{N}^n$ is an *ascent sequence of length* n if and only if it satisfies $x_1 = 0$ and $x_i \in [0, 1 + \mathrm{asc}(x_1, \ldots, x_{i-1})]$ for all $2 \leq i \leq n$. Here, for any integer sequence $(x_1, \ldots, x_i)$, the number of *ascents* of this sequence is

$$\mathrm{asc}(x_1, \ldots, x_i) = |\{\, 1 \leq j < i \,:\, x_j < x_{j+1} \,\}|.$$

For instance, (0, 1, 0, 2, 3, 1, 0, 0, 2) is an ascent sequence. We let $\mathcal{A}$ denote the set of all ascent sequences where we assume the empty word is also an ascent sequence.

To define the bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences, Bousquet-Mélou et al. [1] used a step by step decomposition of a $(\mathbf{2}+\mathbf{2})$-free poset P where are each step one removes a maximal element located on the lowest level together with certain relations. If one records the levels from which one removed such maximal elements and then reads the resulting sequence backwards, one obtains an ascent sequence associated to the poset. We shall give a detailed account to this bijection in Section 2. In the process of decomposing the $(\mathbf{2}+\mathbf{2})$-free poset P, one will reach a point where the remaining poset consists of an antichain, possibly having one element. We define $\mathrm{lds}(P)$ to be the maximum size of such an antichain, which is also equal to the size of the down-set of the last removed element that has a non-trivial down-set. By definition, the value of lds on an antichain is 0 as there are no non-trivial down-sets for such a poset.

Bousquet-Mélou et al. [1] studied a more general generating function $F(t, u, v)$ of $(\mathbf{2}+\mathbf{2})$-free posets according to size="number of elements" (variable t), levels="*number of levels*" (variable u), and minmax="level of minimum maximal element" (variable v). The first few terms of $F(t, u, v)$ are

$$F(t, u, v) = 1 + t + (1 + uv)t^2 + (1 + 2uv + u + u^2v^2)t^3 + O(t^4).$$

An explicit form of $F(t, u, v)$ can be obtained from [1, Lemma 13] and [1, Proposition 14]. The key strategy used by Bousquet-Mélou et al. [1] to derive such formulas was to translate the appropriate statistics on $(\mathbf{2}+\mathbf{2})$-free posets to statistics on ascent sequences since it is much easier to count ascent sequences.

The main result of this paper, Theorem 4, is an explicit form of the generating function $G(t,u,v,z,x)$ for a generalization of $F(t,u,v)$, when two more statistics are taken into account — min="number of minimal elements" in a poset (variable z) and lds="size of *non-trivial last down-set*" (variable x). That is, we shall find an explicit formula for

$$G(t,u,v,z,x)=\sum_{P\in\mathcal{P}} t^{\mathrm{size}(P)}u^{\mathrm{levels}(P)}v^{\mathrm{minmax}(P)}z^{\min(P)}x^{\mathrm{lds}(P)}$$

where, as above, $\mathcal{P}$ is the set of all $(\mathbf{2}+\mathbf{2})$-free posets. As in [1], to find $G(t,u,v,z,x)$, we translate our problem on $(\mathbf{2}+\mathbf{2})$-free posets to an equivalent problem on ascent sequences. That is, we define the following statistics on an ascent sequence: length="the number of elements in the sequence," last="the rightmost element of the sequence," zeros="the number of 0's in the sequence," run="the number of elements in the leftmost run of 0's"="the number of 0's to the left of the leftmost non-zero element." By definition, if there are no non-zero elements in an ascent sequence, the value of run is 0. Then we shall prove the following.

Lemma 1. *The function $G(t,u,v,z,x)$ defined above can alternatively be defined on ascent sequences as*

$$G(t,u,v,z,x)=\sum_{w\in\mathcal{A}} t^{\mathrm{length}(w)}u^{\mathrm{asc}(w)}v^{\mathrm{last}(w)}z^{\mathrm{zeros}(w)}x^{\mathrm{run}(w)}=\sum_{n,a,\ell,m,r\geq 0} G_{n,a,\ell,m,r}t^n u^a v^\ell z^m x^r. \tag{2}$$

Proof. To prove the statement we need to show equidistribution of the statistics involved. All but one case follow from the results in [1]. More precisely, we can use the bijection from $(\mathbf{2}+\mathbf{2})$-free posets to ascent sequences presented in [1] which sends size $\to$ length, levels $\to$ asc, minmax $\to$ last, and min $\to$ zeros. The fact that lds goes to run follows from the bijection. That is, in the process of decomposing the poset, there will be a point where we remove the element, say e, whose down-set gives lds. At that point, we will be left with incomparable elements located on level 0, which gives in the corresponding ascent sequence the initial run of 0's followed by 1 corresponding to e located on level 1. □

We shall also give an explicit form of a specialization of $G(t,u,v,z,x)$, namely $G(t,1,1,z,1)$, which cannot be derived directly from $G(t,u,v,z,x)$ by the substitution. More precisely, let $p_{n,k}$ denote the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements or, equivalently, the number of ascent sequences of length n with k zeros. Then we shall prove that

$$P(t,z)=\sum_{n,k\geq 0} p_{n,k}t^n z^k = 1+\sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{3}$$

Moreover, we will conjecture a simpler form of writing $P(t,z)$ (see Conjecture 1).

A poset P is $(\mathbf{3}+\mathbf{1})$-free if it does not contain, as an induced subposet, a 3-element chain and an element which incomparable to the elements in the 3-element chain. It is known that the number of posets avoiding $(\mathbf{2}+\mathbf{2})$ and $(\mathbf{3}+\mathbf{1})$ is given by the Catalan numbers (see [11, 10]). Define a *restricted ascent sequence* as follows. A sequence $(x_1,\ldots,x_n)\in\mathbb{N}^n$ is a restricted ascent sequence of length n if it satisfies $x_1=0$ and $x_i\in[m-1,1+\mathrm{asc}(x_1,\ldots,x_{i-1})]$ for all $2\leq i\leq n$, where m is the maximum element in $(x_1, \ldots, x_{i-1})$. For instance, (0, 1, 0, 2, 3, 2, 2, 3, 2) is a restricted ascent sequence, whereas (0, 1, 0, 2, 0, 1) is not. Thus, the difference here from the definition of an ascent sequence is 0 substituted by $m - 1$. We shall show that restricted ascent sequences are counted by the Catalan numbers. For $n \leq 6$, the bijection in [1] sends $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets to restricted ascent sequences which lead us to initially conjecture that it always the case that the bijection in [1] sends $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets to restricted ascent sequences. However, this is not true as we shall produce counter examples when $n = 7$.

