Marked mesh patterns in 132-avoiding permutations I

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Abstract

This paper is a continuation of the systematic study of the distributions of simple marked mesh patterns initiated in [6]. We study simple marked mesh patterns on 132-avoiding permutations. We derive generating functions for the number of occurrences of 4-parameter simple marked mesh patterns where only one parameter is allowed to be non-zero or a non-empty set. We show that specializations of such generating functions count a number of classical combinatorial sequences. Generating functions for the number of occurrences of 4-parameter simple marked mesh patterns where two or more of the parameters are allowed to be non-zero are studied in the upcoming paper [7].

Keywords: permutation statistics, marked mesh pattern, distribution, Catalan numbers, Fibonacci numbers, Fine numbers

Contents

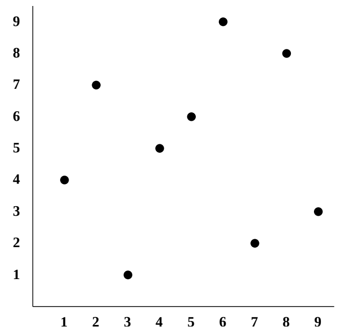
1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [1] to provide explicit expansions for certain permutation statistics as (possibly infinite) linear combinations of (classical) permutation patterns. This notion was further studied in [4, 6, 10]. The present paper, as well as the upcoming paper [7], are continuations of the systematic study of distributions of simple marked mesh patterns on permutations initiated by Kitaev and Remmel [6].

In this paper, we study the number of occurrences of what we call *simple marked mesh patterns*. To start with, let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation written in one-line notation. Then we will consider the graph of σ, $G(\sigma)$, to be the set of points $\{(i, \sigma_i) : 1 \leq i \leq n\}$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers, we say that σ_i matches the simple marked mesh pattern $MMP(a, b, c, d)$ in σ if, in the coordinate system centered at (i, σ_i), $G(\sigma)$ has $\geq a$, $\geq b$, $\geq c$, and $\geq d$ points in Quadrants I, II, III, and IV, respectively. For example, if $\sigma = 471569283$, then $\sigma_4 = 5$ matches the simple marked mesh pattern $MMP(2, 1, 2, 1)$, since relative to the coordinate system with origin $(4, 5)$, $G(\sigma)$ has 3, 1, 2, and 2 points in Quadrants I, II, III, and IV, respectively.

Note that if a coordinate in $MMP(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant. In addition, we shall consider patterns $MMP(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here, when a coordinate of $MMP(a, b, c, d)$ is the empty set, there must be no points in the corresponding quadrant for σ_i to match $MMP(a, b, c, d)$ in σ. For example, if $\sigma = 471569283$, then $\sigma_3 = 1$ matches the marked mesh pattern $MMP(4, 2, \emptyset, \emptyset)$, since relative to the coordinate system with origin $(3, 1)$, $G(\sigma)$ has 6, 2, 0, and 0 points in Quadrants I, II, III, and IV, respectively. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a, b, c, d)$ in σ.

Given a sequence $w = w_1 \ldots w_n$ of distinct integers, let $\mathrm{red}(w)$ be the permutation found by replacing the i-th largest integer that appears in w by i. For example, if $w = 2754$, then $\mathrm{red}(w) = 1432$. Given a permutation $\tau = \tau_1 \ldots \tau_j \in S_j$, we say that the pattern τ *occurs* in $\sigma \in S_n$ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \ldots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. We will let $S_n(\tau)$ denote the



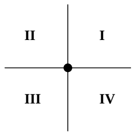


Figure 1: The graph of $\sigma = 471569283$.

set of permutations in S_n that avoid τ. In the theory of permutation patterns, τ is called a *classical pattern*. See [5] for a comprehensive introduction to the area of permutation patterns.

It has been a rather popular direction of research in the literature on permutation patterns to study permutations avoiding a 3-letter pattern subject to extra restrictions (see [5, Subsection 6.1.5]). The main goal of this paper and the upcoming paper [7] is to study the generating functions

$$Q_{132}^{(a,b,c,d)}(t,x) = 1 + \sum_{n\geq 1} t^n Q_{n,132}^{(a,b,c,d)}(x) \tag{1}$$

where for any $a,b,c,d \in \mathbb{N} \cup \{\emptyset\}$,

$$Q_{n,132}^{(a,b,c,d)}(x) = \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}. \tag{2}$$

More precisely, this paper deals with the case when only one of a, b, c and d is allowed to be non-zero or non-empty set while [7] deals with the cases where at least two of a, b, c and d are greater than 0.

For example, here are two tables of statistics for $S_3(132)$ that we will be interested in.

σ	$\mathrm{mmp}^{(1,0,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,1,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,1,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,0,1)}(\sigma)$
123	2	0	2	0
213	2	1	1	1
231	1	1	1	2
312	1	2	1	1
321	0	2	0	2

σ	$\mathrm{mmp}^{(2,0,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,2,0,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,2,0)}(\sigma)$	$\mathrm{mmp}^{(0,0,0,2)}(\sigma)$
123	1	0	1	0
213	0	0	1	0
231	0	1	0	0
312	0	0	0	1
321	0	1	0	1

Note that there is one obvious symmetry in this case. That is, we have the following lemma.

Lemma 1. *For any $a, b, c, d \in \{\emptyset\} \cup \mathbb{N}$,*

$$Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x).$$

Proof. If we start with the graph $G(\sigma)$ of a permutation $\sigma \in S(132)$ and reflect the graph about the line $y = x$, then we get the permutation σ^{-1}, which is also in $S(132)$. It is easy to see that points in Quadrants I, II, III, and IV in the coordinate system with origin (i, σ_i) in $G(\sigma)$ will reflect to points in Quadrants I, IV, III, and II, respectively, in the coordinate system with origin (σ_i, i) in $G(\sigma^{-1})$. It follows that the map $\sigma \to \sigma^{-1}$ shows that $Q_{n,132}^{(a,b,c,d)}(x) = Q_{n,132}^{(a,d,c,b)}(x)$. $\square$

As a matter of fact, *avoidance* of a marked mesh pattern $MMP(a, b, c, d)$ with $a, b, c, d \in \mathbb{N}$ can always be expressed in terms of multi-avoidance of (usually many) classical patterns. Thus, among our results we will re-derive several known facts in the permutation pattern theory. However, our main goals are more ambitious in that we will compute the generating function for the distribution of the occurrences of the pattern in question, not just the generating function for the number of permutations that avoid the pattern. Moreover, we shall show that sequences of the form $(Q_{n,132}^{(a,b,c,d)}(x)|x^r)_{n\geq s}$ count a variety of combinatorial objects that appear in the *On-line Encyclopedia of Integer Sequences* (OEIS) [8]. Thus our results will give new combinatorial interpretations of such classical sequences such as the Fine numbers and the Fibonacci numbers, as well as provide certain sequences that appear in the OEIS with a combinatorial interpretation where none had existed before.