This paper is organized as follows. In Section 2, we follow [1, Section 3] to describe a $(\mathbf{2}+\mathbf{2})$-free posets decomposition that gives a bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences. The bijection allows us to reduce the enumerative problem on posets to that on ascent sequences. In Section 3 we find explicitly the function $G(t, u, v, z, x)$ using the ascent sequences (see Theorem 4). In Section 4, we shall derive our formula for $P(t, z)$ and state a conjecture on a different form for it. We also show in Section 4 how to get $P(t)$ from $P(t, z)$ thus providing an alternative proof for the enumeration in [1]. Finally, in Section 5 we define a subset of ascent sequences counted by the Catalan numbers and discuss its relations to $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets.

2 $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences

In this section, we shall review the bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences given in [1, Section 3]. In order to do this, Bousquet-Mélou et al. [1] introduced two operations on posets in $\mathcal{P}_n$. The first is an addition operation; it adds an element to $P \in \mathcal{P}_n$ that results in $Q \in \mathcal{P}_{n+1}$. The second is a removal operation; it removes a maximal element m_P from $P \in \mathcal{P}_n$ and results in $Q \in \mathcal{P}_{n-1}$. Before giving these operations we need to define some terminology.

Let $D(x)$ be the set of predecessors of x (the strict down-set of x): $D(x) = \{\, y : y < x \,\}$. Clearly, any poset is uniquely specified by listing the sets of predecessors. It is well-known—see for example Khamis [9]—that a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if its sets of predecessors, $\{D(x) : x \in P\}$, can be linearly ordered by inclusion. Let

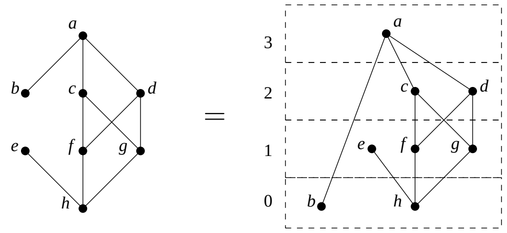
$$D(P) = (D_0, D_1, \ldots, D_{k-1})$$

with $D_0 \subset D_1 \subset \cdots \subset D_{k-1}$ be that chain. In this context we define $D_i(P) = D_i$ and $\ell(P) = k$. We say the element x is at *level* i in P if $D(x) = D_i$ and we write $\ell(x) = i$. The set of all elements at level i we denote $L_i(P) = \{\, x \in P : \ell(x) = i \,\}$ and we let

$$L(P) = \big(L_0(P), L_1(P), \ldots, L_{k-1}(P) \big).$$

For instance, $L_0(P)$ is the set of minimal elements and $L_{k-1}(P)$ is the set of maximal elements whose set of predecessors is also maximal. Let m_P be a maximal element of P whose set of predecessors is smallest. This element may not be unique but the level on which it resides is. Let us write $\ell^\star(P) = \ell(m_P)$.

Example 1. *Consider the* $(\mathbf{2}+\mathbf{2})$*-free poset* P*:*



The diagram on the right shows the poset redrawn according to the level numbers of the sets of predecessors. We have $D(a) = \{b, c, d, f, g, h\}$*,* $D(b) = \emptyset$*,* $D(c) = D(d) = \{f, g, h\}$*,* $D(e) = D(f) = D(g) = \{h\}$ *and* $D(h) = \emptyset$*. These may be ordered by inclusion as*

$$\underbrace{D(h) = D(b)}_{\ell(h)=\ell(b)=0} \subset \underbrace{D(e) = D(f) = D(g)}_{\ell(e)=\ell(f)=\ell(g)=1} \subset \underbrace{D(c) = D(d)}_{\ell(c)=\ell(d)=2} \subset \underbrace{D(a)}_{\ell(a)=3}.$$

Thus $\ell(P) = 4$*. The maximal elements of* P *are* e *and* a*. Since* $D(e) \subset D(a)$ *we have* $m_P = e$ *and* $\ell^\star(P) = 1$*. In addition,* $D_0 = \emptyset$*,* $D_1 = \{h\}$*,* $D_2 = \{f, g, h\}$ *and* $D_3 = \{b, c, d, f, g, h\}$*. With* $L_i = L_i(P)$ *we also have* $L_0 = \{h, b\}$*,* $L_1 = \{e, f, g\}$*,* $L_2 = \{c, d\}$ *and* $L_3 = \{a\}$*.*

Clearly, any $(\mathbf{2}+\mathbf{2})$-free poset P is determined by the pair $\big(D(P), L(P)\big)$. Thus when defining the addition and subtraction operations below it suffices to specify how $D(P)$ and $L(P)$ change.

The addition operation is actually one of three addition operations, which will depend on a parameter of P. These addition operations are, in a sense, disjoint. The first addition operation will result in $\ell(P) = \ell(Q)$ whereas the second two addition operations will result in $\ell(Q) = \ell(P) + 1$.

Given $P \in \mathcal{P}_n$ and $0 \le i \le \ell(P)$, let $\Phi(P, i)$ be the poset Q obtained from P according to the following:

(Add1) If $0 \le i \le \ell^\star(P)$, then introduce a new maximal element z on level i which covers the same elements as the other elements on level i. In terms of predecessors and levels, $D(Q) = D(P)$ and

$$L_j(Q) = \begin{cases} L_j(P) & \text{if } j \ne i, \\ L_i(P) \cup \{z\} & \text{if } j = i. \end{cases}$$

(Add2) If $i = 1 + \ell(P)$, then add a new element covering all the maximal elements of P.