Given a permutation $\sigma \in S_n$, we say that σ_i is a *right-to-left maximum* of σ if $\sigma_i > \sigma_j$ for all $j > i$. Similarly, one can define a *right-to-left minimum*, a *left-to-right minimum*, and a *left-to-right maximum*.

2 Connections with other combinatorial objects

It is well-known that the cardinality of $S_n(132)$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. There are many combinatorial interpretations of the Catalan numbers. For example, in his book [9], Stanley lists 66 different interpretations and there are many more such interpretations that can be found on his web site. Hence, any time one has a natural bijection from $S_n(132)$ into a set of combinatorial objects O_n counted by the n-th Catalan number, one can use the bijection to transfer our statistics $\mathrm{mmp}^{(a,b,c,d)}$ to corresponding statistics on the elements of O_n. In this section, we shall briefly describe some of these statistics in two of the most well-known interpretations of the Catalan numbers, namely, Dyck paths and binary trees.

A Dyck path of length $2n$ is a path that starts at $(0,0)$ and ends at the point $(2n,0)$ that consists of a sequence of up-steps $(1,1)$ and down-steps $(1,-1)$ such that the path always stays on or above the x-axis. We will generally encode a Dyck path by its sequence of up-steps and down-steps. Let $\mathcal{D}_{2n}$ denote the set of Dyck paths of length $2n$. Then it is easy to construct a bijection $\phi_n : S_n(132) \to \mathcal{D}_{2n}$ by induction. To define ϕ_n, we need to define the lifting of a path $P \in \mathcal{D}_{2n}$ to a path $L(P) \in \mathcal{D}_{2n+2}$. Here $L(P)$ is constructed by simply appending an up-step at the start of P and a down-step at the end of P. That is, if $P = (p_1, \ldots, p_{2n})$, then $L(P) = ((1,1), p_1, \ldots, p_{2n}, (1,-1))$. An example of this map is pictured in Figure 2. If $P_1 \in \mathcal{D}_{2k}$ and $P_2 \in \mathcal{D}_{2n-2k}$, we let P_1P_2 denote the element of $\mathcal{D}_{2n}$ which consists of the path P_1 followed by the path P_2.

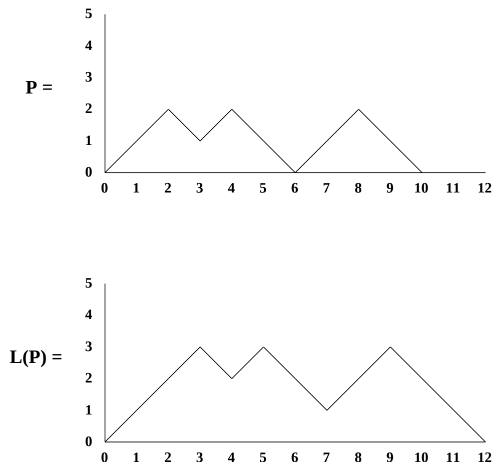


Figure 2: The lifting of a Dyck path.

Then we let $\phi_1(1) = ((1,1),(1,-1))$. For any $n > 1$ and any $\sigma \in S_n(132)$, we define $\phi_n(\sigma)$ by cases as follows.

Case 1. $\sigma_n = n$.
Then $\phi_n(\sigma) = L(\phi_{n-1}(\sigma_1 \ldots \sigma_{n-1}))$.

Case 2. $\sigma_i = n$ where $1 \leq i < n$. In this case, $\phi_n(\sigma) = P_1P_2$ where
$P_1 = \phi_i(\mathrm{red}(\sigma_1 \ldots \sigma_i))$ and $P_2 = \phi_{n-i}(\mathrm{red}(\sigma_{i+1} \ldots \sigma_n)) = \phi_{n-i}(\sigma_{i+1} \ldots \sigma_n)$.

We have pictured the first few values of this map by listing the permutation σ on the left and the value of $\phi_n(\sigma)$ on the right in Figure 3.

Suppose we are given a path $P = (p_1, \ldots, p_{2n}) \in \mathcal{D}_{2n}$. Then we say that a step p_i has height s if p_i is an up-step and the right-hand end point of p_i is (i,s) or p_i is a down-step and the left-hand end point of p_i is $(i-1,s)$. We say that $(p_i, \ldots, p_{i+2k-1})$ is an *interval of length* $2k$ if p_i is an up-step, p_{i+2k-1} is a down-step, p_i and p_{i+2k-1} have height 1, and, for

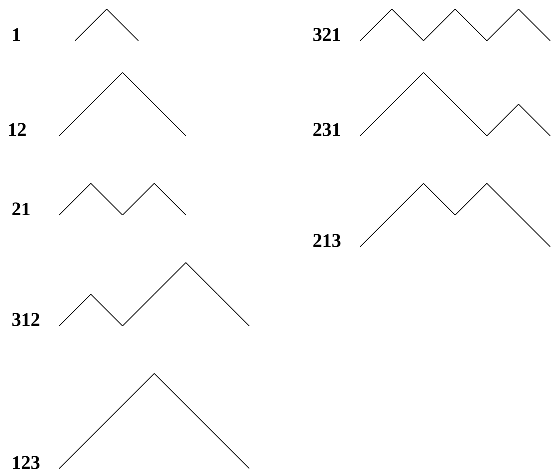


Figure 3: Some initial values of the map ϕ_n.

all $i < j < 2k-1$, the height of p_j is strictly greater than 1. Thus an interval is a segment of the path which starts and ends on the x-axis but does not hit the x-axis in between. For example, if we consider the path $\phi_3(312) = (p_1, \ldots, p_6)$ pictured in Figure 3, then the heights of the steps reading from left to right are $1, 1, 1, 2, 2, 1$ and there are two intervals, one of length 2 consisting of (p_1, p_2) and one of length 4 consisting of (p_3, p_4, p_5, p_6).

This given, the following theorem is straightforward to prove by induction.

Theorem 2. *Let $k \geq 1$.*

1. *For any $\sigma \in S_n(132)$, $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ is equal to the number of up-steps (equivalently, to the number of down-steps) of height $\geq k+1$ in $\phi_n(\sigma)$.*

2. *For any $\sigma \in S_n(132)$, 1 plus the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$ is equal to one half the maximum length in an interval in $\phi_n(\sigma)$.*

Proof. We proceed by induction on n. Clearly the theorem is true for $n = 1$. Now suppose that $n > 1$ and the theorem is true for all $m < n$. Let $\sigma \in S_n(132)$ with $\sigma_i = n$. Then it must be the case that $\sigma_1, \ldots, \sigma_{i-1}$ are all strictly bigger than all the elements in $\{\sigma_{i+1}, \ldots, \sigma_n\}$, so $\{1, \ldots, n-i\} = \{\sigma_{i+1}, \ldots, \sigma_n\}$ and $\{n-i+1, \ldots, n\} = \{\sigma_1, \ldots, \sigma_i\}$. Now consider the two cases in the definition of ϕ_n.

Case 1. $\sigma_n = n$.
In this case, $\phi_n(\sigma) = L(P)$ where $P = \phi_{n-1}(\sigma_1 \ldots \sigma_{n-1})$. Thus for $k \geq 2$, the number of up-steps of height $> k$ in $\phi_n(\sigma)$ equals the number of up-steps of height $\geq k$ in $\phi_{n-1}(\sigma_1 \ldots \sigma_{n-1})$, which equals $\mathrm{mmp}^{(k-1,0,0,0)}(\sigma_1 \ldots \sigma_{n-1})$ by induction. But since $\sigma_n = n$, it is clear that for $k \geq 2$, $\mathrm{mmp}^{(k-1,0,0,0)}(\sigma_1 \ldots \sigma_{n-1}) = \mathrm{mmp}^{(k,0,0,0)}(\sigma)$. Thus $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ equals the number of up-steps of height $> k$ in $\phi_n(\sigma)$. Finally, $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-1$ and there are $n-1$ up-steps of height ≥ 2 in $\phi_n(\sigma)$.

In this case, the maximum length of an interval in $\phi_n(\sigma)$ equals $2n$ and $\sigma_n = n$ shows that $\mathrm{mmp}^{(0,0,n-1,0)}(\sigma) = 1$, so one half of the maximum length interval in $\phi_n(\sigma)$ equals 1 plus the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$.

Case 2. $\sigma_i = n$ where $1 \leq i \leq n-1$.
In this case, $\phi_n(\sigma) = P_1P_2$ where $P_1 = \phi_i(\mathrm{red}(\sigma_1 \ldots \sigma_i))$ and $P_2 = \phi_{n-i}(\sigma_{i+1} \ldots \sigma_n)$. It follows that for any $k \geq 1$, the number of up-steps of height $> k$ in $\phi_n(\sigma)$ equals the number of up-steps of height $> k$ in P_1 plus the number of up-steps of height $> k$ in P_2, which by induction is equal to

$$\mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \ldots \sigma_i)) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \ldots \sigma_n).$$

But clearly

$$\mathrm{mmp}^{(k,0,0,0)}(\sigma) = \mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \ldots \sigma_i)) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \ldots \sigma_n)$$

so that $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ is equal to the number of up-steps of height $> k$ in $\phi_n(\sigma)$.

Finally, in this case the maximum length of an interval in $\phi_n(\sigma)$ is the maximum of the maximum length intervals in P_1 and P_2. On the other hand, the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$ is the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\mathrm{red}(\sigma_1 \ldots \sigma_i)) \neq 0$ or $\mathrm{mmp}^{(0,0,k,0)}(\sigma_{i+1} \ldots \sigma_n) \neq 0$. Thus it follows from the induction hypothesis that one half of the maximum length of an interval in $\phi_n(\sigma)$ is 1 plus the maximum k such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) \neq 0$. □

We have the following corollary to Theorem 2.

Corollary 1. *Let $k \geq 1$.*

1. *The number of permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(k,0,0,0)}(\sigma) = 0$ equals the number of Dyck paths $P \in \mathcal{D}_{2n}$ such that all steps have height $\leq k$.*

2. *The number of permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = 0$ equals the number of Dyck paths $P \in \mathcal{D}_{2n}$ such that the maximum length of an interval is $\leq 2k$.*

Another set counted by the Catalan numbers is the set of rooted binary trees on n nodes where each node is either a leaf, a node with a left child, a node with a right child, or a node with both a right and a left child. Let $\mathcal{B}_n$ denote the set of rooted binary trees with n nodes. Then it is well-known that $|\mathcal{B}_n| = C_n$. Again it is easy to define a bijection $\theta_n : S_n(132) \to \mathcal{B}_n$ by induction. Start with a single node, denoted the root, and let i be such that $\sigma_i = n$. Then, if $i > 1$, the root will have a left child, and the subtree above that child is $\theta_{i-1}(\mathrm{red}(\sigma_1 \ldots \sigma_{i-1}))$. If $i < n$, then the root will have a right child, and the subtree above that child is $\theta_{n-i}(\sigma_{i+1} \ldots \sigma_n)$. We have pictured the first few values of this map by listing a permutation σ on the left and the value of $\theta_n(\sigma)$ on the right in Figure 4.

If $T \in \mathcal{B}_n$ and η is a node of T, then the left subtree of η is the subtree of T whose root is the left child of η and the right subtree of η is the subtree of T whose root is the right child of η. The edge that connects η to its left child will be called a *left edge* and the edge that connects η to its right child will be called a *right edge*. This given, the following theorem is straightforward to prove by induction.

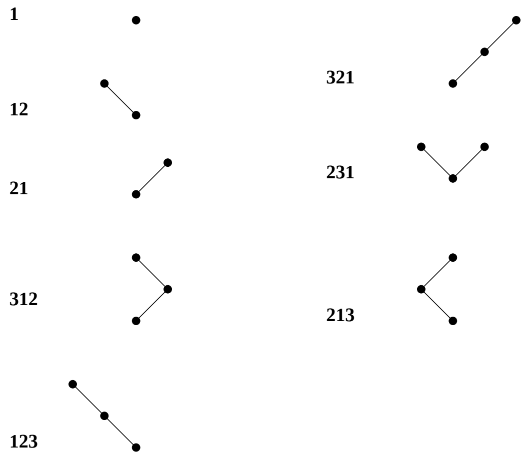


Figure 4: Some initial values of the map θ_n.