(Add3) If $\ell^\star(P) < i \le \ell(P)$, then let $\mathcal{M}$ be the set of maximal elements $x \in P$ with $\ell(x) < i$. Introduce a new element z and set $D(z) = D_i(P)$. For all elements of P on level i and above, ensure they are greater than every element in $\mathcal{M}$. In terms of predecessors and levels,

$$D_j(Q) = \begin{cases} D_j(P) & \text{if } 0 \le j \le i, \\ D_{j-1}(P) \cup \mathcal{M} & \text{if } i < j \le \ell(P), \end{cases}$$

and

$$L_j(Q) = \begin{cases} L_j(P) & \text{if } 0 \le j < i, \\ \{z\} & \text{if } j = i, \\ L_{j-1}(P) & \text{if } i < j \le \ell(P). \end{cases}$$

An important property of the above addition operations is that $\ell^\star(\Phi(P,i)) = i$, since all maximal elements below level i are covered and therefore not maximal in $\Phi(P,i)$. This allows us to give the three rules for reversing each of the addition rules above.

As before, let m_P be a maximal element of P whose set of predecessors is smallest. For non-empty $P \in \mathcal{P}_n$, let $\Psi(P) = (Q,i)$ where $i = \ell^\star(P)$ and Q is the poset that results from applying:

(Sub1) If m_P is not alone on level i, then remove m_P. In terms of predecessors and levels, $D(Q) = D(P)$ and

$$L_j(Q) = \begin{cases} L_j(P) & \text{if } j \neq i, \\ L_i(P) - \{m_P\} & \text{if } j = i. \end{cases}$$

(Sub2) If m_P is alone on level $i = \ell(P)$, then remove the unique element of level i.

(Sub3) If m_P is alone on level $i \leq \ell(P) - 1$, then set $\mathbb{N} = D_{i+1}(P) - D_i(P)$. Make each element in $\mathbb{N}$ a maximal element of the poset by removing any covers. Finally, remove the element m_P. In terms of predecessors and levels,

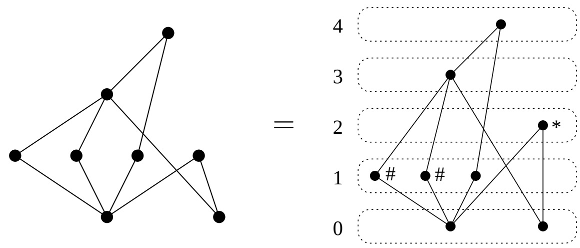
$$D_j(Q) = \begin{cases} D_j(P) & \text{if } 0 \leq j < i, \\ D_{j+1}(P) - \mathbb{N} & \text{if } i \leq j < \ell(P) - 1, \end{cases}$$

and

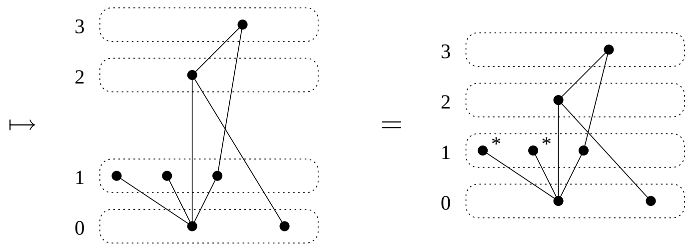
$$L_j(Q) = \begin{cases} L_j(P) & \text{if } 0 \leq j < i, \\ L_{j+1}(P) & \text{if } i \leq j < \ell(P) - 1. \end{cases}$$

We provide an example showing how to find the ascent sequence corresponding to a given unlabeled poset. See [1, Section 3] for an example of how to construct an unlabeled $(\mathbf{2}+\mathbf{2})$-free poset corresponding to a given ascent sequence.

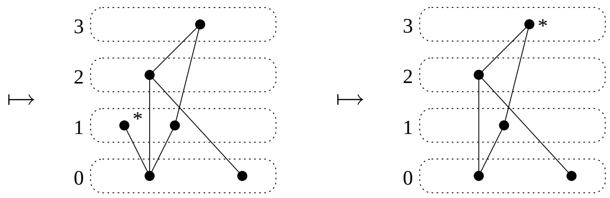
Example 2. *Let P be the unlabeled $(\mathbf{2}+\mathbf{2})$-free poset with this Hasse diagram:*



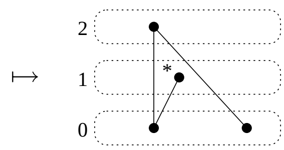
The diagram on the right shows the poset redrawn according to the level numbers of the sets of predecessors. The element m_P is marked by $*$ and $\ell(m_P) = \ell^\star(P) = 2$ so $x_8 = 2$. Since m_P is alone on level $2 < \ell(P)$ we apply rule Sub2 *to remove it. The elements corresponding to $\mathbb{N}$ are indicated by #'s. For each of the elements above level 2, destroy any relations to these # elements. Remove m_P. Adjust the level numbers accordingly.*



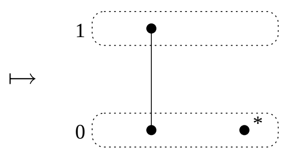
There are now two elements marked by $*$ that may be considered m_P, however only the level number is important. Thus $x_7 = 1$ and remove either of the points according to rule Sub1 *since there is more than one element on level 1. Repeat again to give $x_6 = 1$.*



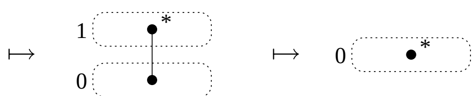
Next, the m_P element is the single maximal element of the poset so $x_5 = 3 = \ell(P) - 1$ and so we apply rule Sub2*:*



The marked element is now alone on level $1 < \ell(P) - 1$ so $x_4 = 1$ and we apply Sub2 *to get:*



The marked element is at level 0 so $x_3 = 0$ and apply Sub1 *(since it is not alone on its level) to get*



The final 2 values are easily seen to be $x_2 = 1$ and $x_1 = 0$. The ascent sequence which encodes this $(\mathbf{2}+\mathbf{2})$-free poset is $x = (0, 1, 0, 1, 3, 1, 1, 2)$.