Theorem 3. *Let $k \geq 1$.*

1. *For any $\sigma \in S_n(132)$, $\mathrm{mmp}^{(k,0,0,0)}(\sigma)$ is equal to the number of nodes η in $\theta_n(\sigma)$ such that there are $\geq k$ left edges on the path from η to the root of $\theta_n(\sigma)$.*

2. *For any $\sigma \in S_n(132)$, $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$ is the number of nodes η in $\theta_n(\sigma)$ whose left subtree has size $\geq k$.*

Proof. We proceed by induction on n. Clearly the theorem is true for $n = 1$. Now suppose that $n > 1$ and the theorem is true for all $m < n$. Let $\sigma \in S_n(132)$ with $\sigma_i = n$, r be the root of $\theta_n(\sigma)$, and η be a node in $\theta_n(\sigma)$.

If η is in the left subtree of the root, then η has $\geq k$ left edges on the path to r if and only if it has $\geq k-1$ left edges on the path to the root of the left subtree of r. If η is in the right subtree of the root, then η has $\geq k$ left edges on the path to r if and only if it has $\geq k$ left edges on the path to the root of the right subtree of r. Therefore, by the induction hypothesis the number of nodes with $\geq k$ left edges on the path to the root is $\mathrm{mmp}^{(k-1,0,0,0)}(\mathrm{red}(\sigma_1 \dots \sigma_{i-1})) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \dots \sigma_n)$, regarding each term as 0 if there is no corresponding subtree. However, since each term in $\sigma_1 \dots \sigma_{i-1}$ has n to the right of it and n never matches $MMP(k,0,0,0)$, we see that $\mathrm{mmp}^{(k-1,0,0,0)}(\mathrm{red}(\sigma_1 \dots \sigma_{i-1})) = \mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \dots \sigma_i))$. Thus, the number of nodes with $\geq k$ left edges on the path to the root is $\mathrm{mmp}^{(k,0,0,0)}(\mathrm{red}(\sigma_1 \dots \sigma_i)) + \mathrm{mmp}^{(k,0,0,0)}(\sigma_{i+1} \dots \sigma_n) = \mathrm{mmp}^{(k,0,0,0)}(\sigma)$.

It is clear that the number of nodes with left subtrees of size $\geq k$ is equal to the sum of those from each subtree of the root, possibly plus one for the root itself. In other words, if $\chi(\text{statement})$ equals 1 if the statement is true and 0 otherwise, then by the induction hypothesis, the number of such nodes is $\mathrm{mmp}^{(0,0,k,0)}(\mathrm{red}(\sigma_1 \dots \sigma_{i-1})) + \mathrm{mmp}^{(0,0,k,0)}(\sigma_{i+1} \dots \sigma_n) + \chi(i > k)$, again regarding each term as 0 if there is no corresponding subtree. However, since n does not affect whether any other point matches $MMP(0,0,k,0)$ and matches itself whenever $i > k$, we see this is precisely equal to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$. $\square$

Thus we have the following corollary.

Corollary 2. *Let $k \geq 1$.*

1. *The number of permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(k,0,0,0)}(\sigma) = 0$ equals the number of rooted binary trees $T \in \mathcal{B}_n$ which have no nodes η with $\geq k$ left edges on the path from η to the root of T.*

2. *The number of permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = 0$ equals the number of rooted binary trees $T \in \mathcal{B}_n$ such that there is no node η of T whose left subtree has size $\geq k$.*

3 The function $Q_{n,132}^{(k,0,0,0)}(x)$

Throughout this paper, we shall classify the 132-avoiding permutations $\sigma = \sigma_1 \ldots \sigma_n$ by the position of n in σ. Let $S_n^{(i)}(132)$ denote the set of $\sigma \in S_n(132)$ such that $\sigma_i = n$.

Clearly each $\sigma \in S_n^{(i)}(132)$ has the structure pictured in Figure 5. That is, in the graph of σ, the elements to the left of n, $A_i(\sigma)$, have the structure of a 132-avoiding permutation, the elements to the right of n, $B_i(\sigma)$, have the structure of a 132-avoiding permutation, and all the elements in $A_i(\sigma)$ lie above all the elements in $B_i(\sigma)$. As mentioned above, the number of 132-avoiding permutations in S_n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and the generating function for these numbers is given by

$$C(t) = \sum_{n \geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}}. \tag{3}$$

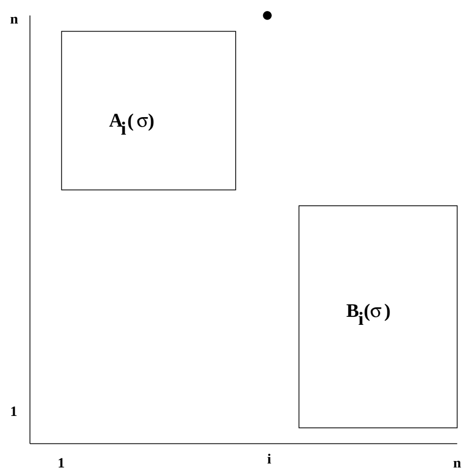


Figure 5: The structure of 132-avoiding permutations.

Clearly,

$$Q_{132}^{(0,0,0,0)}(t,x) = \sum_{n \geq 0} C_n x^n t^n = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt}.$$

Next we consider $Q_{132}^{(k,0,0,0)}(t,x)$. If $k \geq 1$, then it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(k-1,0,0,0)}(x)$ to $Q_{n,132}^{(k,0,0,0)}(x)$, since each of the elements to the left of n will match the pattern $MMP(k,0,0,0)$ if it matches the pattern $MMP(k-1,0,0,0)$ in the graph of $A_i(\sigma)$, and our choices for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,0,0)}(x)$ to $Q_{n,132}^{(k,0,0,0)}(x)$ because the elements to the left of $B_i(\sigma)$ have no effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(k,0,0,0)$. It follows that

$$Q_{n,132}^{(k,0,0,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,0,0)}(x) Q_{n-i,132}^{(k,0,0,0)}(x). \tag{4}$$

Multiplying both sides of (4) by t^n and summing for $n \geq 1$, we see that

$$-1 + Q_{132}^{(k,0,0,0)}(t,x) = tQ_{132}^{(k-1,0,0,0)}(t,x)\ Q_{132}^{(k,0,0,0)}(t,x).$$

Hence for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}.$$

Thus we have the following theorem.

Theorem 4.

$$Q_{132}^{(0,0,0,0)}(t,x) = C(xt) = \frac{1-\sqrt{1-4xt}}{2xt} \tag{5}$$

and, for $k \geq 1$,

$$Q_{132}^{(k,0,0,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,x)}. \tag{6}$$

Theorem 4 immediately implies the following corollary.

Corollary 3.

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t} \tag{7}$$

and, for $k \geq 2$,

$$Q_{132}^{(k,0,0,0)}(t,0) = \frac{1}{1 - tQ_{132}^{(k-1,0,0,0)}(t,0)}. \tag{8}$$

Thus one can easily compute that

$$\begin{aligned}
Q_{132}^{(2,0,0,0)}(t,0) &= \frac{1-t}{1-2t}, \\
Q_{132}^{(3,0,0,0)}(t,0) &= \frac{1-2t}{1-3t+t^2}, \\
Q_{132}^{(4,0,0,0)}(t,0) &= \frac{1-3t+t^2}{1-4t+3t^2}, \\
Q_{132}^{(5,0,0,0)}(t,0) &= \frac{1-4t+3t^2}{1-5t+6t^2-t^3}, \\
Q_{132}^{(6,0,0,0)}(t,0) &= \frac{1-5t+6t^2-t^3}{1-6t+10t^2-4t^3}, \text{ and} \\
Q_{132}^{(7,0,0,0)}(t,0) &= \frac{1-6t+10t^3-4t^3}{1-7t+15t^2-10t^3+t^4}.
\end{aligned}$$

By Corollary 1, $Q_{132}^{(k,0,0,0)}(t,0)$ is also the generating function for Dyck paths whose maximum height is less than or equal to k. For example, this interpretation is given to sequence A080937 in the OEIS, which is the sequence $(Q_{n,132}^{(5,0,0,0)}(0))_{n\geq 0}$, and to sequence A080938 in the OEIS, which is the sequence $(Q_{n,132}^{(7,0,0,0)}(0))_{n\geq 0}$. However, similar interpretations are not given to $(Q_{n,132}^{(k,0,0,0)}(0))_{n\geq 0}$ where $k \notin \{5,7\}$. For example, such an interpretation is not found for $(Q_{n,132}^{(2,0,0,0)}(0))_{n\geq 0}$, $(Q_{n,132}^{(3,0,0,0)}(0))_{n\geq 0}$, $(Q_{n,132}^{(4,0,0,0)}(0))_{n\geq 0}$, or $(Q_{n,132}^{(6,0,0,0)}(0))_{n\geq 0}$, which are sequences A011782, A001519, A124302, and A024175 in the OEIS, respectively. Similarly, by Corollary 2, the generating function $Q_{132}^{(k,0,0,0)}(t,0)$ is the generating function of rooted binary trees T which have no nodes η such that there are $\geq k$ left edges on the path from η to the root of T.

We can easily compute the first few terms of $Q_{132}^{(k,0,0,0)}(t,x)$ for small k using Mathematica or Maple. For example, we have computed the following.

$$\begin{aligned}
&Q_{132}^{(1,0,0,0)}(t,x) = 1+t+(1+x)t^2+\left(1+2x+2x^2\right)t^3+\left(1+3x+5x^2+5x^3\right)t^4 \\
&\left(1+4x+9x^2+14x^3+14x^4\right)t^5+\left(1+5x+14x^2+28x^3+42x^4+42x^5\right)t^6+ \\
&\left(1+6x+20x^2+48x^3+90x^4+132x^5+132x^6\right)t^7+ \\
&\left(1+7x+27x^2+75x^3+165x^4+297x^5+429x^6+429x^7\right)t^8+ \\
&\left(1+8x+35x^2+110x^3+275x^4+572x^5+1001x^6+1430x^7+1430x^8\right)t^9+\cdots.
\end{aligned}$$

Note in this case, it is quite easy to explain some of the coefficients that appear in the polynomials $Q_{n,132}^{(1,0,0,0)}(x)$. That is, we have the following theorem.

Theorem 5. *1.* $Q_{n,132}^{(1,0,0,0)}(0) = 1$ *for* $n \geq 1$,

2. $Q_{n,132}^{(1,0,0,0)}(x)|_x = n-1$ *for* $n \geq 2$,

3. $Q_{n,132}^{(1,0,0,0)}(x)|_{x^2} = \binom{n}{2} - 1$ *for* $n \geq 3$,

4. $Q^{(1,0,0,0)}_{n,132}(x)|_{x^{n-1}} = C_{n-1}$ *for* $n \geq 1$, *and*

5. $Q^{(1,0,0,0)}_{n,132}(x)|_{x^{n-2}} = C_{n-1}$ *for* $n \geq 2$.

Proof. There is only one permutation $\sigma \in S_n$ with $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 0$, namely, $\sigma = n(n-1)\dots 1$. Thus the constant term in $Q^{(1,0,0,0)}_{n,132}(x)$ is always 1. Also the only way to get a permutation $\sigma \in S_n$ that has $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-1$ is to have $\sigma_n = n$. It follows that the coefficient of x^{n-1} in $Q^{(1,0,0,0)}_{n,132}(x)$ is the number of permutations $\sigma \in S_n(132)$ such that $\sigma_n = n$ which is clearly C_{n-1}. It is also easy to see that the only permutations $\sigma \in S_n(132)$ with $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 1$ are the permutations of the form

$$\sigma = n(n-1)\dots(i+1)(i-1)i(i-2)\dots 21.$$

Thus the coefficient of x in $Q^{(1,0,0,0)}_{n,132}(x)$ is always $n-1$.

For (3), note that we have $Q^{(1,0,0,0)}_{3,132}(x)|_{x^2} = 2 = \binom{3}{2} - 1$. For $n \geq 4$, let $a(n)$ denote the coefficient of x^2 in $Q^{(1,0,0,0)}_{n,132}(x)$. The permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$ must have either $\sigma_1 = n$, $\sigma_2 = n$, or $\sigma_3 = n$. If $\sigma_3 = n$, it must be the case that $\{\sigma_1, \sigma_2\} = \{n-1, n-2\}$ and that $\mathrm{mmp}^{(1,0,0,0)}(\sigma_4 \dots \sigma_n) = 0$. Thus $\sigma_4 \dots \sigma_n$ must be decreasing, so there are exactly two permutations $\sigma \in S_n(132)$ such that $\sigma_3 = n$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$. If $\sigma_2 = n$, it must be the case that $\sigma_1 = n-1$ and that $\mathrm{mmp}^{(1,0,0,0)}(\sigma_3 \dots \sigma_n) = 1$. In that case, we know that there are $n-3$ choices for $\sigma_3 \dots \sigma_n$, so there are $n-3$ permutations $\sigma \in S_n(132)$ such that $\sigma_2 = n$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$. Finally, it is clear that if $\sigma_1 = n$, then we must have that $\mathrm{mmp}^{(1,0,0,0)}(\sigma_2 \dots \sigma_n) = 2$, so there are $a(n-1)$ permutations $\sigma \in S_n(132)$ such that $\sigma_1 = n$ and $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = 2$. Thus we have shown that $a(n) = a(n-1) + n - 1$ from which it easily follows by induction that $a(n) = \binom{n}{2} - 1$.