3 Main results

For $r \geq 1$, let $G_r(t, u, v, z)$ denote the coefficient of x^r in $G(t, u, v, z, x)$. Thus $G_r(t, u, v, z)$ is the generating function of those ascent sequences that begin with $r \geq 1$ 0's followed by 1. We let $G^r_{a,l,m,n}$ denote the number of ascent sequences of length n which begin with r 0's followed by 1, have a ascents, last letter ℓ, and a total of m zeros. We then let

$$G_r := G_r(t, u, v, z) = \sum_{a,\ell,m \geq 0, n \geq r+1} G^r_{a,l,m,n} t^n u^a v^\ell z^m. \tag{4}$$

Clearly, since the sequence $0\ldots0$ has no ascents and no initial run of 0's (by definition), we have that the generating function for such sequences is

$$1 + tz + (tz)^2 + \cdots = \frac{1}{1-tz}$$

where 1 corresponds to the empty word. Thus, we have the following relation between G and G_r:

$$G = \frac{1}{1-tz} + \sum_{r\geq1} G_r\, x^r. \tag{5}$$

Lemma 2. *For $r \geq 1$, the generating function $G_r(t,u,v,z)$ satisfies*

$$(v-1-tv(1-u))G_r = (v-1)t^{r+1}uvz^r + t((v-1)z-v)G_r(t,u,1,z) + tuv^2G_r(t,uv,1,z). \tag{6}$$

Proof. Our proof follows the same steps as in Lemma 13 in [1]. Fix $r \geq 1$. Let $x' = (x_1,\ldots,x_{n-1})$ be an ascent sequence beginning with r 0's followed by 1, with a ascents and m zeros where $x_{n-1} = \ell$. Then $x = (x_1,\ldots,x_{n-1},i)$ is an ascent sequence if and only if $i \in [0, a+1]$. Clearly x also begins with r 0's followed by 1. Now, if $i = 0$, the sequence x has a ascents and $m+1$ zeros. If $1 \leq i \leq \ell$, x has a ascents and m zeros. Finally if $i \in [\ell+1, a+1]$, then x has $a+1$ ascents and m zeros. Counting the sequence $0\ldots01$ with r 0's separately, we have

$$\begin{aligned}
G_r &= t^{r+1}u^1v^1z^r + \sum_{\substack{a,\ell,m\geq0\\ n\geq r+1}} G^r_{a,\ell,m,n}t^{n+1}\left(u^av^0z^{m+1} + \sum_{i=1}^{\ell}u^av^iz^m + \sum_{i=\ell+1}^{a+1}u^{a+1}v^iz^m\right)\\
&= t^{r+1}uvz^r + t\sum_{\substack{a,\ell,m\geq0\\ n\geq r+1}} G_{a,\ell,m,n}t^nu^az^m\left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+2}-v^{\ell+1}}{v-1}\right)\\
&= t^{r+1}uvz^r + tzG_r(t,u,1,z) + tv\frac{G_r - G_r(t,u,1,z)}{v-1} + tuv\frac{vG_r(t,uv,1,z) - G_r}{v-1}.
\end{aligned}$$

The result follows. □

Next just like in Subsection 6.2 of [1], we use the kernel method to proceed. Setting $(v-1-tv(1-u)) = 0$ and solving for v, we obtain that the substitution $v = 1/(1+t(u-1))$ will kill the left-hand side of (6). We can then solve for $G_r(t,u,1,z)$ to obtain that

$$G_r(t,u,1,z) = \frac{(1-u)t^{r+1}uz^r + uG_r\left(t, \frac{u}{1+t(u-1)}, 1, z\right)}{(1+zt(u-1))(1+t(u-1))}. \tag{7}$$

Next we define

$$\begin{aligned}
\delta_k &= u - (1-t)^k(u-1) \text{ and} &(8)\\
\gamma_k &= u - (1-zt)(1-t)^{k-1}(u-1) &(9)
\end{aligned}$$

for $k \geq 1$. We also set $\delta_0 = \gamma_0 = 1$. Observe that $\delta_1 = u - (1-t)(u-1) = 1 + t(u-1)$ and $\gamma_1 = u - (1-zt)(u-1) = 1 + zt(u-1)$. Thus we can rewrite (7) as

$$G_r(t,u,1,z) = \frac{t^{r+1}z^ru(1-u)}{\delta_1\gamma_1} + \frac{u}{\delta_1\gamma_1}G_r(t, \frac{u}{\delta_1}, 1, z). \tag{10}$$

For any function of $f(u)$, we shall write $f(u)|_{u=\frac{u}{\delta_k}}$ for $f(u/\delta_k)$. It is then easy to check that

1. $(u-1)|_{u=\frac{u}{\delta_k}} = \frac{(1-t)^k(u-1)}{\delta_k}$,

2. $\delta_s|_{u=\frac{u}{\delta_k}} = \frac{\delta_{s+k}}{\delta_k}$,

3. $\gamma_s|_{u=\frac{u}{\delta_k}} = \frac{\gamma_{s+k}}{\delta_k}$, and

4. $\frac{u}{\delta_s}|_{u=\frac{u}{\delta_k}} = \frac{u}{\delta_{s+k}}$.

Using these relations, one can iterate the recursion (10) to prove by induction that for all $n \geq 1$,

$$\begin{aligned} G_r(t,u,1,z) &= \frac{t^{r+1}z^r u(1-u)}{\delta_1\gamma_1} + \left(t^{r+1}z^r u(1-u)\sum_{s=2}^{2^n-1}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i}\right) + \\ & \frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i} G_r(t,\frac{u}{\delta_{2^n}},1,z). \end{aligned} \tag{11}$$

Since $\delta_0 = 1$, it follows that as a power series in u,

$$G_r(t,u,1,z) = t^{r+1}z^r u(1-u)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i}. \tag{12}$$

We have used Mathematica to compute that

$$\begin{aligned} & G_1(t,u,1,z) = uzt^2 + \left(uz+u^2z+uz^2\right)t^3 \\ & + \left(uz+3u^2z+u^3z+uz^2+3u^2z^2+uz^3\right)t^4 \\ & + \left(uz+6u^2z+7u^3z+u^4z+uz^2+8u^2z^2+7u^3z^2+uz^3+5u^2z^3+uz^4\right)t^5 + O[t]^6. \end{aligned}$$

For example, the coefficient of t^4u^2, $3z+3z^2$ makes sense as there are 3 ascent sequences of length 4 with 2 ascents and 1 zero, namely, 0112, 0121, and 0122, while there are 3 ascent sequences of length 4 with 2 ascents and 2 zeros, namely, 0101, 0102, and 0120 (there are no other ascents sequences of length 4 with 2 ascents).