Finally, for (5), let $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$ be such that $\mathrm{mmp}^{(1,0,0,0)}(\sigma) = n-2$. We clearly cannot have $\sigma_n = n$ so that n and σ_n must be the two elements of σ that do not match the pattern $MMP(1,0,0,0)$ in σ. Now if $\sigma_i = n$, then $B_i(\sigma)$ consists of the elements $1, \dots, n-i$. But then it must be the case that $\sigma_n = n-i$. Note that this implies that σ_n can be removed from σ in a completely reversible way. That is, $\sigma \to \mathrm{red}(\sigma_1 \dots \sigma_{n-1})$ is a bijection onto $S_{n-1}(132)$. Hence there are C_{n-1} such σ. $\square$

$$\begin{aligned}
&Q^{(2,0,0,0)}_{132}(t,x) = 1 + t + 2t^2 + (4+x)t^3 + \left(8 + 4x + 2x^2\right)t^4 + \\
&\left(16 + 12x + 9x^2 + 5x^3\right)t^5 + \left(32 + 32x + 30x^2 + 24x^3 + 14x^4\right)t^6 + \\
&\left(64 + 80x + 88x^2 + 85x^3 + 70x^4 + 42x^5\right)t^7 + \\
&\left(128 + 192x + 240x^2 + 264x^3 + 258x^4 + 216x^5 + 132x^6\right)t^8 + \\
&\left(256 + 448x + 624x^2 + 760x^3 + 833x^4 + 819x^5 + 693x^6 + 429x^7\right)t^9 + \cdots.
\end{aligned}$$

Again it is easy to explain some of these coefficients. That is, we have the following theorem.

Theorem 6. *1.* $Q^{(2,0,0,0)}_{n,132}(0) = 2^{n-1}$ *if* $n \geq 3$,

2. *for $n \geq 3$, the highest power of x which appears in $Q_{n,132}^{(2,0,0,0)}(x)$ is x^{n-2} and $Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-2}} = C_{n-2}$, and*

3. $Q_{n,132}^{(2,0,0,0)}(x)|_x = (n-2)2^{n-3}$ *for $n \geq 3$.*

Proof. It is easy to see that the only $\sigma \in S_n(132)$ that have $\mathrm{mmp}^{(2,0,0,0)}(\sigma) = n-2$ must have $\sigma_{n-1} = n-1$ and $\sigma_n = n$. Note that if $\sigma_{n-1} = n$ and $\sigma_n = n-1$ then we have an occurrence of 132 for $n \geq 3$. Thus the coefficient of x^{n-2} in $Q_{n,132}^{(2,0,0,0)}(x)$ is C_{n-2} if $n \geq 3$.

The fact that $Q_{n,132}^{(2,0,0,0)}(x) = 2^{n-1}$ for $n \geq 1$ is an immediate consequence of the fact that $Q^{(2,0,0,0)}(t,0) = \frac{1-t}{1-2t}$. In fact, this is a known result [5] since avoidance of the pattern $MMP(2,0,0,0)$ is equivalent to avoiding simultaneously the (classical) patterns 132 and 123. One can also give a simple combinatorial proof of this fact. Clearly it is true for $n=1$. For $n \geq 2$, note that σ_1 must be either n or $n-1$. Also, $\mathrm{red}(\sigma_2 \ldots \sigma_n)$ must avoid the pattern $MMP(2,0,0,0)$. Since every permutation $\mathrm{red}(\sigma_2 \ldots \sigma_n)$ avoiding $MMP(2,0,0,0)$ can be obtained in this manner in exactly two ways, once with $\sigma_1 = n$ and once with $\sigma_n = n-1$, we see that there are $2 \cdot 2^{n-2} = 2^{n-1}$ such σ.

The sequence $(Q_{n,132}^{(2,0,0,0)}(x)|_x)_{n\geq 3}$ is the sequence

$$1, 4, 12, 32, 80, 192, 448, \ldots,$$

which is sequence A001787 in OEIS, whose n-th term is $a_n = n2^{n-1}$. Now a_n has many combinatorial interpretations including the number of edges in the n-dimensional hypercube and the number of permutations in $S_{n+2}(132)$ with exactly one occurrence of the pattern 123. The ordinary generating function of the sequence is $\frac{x}{(1-2x)^2}$, which implies that

$$Q_{132}^{(2,0,0,0)}(t,x)|_x = \frac{t^3}{(1-2t)^2}. \tag{9}$$

This can be proved in two different ways. That is, for any $k \geq 2$,

$$\begin{aligned}
Q_{132}^{(k,0,0,0)}(t,x)|_x &= \left(\frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,x)}\right)\Big|_x \\
&= \left(1 + \sum_{n\geq 1} t^n (Q_{132}^{(k-1,0,0,0)}(t,x))^n\right)\Big|_x \\
&= \sum_{n\geq 1} nt^n (Q_{132}^{(k-1,0,0,0)}(t,0))^{n-1} Q_{132}^{(k-1,0,0,0)}(t,x)|_x \\
&= Q_{132}^{(k-1,0,0,0)}(t,x)|_x \sum_{n\geq 1} nt^n (Q_{132}^{(k-1,0,0,0)}(t,0))^{n-1}. \qquad (10)
\end{aligned}$$

However

$$\begin{aligned}
\frac{d}{dt} Q^{(k,0,0,0)}(t,0) &= \frac{d}{dt}\left(\frac{1}{1-tQ_{132}^{(k-1,0,0,0)}(t,0)}\right) \\
&= \sum_{n\geq 1} n(tQ_{132}^{(k-1,0,0,0)}(t,0))^{n-1} \frac{d}{dt}\left(tQ_{132}^{(k-1,0,0,0)}(t,0)\right)
\end{aligned}$$

so that

$$\frac{t\frac{d}{dt}Q^{(k,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(k-1,0,0,0)}(t,0)\right)} = \sum_{n\geq 1} nt^n (Q_{132}^{(k-1,0,0,0)}(t,0))^{n-1}. \tag{11}$$