Note that we can rewrite (6) as

$$G_r(t,u,v,z) = \frac{t^{r+1}z^r uv(1-v)}{v\delta_1 - 1} + \frac{t(z(v-1)-v)}{v\delta_1-1}G_r(t,u,1,z) + \frac{uv^2t}{v\delta_1-1}G_r(t,uv,1,z). \tag{13}$$

For $s \geq 1$, we let

$$\begin{aligned} \bar{\delta}_s &= \delta_s|_{u=uv} = uv-(1-t)^s(uv-1) \text{ and} \\ \bar{\gamma}_s &= \gamma_s|_{u=uv} = uv-(1-zt)(1-t)^{s-1}(uv-1) \end{aligned}$$

and set $\bar{\delta}_0 = \bar{\gamma}_0 = 1$. Then using (13) and (12), we have the following theorem.

Theorem 3. *For all $r \geq 1$,*

$$G_r(t,u,v,z) = \frac{t^{r+1}z^r u}{v\delta_1 - 1}\left(v(v-1) + t(1-u)(z(v-1)-v)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i} \right.$$
$$\left. +uv^3t(1-uv)\sum_{s\geq 0}\frac{(uv)^s(1-t)^s}{\bar{\delta}_s\bar{\delta}_{s+1}\prod_{i=1}^{s+1}\bar{\gamma}_i}\right) \quad (14)$$

It is easy to see from Theorem 3 that

$$G_r(t,u,v,z) = t^{r-1}z^{r-1}G_1(t,u,v,z). \quad (15)$$

This is also easy to see combinatorially since every ascent sequence counted by $G_r(t,u,v,z)$ is of the form $0^{r-1}a$ where a is an ascent sequence a counted by $G_1(t,u,v,z)$.

We have used Mathematica to compute that

$$\begin{aligned}
&G_1(t,u,v,z) = uvzt^2 + \left(uvz + u^2v^2z + uz^2\right)t^3 \\
&+ \left(uvz + u^2vz + 2u^2v^2z + u^3v^3z + uz^2 + u^2z^2 + u^2vz^2 + u^2v^2z^2 + uz^3\right)t^4 \\
&+ \left(uvz + 3u^2vz + u^3vz + 3u^2v^2z + 2u^3v^2z + 4u^3v^3z + u^4v^4z + uz^2 + 3u^2z^2 + u^3z^2 + 3u^2vz^2 \right.\\
&\left. +u^3vz^2 + 2u^2v^2z^2 + 2u^3v^2z^2 + 3u^3v^3z^2 + uz^3 + 3u^2z^3 + u^2vz^3 + u^2v^2z^3 + uz^4\right)t^5 + O[t]^6.
\end{aligned}$$

For example, the coefficient of t^4u is $zv + z^2 + z^3$ which makes sense since the sequences counted by the terms are 0111, 0110, and 0100, respectively.

Note that

$$\begin{aligned}
G(t,u,v,z,x) &= \frac{1}{(1-tz)} + \sum_{r\geq 1} G_r(t,u,v,z)x^r \\
&= \frac{1}{(1-tz)} + \sum_{r\geq 1} t^{r-1}z^{r-1}G_1(t,u,v,z)x^r \\
&= \frac{1}{(1-tz)} + \frac{1}{1-tzx}xG_1(t,u,v,z)
\end{aligned}$$

Thus we have the following theorem.

Theorem 4.

$$G(t,u,v,z,x) = \frac{1}{(1-tz)} + \frac{t^2zxu}{(1-tzx)(v\delta_1-1)}\left(v(v-1)\right.$$
$$\left. + t(1-u)(z(v-1)-v)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i} + uv^3t(1-uv)\sum_{s\geq 0}\frac{(uv)^s(1-t)^s}{\bar{\delta}_s\bar{\delta}_{s+1}\prod_{i=1}^{s+1}\bar{\gamma}_i}\right). \quad (16)$$

Again, we have used Mathematica to compute the first few terms of this series:

$$G(t,u,v,z,x) = 1 + zt + \left(uvxz + z^2\right)t^2 + \left(uvxz + u^2v^2xz + uxz^2 + uvx^2z^2 + z^3\right)t^3$$
$$+ \left(uvxz + u^2vxz + 2u^2v^2xz + u^3v^3xz + uxz^2 + u^2xz^2 + u^2vxz^2\right.$$
$$\left.+u^2v^2xz^2 + uvx^2z^2 + u^2v^2x^2z^2 + uxz^3 + ux^2z^3 + uvx^3z^3 + z^4\right)t^4$$
$$\left(uvxz + 3u^2vxz + u^3vxz + 3u^2v^2xz + 2u^3v^2xz + 4u^3v^3xz + u^4v^4xz\right.$$
$$+uxz^2 + 3u^2xz^2 + u^3xz^2 + 3u^2vxz^2 + u^3vxz^2 + 2u^2v^2xz^2 + 2u^3v^2xz^2 + 3u^3v^3xz^2$$
$$+uvx^2z^2 + u^2vx^2z^2 + 2u^2v^2x^2z^2 + u^3v^3x^2z^2 + uxz^3 + 3u^2xz^3 + u^2vxz^3 + u^2v^2xz^3$$
$$+ux^2z^3 + u^2x^2z^3 + u^2vx^2z^3 + u^2v^2x^2z^3 + uvx^3z^3 + u^2v^2x^3z^3 + uxz^4$$
$$\left.+ux^2z^4 + ux^3z^4 + uvx^4z^4 + z^5\right)t^5 + O[t]^6.$$

One can check that, for instance, the 3 sequences corresponding to the term $3u^2v^2xzt^5$ are 01112, 01122 and 01222.

4 Counting $(\mathbf{2}+\mathbf{2})$-free posets by size and number of minimal elements

In this section, we shall compute the generating function of $(\mathbf{2}+\mathbf{2})$-free posets by size and the number of minimal elements which is equivalent to finding the generating function for ascent sequences by length and the number of zeros.

For $n \geq 1$, let $H_{a,b,\ell,n}$ denote the number of ascent sequences of length n with a ascents and b zeros which have last letter ℓ. Then we first wish to compute

$$H(u,z,v,t) = \sum_{n\geq 1, a,b,\ell \geq 0} H_{a,b,\ell,n} u^a z^b v^\ell t^n. \tag{17}$$

Using the same reasoning as in the previous section, we see that

$$\begin{aligned}
H(u,z,v,t) &= tz + \sum_{\substack{a,b,\ell\geq 0\\ n\geq 1}} H_{a,b,\ell,n} t^{n+1} \left(u^a v^0 z^{b+1} + \sum_{i=1}^{\ell} u^a v^i z^b + \sum_{i=\ell+1}^{a+1} u^{a+1} v^i z^b \right) \\
&= tz + t \sum_{\substack{a,b,\ell\geq 0\\ n\geq r+1}} H_{a,b\ell,n} t^n u^a z^b \left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+2}-v^{\ell+1}}{v-1} \right) \\
&= tz + \frac{tv(1-u)}{v-1} H(u,v,z,t) + \frac{t(z(v-1)-v)}{v-1} H(u,1,z,t) + \frac{tuv^2}{v-1} H(uv,1,z,t).
\end{aligned}$$

Thus we have the following lemma.

Lemma 5.

$$(v-1-tv(1-u))H(u,v,z,t) = tz(v-1) + t(z(v-1)-v)H(u,1,z,t) + tuv^2 H(uv,1,z,t). \tag{18}$$

Setting $(v-1-t(1-u)) = 0$, we see that the substitution $v = 1 + t(u-1) = \delta_1$ kills the left-hand side of (18). We can then solve for $H(u,1,z,t)$ to obtain the recursion

$$H(u,1,z,t) = \frac{zt(1-u)}{\gamma_1} + \frac{u}{\delta_1\gamma_1} H(uv,1,z,t). \tag{19}$$

By iterating (19), we can prove by induction that for all $n \geq 1$,

$$H(u,1,z,t) = \frac{zt(1-u)}{\gamma_1} + \left(\sum_{s=1}^{2^n-1} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i} H(\frac{u}{\delta_{2^n}},1,z,t). \quad (20)$$

Since $\delta_0 = 1$, we can rewrite (20) as

$$H(u,1,z,t) = \left(\sum_{s=0}^{2^n-1} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i} H(\frac{u}{\delta_{2^n}},1,z,t). \quad (21)$$

Thus as a power series in u, we can conclude the following.

Theorem 6.

$$H(u,1,z,t) = \sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}. \quad (22)$$

We would like to set $u = 1$ in the power series $\sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1}\gamma_i}$, but the factor $(1-u)$ in the series does not allow us to do that in this form. Thus our next step is to rewrite the series in a form where it is obvious that we can set $u = 1$ in the series. To that end, observe that for $k \geq 1$,

$$\delta_k = u - (1-t)^k(u-1) = 1 + u - 1 - (1-t)^k(u-1) = 1 - (1-t)^k - 1)(u-1)$$

so that

$$\frac{1}{\delta_k} = \sum_{n\geq 0}((1-t)^k - 1)^n(u-1)^n \sum_{n\geq 0}(u-1)^n \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{km}. \quad (23)$$

Substituting (23) into (22), we see that

$$\begin{aligned} H(u,1,z,t) &= \frac{zt(1-u)}{\gamma_1} + \sum_{k\geq 1} \frac{zt(1-u)u^k(1-t)^k}{\prod_{i=1}^{k+1}\gamma_i} \sum_{n\geq 0}(u-1)^n \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{km} \\ &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m} zt \sum_{k\geq 1}\frac{(u-1)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} \\ &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m} \frac{zt}{(1-zt)^{m+1}} \times \\ &\quad \sum_{k\geq 1}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}. \end{aligned}$$

Next we need to study the series

$$\sum_{k\geq 1}\frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}$$

where $m \geq 0$. We can rewrite this series in the form

$$-\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} + \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}.$$

We let

$$\psi_{m+1}(u) = \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}. \tag{24}$$

We shall show that $\psi_{m+1}(u)$ is in fact a polynomial for all $m \geq 0$. First, we claim that $\psi_{m+1}(u)$ salsifies the following recursion:

$$\psi_{m+1}(u) = \frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} + \frac{u\delta_1^m}{\gamma_1}\psi_{m+1}\left(\frac{u}{\delta_1}\right). \tag{25}$$

That is, one can easily iterate (25) to prove by induction that for all $n \geq 1$,

$$\psi_{m+1}(u) = \left(\sum_{s=0}^{2^n-1} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^s(1-t)^{s(m+1)}}{\prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2n}(\delta_{2^n})^m}{\prod_{i=1}^{2^n}\gamma_i}\psi_{m+1}(\frac{u}{\delta_{2^n}}). \tag{26}$$

Hence it follows that if $\psi_{m+1}(u)$ satisfies the recursion (25), then $\psi_{m+1}(u)$ is given by the power series in (24). However, it is routine to check that the polynomial

$$\phi_{m+1}(u) = -\sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j} \prod_{i=j+1}^{m}(1-((1-t)^i) \tag{27}$$

satisfies the recursion that

$$\gamma_1\phi_{m+1}(u) = (u-1)^{m+1}(1-zt)^{m+1} + u\delta_1^m\phi_{m+1}\left(\frac{u}{\delta_1}\right). \tag{28}$$

Thus we have proved the following lemma.