Combining (10) and (11), we obtain the recursion

$$Q_{132}^{(k,0,0,0)}(t,x)|_x = Q_{132}^{(k-1,0,0,0)}(t,x)|_x \frac{t\frac{d}{dt}Q^{(k,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(k-1,0,0,0)}(t,0)\right)}. \tag{12}$$

But then

$$Q_{132}^{(1,0,0,0)}(t,x)|_x = \sum_{n\geq 2}(n-1)t^n = \frac{t^2}{(1-t)^2}$$

and

$$Q_{132}^{(1,0,0,0)}(t,0) = \frac{1}{1-t} \text{ and } Q_{132}^{(2,0,0,0)}(t,0) = \frac{1-t}{1-2t}.$$

Thus

$$\begin{aligned} Q_{132}^{(2,0,0,0)}(t,x)|_x &= Q_{132}^{(1,0,0,0)}(t,x)|_x \frac{t\frac{d}{dt}Q^{(2,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ_{132}^{(1,0,0,0)}(t,0)\right)} \\ &= \frac{t^2}{(1-t)^2}\frac{t\frac{d}{dt}\left(\frac{1-t}{1-2t}\right)}{\frac{d}{dt}\frac{t}{1-t}} \\ &= \frac{t^3}{(1-2t)^2}. \end{aligned} \tag{13}$$

We can also give a direct proof of this result. That is, we can give a direct proof of the fact that for $n \geq 3$, $b(n) = Q_{n,132}^{(2,0,0,0)}(x)|_x = (n-2)2^{n-3}$. Note that $b(3) = 1 = (3-2)2^{3-3}$ and $b(4) = (4-2)2^{4-3} = 4$ so that our claim holds for $n = 3, 4$. Then let $n \geq 5$ and assume by induction that $b(k) = (k-2)2^{k-3}$ for $3 \leq k < n$. Now suppose that $\sigma \in S_n^{(i)}(132)$ and $\mathrm{mmp}^{(2,0,0,0)} = 1$. If the element of σ that matches $MMP(2,0,0,0)$ occurs in $A_i(\sigma)$, then it must be the case that $\mathrm{mmp}^{(1,0,0,0)}(A_i(\sigma)) = 1$ and $\mathrm{mmp}^{(2,0,0,0)}(B_i(\sigma)) = 0$. By our previous results, we have $(i-2)$ choices for $A_i(\sigma)$ and $a(n-i) = 2^{n-i-1}$ choices for $B_i(\sigma)$. Note that this can happen only for $3 \leq i \leq n-1$ so that such permutations contribute

$$\sum_{i=3}^{n-1}(i-2)2^{n-i-1} = \sum_{j=1}^{n-3} j2^{n-3-j} = \sum_{k=0}^{n-4} 2^k(n-3-k)$$

to $b(n)$. If the element of σ which matches $MMP(2,0,0,0)$ occurs in $B_i(\sigma)$, then it must be the case that $\mathrm{mmp}^{(1,0,0,0)}(A_i(\sigma)) = 0$ which means that $A_i(\sigma)$ is decreasing and $\mathrm{mmp}^{(2,0,0,0)}(B_i(\sigma)) = 1$. This can happen only for $1 \leq i \leq n-3$. Thus such permutations will contribute

$$b(3) + \cdots + b(n-1) = \sum_{i=3}^{n-1}(i-2)2^{(i-3)} = \sum_{k=0}^{n-4} 2^k(k+1)$$

to $b(n)$. The only permutations that we have not accounted for are the permutations $\sigma = \sigma_1 \dots \sigma_n \in S_n(132)$ where $\sigma_n = n$ and $\mathrm{mmp}^{(2,0,0,0)}(\sigma_1 \dots \sigma_{n-1}) = 1$, and there are $n-2$ such permutations. Thus

$$\begin{aligned} b(n) &= (n-2) + \sum_{k=0}^{n-4} 2^k (n-3-k+k+1) \\ &= (n-2)(1+\sum_{k=0}^{n-4} 2^k) \\ &= (n-2)(1+2^{n-3}-1) = (n-2)2^{n-3}. \end{aligned}$$

□

Problem 1. Comparing the sequence $(Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-3}})_{n\geq 4}$ with sequence A038629 in the OEIS, it seems that $Q_{n,132}^{(2,0,0,0)}(x)|_{x^{n-3}} = C_{n-2} + 2C_{n-3}$. Is this the case?

$$\begin{aligned} &Q_{132}^{(3,0,0,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (13+x)t^4 + \left(34+6x+2x^2\right)t^5 + \\ &\left(89+25x+13x^2+5x^3\right)t^6 + \left(233+90x+58x^2+34x^3+14x^4\right)t^7 + \\ &\left(610+300x+222x^2+158x^3+98x^4+42x^5\right)t^8 + \\ &\left(1597+954x+783x^2+628x^3+468x^4+300x^5+132x^6\right)t^9 + \cdots. \end{aligned}$$

The sequence $(Q_n^{(3,0,0,0)}(0))_{n\geq 0}$ is sequence A001519 in the OEIS whose terms satisfy the recursion $a(n) = 3a(n-1) - a(n-2)$ with $a(0) = a(1) = 1$. That is, since $Q_{132}^{(3,0,0,0)}(t,0) = \frac{1-2t}{1-3t+t^2}$, it is easy to see that for $n \geq 2$,

$$Q_n^{(3,0,0,0)}(0) = 3Q_{n-1}^{(3,0,0,0)}(0) - Q_{n-2}^{(3,0,0,0)}(0) \tag{14}$$

with $Q_0^{(3,0,0,0)}(0) = Q_1^{(3,0,0,0)}(0) = 1$. This is a known fact [5], since the avoidance of the pattern $MMP(3,0,0,0)$ is equivalent to avoiding the six (classical) patterns of length 4 beginning with the smallest element plus the pattern 132. This is equivalent to simultaneously avoiding 132 and 1234 which is one of the combinatorial interpretations given to sequence A001519 in the OEIS. We note that the OEIS also gives another combinatorial interpretation of this sequence as the number of permutations $\sigma \in S_{n+1}$ that avoid the patterns 321 and 3412.

Problem 2. Can one give a combinatorial proof of (14)?

Problem 3. Do any of the known bijections between $S_n(132)$ and $S_n(321)$ (see [5]) send $(132, 1234)$-avoiding permutations to $(321, 3412)$-avoiding permutations? If not, find such a bijection.

The sequence $(Q^{(3,0,0,0)}_{n,132}(x)|_x)_{n\geq 4}$ is sequence A001871 in the OEIS, which has the generating function $\frac{1}{(1-3x+x^2)^2}$. The n-th term of this sequence counts the number of 3412-avoiding permutations containing exactly one occurrence of the pattern 321.