Lemma 7.

$$\begin{aligned}\psi_{m+1}(u) &= \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} \\ &= -\sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j} \prod_{i=j+1}^{m}(1-((1-t)^i).\end{aligned} \tag{29}$$

It thus follows that

$$\begin{aligned}H(u,1,z,t) &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}\frac{zt}{(1-zt)^{m+1}} \times \\ &\quad -\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} - \sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j} \prod_{i=j+1}^{m}(1-((1-t)^i).\end{aligned}$$

There is no problem in setting $u = 1$ in this expression to obtain that

$$H(1, 1, z, t) = \sum_{n \geq 0} \frac{zt}{(1 - zt)^{n+1}} \prod_{i=1}^{n} (1 - (1 - t)^i). \tag{30}$$

Clearly our definitions ensure that $1 + H(1, 1, z, t) = P(t, z)$ as defined in the introduction so that we have the following theorem.

Theorem 8.

$$P(t, z) = \sum_{n,k \geq 0} p_{n,k} t^n z^k = 1 + \sum_{n \geq 0} \frac{zt}{(1 - zt)^{n+1}} \prod_{i=1}^{n} (1 - (1 - t)^i). \tag{31}$$

For example, we have used Mathematica to compute the first few terms of $P(t, z)$ as

$$P(t,z) = 1 + zt + \left(z + z^2\right) t^2 + \left(2z + 2z^2 + z^3\right) t^3 + \left(5z + 6z^2 + 3z^3 + z^4\right) t^4 \\ + \left(15z + 21z^2 + 12z^3 + 4z^4 + z^5\right) t^5 + \left(53z + 84z^2 + 54z^3 + 20z^4 + 5z^5 + z^6\right) t^6 + O[t]^7.$$

Next we observe that one can easily derive the ordinary generating function for the number of $(\mathbf{2}+\mathbf{2})$-free posets or, equivalently, for the number of ascent sequences proved by Bousquet-Mélou et al. [1] from Theorem 8. That is, for any sequence of natural numbers $a = a_1 \ldots a_n$, let $a^+ = (a_1 + 1) \ldots (a_n + 1)$ be the result of adding one from each element of the sequence. Moreover, if all the elements of $a = a_1 \ldots a_n$ are positive, then we let $a^- = (a_1 - 1) \ldots (a_n - 1)$ be the result of subtracting one to each element of the sequence. It is easy to see that if $a = a_1 \ldots a_n$ is an ascent sequence, then $0a^+$ is also an ascent sequence. Vice versa, if $b = 0a$ is an ascent sequence with only one zero where $a = a_1 \ldots a_n$, then a^- is an ascent sequence. It follows that the number of ascent sequences of length n is equal to the number of ascent sequences of length $n + 1$ which have only one zero. Hence

$$\begin{aligned} P(t) &= \sum_{n \geq 0} p_n t^n = \frac{1}{t} \frac{\partial P(t, z)}{\partial z}\Big|_{z=0} \\ &= \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1 - t)^i). \end{aligned}$$

Results in [1, 2, 3] show that $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements are in bijection with the following objects. (See [1, 2, 3] for the precise definitions.)

- ascent sequences of length n with k zeros;
- permutations of length n *avoiding*  whose *leftmost-decreasing run* is of size k;
- *regular linearized chord diagrams* on $2n$ points with *initial run of openers* of size k;
- *upper triangular matrices* whose non-negative integer entries sum up to n, each row and column contains a non-zero element, and the sum of entries in the first row is k.

Thus (31) provides generating functions for  -avoiding permutations by the size of the leftmost-decreasing run, for regular linearized chord diagrams by the size of the initial run of openers, and for the upper triangular matrices by the sum of entries in the first row. Moreover, Theorem 4, together with bijections in [1, 2, 3] can be used to enumerate the permutations, diagrams, and matrices subject to 4 statistics. However, we have chosen not to state explicit generating functions related to the permutations and diagrams.

Finally, we conjecture that $P(t,z)$ given in Theorem 8 can be written in a different form.

Conjecture 1.

$$P(t,z) = \sum_{n,k\geq 0} p_{n,k} t^n z^k = \sum_{n\geq 0} \prod_{i=1}^{n} (1-(1-t)^{i-1}(1-zt)).$$

5 Restricted ascent sequences and the Catalan numbers

Recall[1] that a sequence $(x_1,\dots,x_n)\in\mathbb{N}^n$ is a restricted ascent sequence of length n if it satisfies $x_1=0$ and $x_i\in[m-1,1+\mathrm{asc}(x_1,\dots,x_{i-1})]$ for all $2\leq i\leq n$, where m is the maximum element in $(x_1,\dots,x_{i-1})$.

Theorem 9. *The number of restricted ascent sequences of length n is given by the n-th Catalan number.*

Proof. Lets R_n denote the number of restricted ascents sequences of length n. The Catalan numbers C_n can be defined by the recursion

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

with the initial condition that $C_0=1$. It is easy to see that $R_0=1$ since the empty sequence is a restricted ascent sequences. We must show that

$$R_{n+1} = \sum_{k=0}^{n} R_k R_{n-k}. \tag{32}$$

Thus we need a procedure to take a restricted ascent sequence D_1 of length k and a restricted ascent sequence D_2 of length $n-k$ and produce a restricted ascent sequence D of lenght $n+1$. We shall describe a procedure "gluing" two ascent sequences, D_1 and D_2 which is equivalent to gluing two Dyck paths together. To define our gluing procedure we first need the concept of the "rightmost maximum" in an ascent sequence, defined as a left-to-right maximum x such that x is one more than the number of ascents to the left of x, and none of the left-to-right maxima to the right of x has this property (in other words, this is the last time we use the maximum option in the interval $[m-1,1+\mathrm{asc}]$ among the left-to-right maxima). The sequence $00\dots0$ is the only one that does not have the rightmost maximum. For example, 0010101003 has the rightmost maximum (the leftmost) 1, whereas 0010103323234 has the rightmost maximum (the leftmost) 3. Then procedure of "gluing" two ascent sequences, D_1 and D_2 together can be described as follows.

[1]We would like to thank Anders Claesson for sharing with us his software to work with posets. Special thanks go to Hilmar Haukur Gudmundsson for providing us the main ideas, and essentially a solution to Theorem 9

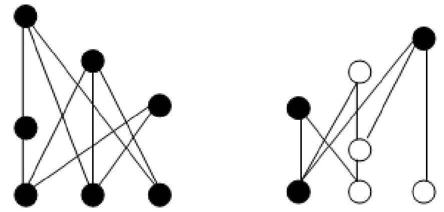


Figure 1: Counterexamples to the statement that restricted ascent sequences correspond to $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets under the bijection in [1].

1. For $D_1 \neq \emptyset$, define $D_1 + D_2 := D_1(1 + asc(D_1))(D_2{+}{+}(asc(D_1)))$ where "++" means increasing *each* element of D_2 by the number $asc(D_1)$. For example, if $D_1 = 01021$ and $D_2 = 01212$, then $D_1 + D_2 = 01021323434$.
2. For $D_1 = \emptyset$ define $D_1 + D_2 := D_2$ with the rightmost maximum element duplicated (add extra 0 if $D_2 = 00\ldots0$). For example, $\epsilon + 01212 = 012212$.

It is easy to see that in Case 1, the element $(1 + asc(D_1)$ is the the rightmost maximum element of $D_1(1 + asc(D_1))(D_2{+}{+}(asc(D_1)))$ which is either the rightmost element if $D_2 = \epsilon$ or is followed by $asc(D_1)$ if $D_2 \neq \epsilon$ since D_2 must start with 0 in that case. It follows that the rightmost maximal element is not duplicated in $D_1 + D + 2$ in Case 1 and, hence, it is easy to recover D_1 and D_2 from $D_1 + D_2$. Clearly, in Case 2, the rightmost maximal element of $D_1 + D_2$ is dulplicated so that we can distinguish Case 1 from Case 2. Moreover, it is easy to see that we can recover D_2 from $D_1 + D_2$ in Case 2. This proves that (32) holds and hence $R_n = C_n$ for all n.

Here are examples of decompositions for $n = 3$ and $n = 4$ (ϵ stays for the empty word):

$000 = \epsilon + 00$	$0000 = \epsilon + 000$	$0100 = 0 + 00$	$0112 = 011 + \epsilon$
$001 = 00 + \epsilon$	$0001 = 000 + \epsilon$	$0101 = 0 + 01$	$0121 = 01 + 0$
$010 = 0 + 0$	$0010 = 00 + 0$	$0102 = 010 + \epsilon$	$0122 = \epsilon + 012$
$011 = \epsilon + 01$	$0011 = \epsilon + 001$	$0110 = \epsilon + 010$	$0123 = 012 + \epsilon$
$012 = 01 + \epsilon$	$0012 = 001 + \epsilon$	$0111 = \epsilon + 011$	

□

Recall that posets avoiding $(\mathbf{3}+\mathbf{1})$ are those that do not contain, as an induced subposet, a 3-element chain together with another element which is incomparable to all elements in the 3-element chain. As we mentioned in the introduction, the number of posets avoiding $(\mathbf{2}+\mathbf{2})$ and $(\mathbf{3}+\mathbf{1})$ is given by the Catalan numbers (see [11, 10]). Using the bijection [1] applied to small restricted ascent sequences, one would be tempted to conjecture that restricted ascent sequences are bijectively mapped to $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets as both of the objects are counted by the Catalan numbers. Indeed, this is true for posets of size less than or equal to six.

Moreover, we can show that the first time one violates that restricted ascent sequence condition, then the corresponding $(\mathbf{2}+\mathbf{2})$-free poset contains an induced copy of $(\mathbf{3}+\mathbf{1})$. That is, suppose that $a = a_1 \ldots a_n$ is a restricted ascent sequence, $m = \max(a_1, \ldots, a_n) \geq 2$, and $x < m - 1$. Then we claim that the poset corresponding to ax, must contain an induced copy of $(\mathbf{3}+\mathbf{1})$. That is, let r be the element on level x that corresponds to x under the bijection of Section 2. Now in ax, x is preceded by a larger element, and thus r has a neighbor, say s, on its level, level x. Because the first time we encounter m in a, its corresponding element z in the poset covers all maximal elements, it follows that there must be at least one non-maximal element, say u, on level $m-1$. Next, since $x < m-1$, there exists an element e in the poset such that $e < c$ and $e \not< b$. That is, c is on a higher level than b and the down-sets are linearly ordered by inclusion according to their levels. Since r copies relations of s, $e \not< r$. Since r is a maximal element, also $r \not< e$ and $r \not< u$. Finally, u is a non-maximal element, thus there exists $v > u$. Finally $v \not< r$ since r is maximal so that the four elements $e < u < v$ and r form a $(\mathbf{3}+\mathbf{1})$ configuration.

If it was the case that our addition operations preserved the property of containing $(\mathbf{3}+\mathbf{1})$ configuration, then it would be the case that the bijection in Section 2 would send $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets to restricted ascent sequences. However, this is is not the case. For example, consider the poset on the left in Figure 1 which corresponds to $(\mathbf{2}+\mathbf{2})$-free poset corresponding to the ascent sequence 0101202. One can check that there is an induced $(\mathbf{3}+\mathbf{1})$ in the poset corresponding to the non-restricted ascent sequence 010120, but clearly there is no induced $(\mathbf{3}+\mathbf{1})$ in the $(\mathbf{2}+\mathbf{2})$-free poset corresponding to the ascent sequence 0101202. This means that there must be a restricted ascent sequence of length seven whose corresponding $(\mathbf{2}+\mathbf{2})$-free poset does contain an induced copy of $(\mathbf{3}+\mathbf{1})$. Such a sequence and its corresponding $(\mathbf{2}+\mathbf{2})$-free poset is shown on the right in Figure 1.

We leave it as open problem to characterize $(\mathbf{2}+\mathbf{2})$-free posets corresponding to restricted ascent sequences under the bijection in [1] and to characterize ascent sequences corresponding to $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets under the same bijection.

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