We can use the recursion (12) to prove this. That is,

$$\begin{aligned}
Q^{(3,0,0,0)}_{132}(t,x)|_x &= Q^{(2,0,0,0)}_{132}(t,x)|_x \frac{t\frac{d}{dt}Q^{(3,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ^{(2,0,0,0)}_{132}(t,0)\right)} \\
&= \frac{t^3}{(1-2t)^2}\cdot\frac{t\frac{d}{dt}\left(\frac{1-2t}{1-3t+t^2}\right)}{\frac{d}{dt}\frac{t(1-t)}{1-2t}} \\
&= \frac{t^4}{(1-3t-t^2)^2}. \qquad (15)
\end{aligned}$$

$$Q^{(4,0,0,0)}_{132}(t,x) = 1+t+2t^2+5t^3+14t^4+(41+x)t^5+\left(122+8x+2x^2\right)t^6+ \left(365+42x+17x^2+5x^3\right)t^7+\left(1094+184x+94x^2+44x^3+14x^4\right)t^8+ \left(3281+731x+431x^2+251x^3+126x^4+42x^5\right)t^9+\cdots.$$

The sequence $(Q^{(4,0,0,0)}(t,0))_{n\geq 1}$ is A007051 in the OEIS. It is easy to compute that

$$\begin{aligned}
Q^{(4,0,0,0)}(t,0) &= \frac{1-3t+t^2}{1-4t+3t^2} \\
&= \frac{1-3t+t^2}{(1-t)(1-3t)} \\
&= 1+\sum_{n\geq 1}\frac{3^{n-1}+1}{2}t^n.
\end{aligned}$$

Thus for $n\geq 1$, $Q^{(4,0,0,0)}_n(0)=\frac{3^{n-1}+1}{2}$, which also counts the number of ordered trees with $n-1$ edges and height at most 4.

The sequence $(Q^{(4,0,0,0)}(t,x)|_x)_{n\geq 5}$, which is

$$1, 8, 42, 184, 731, \ldots,$$

does not appear in the OEIS. However, we can use the recursion (12) to find its generating function. That is,

$$\begin{aligned}
Q^{(4,0,0,0)}_{132}(t,x)|_x &= Q^{(3,0,0,0)}_{132}(t,x)|_x \frac{t\frac{d}{dt}Q^{(4,0,0,0)}(t,0)}{\frac{d}{dt}\left(tQ^{(3,0,0,0)}_{132}(t,0)\right)} \\
&= \frac{t^4}{(1-3t+t^2)^2}\frac{t\frac{d}{dt}\left(\frac{1-3t+t^2}{1-4t+3t^2}\right)}{\frac{d}{dt}\frac{t(1-2t)}{1-3t+t^2}} \\
&= \frac{t^5}{(1-4t+3t^2)^2}. \qquad (16)
\end{aligned}$$

4 The function $Q_{n,132}^{(0,0,k,0)}(x)$

Fix $k \geq 1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,0,k,0)}(x)$ to $Q_{n,132}^{(0,0,k,0)}(x)$ since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,0,k,0)$ in σ. Similarly our choices for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(0,0,k,0)}(x)$ to $Q_{n,132}^{(0,0,k,0)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(0,0,k,0)$ in σ. Note that n will contribute a factor of x to $Q_{n,132}^{(0,0,k,0)}(x)$ if and only if $k < i$.

It follows that

$$Q_{n,132}^{(0,0,k,0)}(x) = \sum_{i=1}^{k} Q_{i-1,132}^{(0,0,k,0)}(x) Q_{n-i,132}^{(0,0,k,0)}(x) + x \sum_{i=k+1}^{n} Q_{i-1,132}^{(0,0,k,0)}(x) Q_{n-i,132}^{(0,0,k,0)}(x). \tag{17}$$

Note that if $i \leq k$, $Q_{i-1,132}^{(0,0,k,0)}(x) = C_{i-1}$. Thus

$$Q_{n,132}^{(0,0,k,0)}(x) = \sum_{i=1}^{k} C_{i-1} Q_{n-i,132}^{(0,0,k,0)}(x) + x \sum_{i=k+1}^{n} Q_{i-1,132}^{(0,0,k,0)}(x) Q_{n-i,132}^{(0,0,k,0)}(x). \tag{18}$$

Multiplying both sides of (18) by t^n and summing for $n \geq 1$, will show that

$$\begin{aligned} -1 + Q_{132}^{(0,0,k,0)}(t,x) &= t(C_0 + C_1 t + \cdots + C_{k-1}t^{k-1}) Q_{132}^{(0,0,k,0)}(t,x) + \\ &\quad tx Q_{132}^{(0,0,k,0)}(t,x)(Q_{132}^{(0,0,k,0)}(t,x) - (C_0 + C_1 t + \cdots + C_{k-1}t^{k-1})). \end{aligned}$$

Hence we obtain the quadratic equation

$$0 = 1 - (-1 + (t - tx)(C_0 + C_1 t + \cdots + C_{k-1}t^{k-1})) Q_{132}^{(0,0,k,0)}(t,x) + tx(Q_{132}^{(0,0,k,0)}(t,x))^2. \tag{19}$$

This implies the following theorem.

Theorem 7. *For $k \geq 1$,*

$$\begin{aligned} Q_{132}^{(0,0,k,0)}(t,x) &= \frac{1 + (tx - t)(\sum_{j=0}^{k-1} C_j t^j) - \sqrt{(1 + (tx - t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}}{2tx} \\ &= \frac{2}{1 + (tx - t)(\sum_{j=0}^{k-1} C_j t^j) + \sqrt{(1 + (tx - t)(\sum_{j=0}^{k-1} C_j t^j))^2 - 4tx}} \end{aligned} \tag{20}$$

and

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1 - t(C_0 + C_1 t + \cdots + C_{k-1}t^{k-1})}. \tag{21}$$

By Corollary 1, $Q_{132}^{(0,0,k,0)}(t,0)$ is also the generating function of all Dyck paths that have no interval of length $\geq 2k$ and the generating function of all rooted binary trees T such that T has no node η whose left subtree has size $\geq k$.

It is easy to explain the highest power and the second highest power of x which occurs in $Q_{n,132}^{(0,0,k,0)}(x)$ for any $k \geq 1$.

Theorem 8. *1. For all $k \geq 1$ and $n > k$, the highest power of x which occurs in $Q_{n,132}^{(0,0,k,0)}(x)$ is x^{n-k} and $Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k}} = C_k$ and*

2. $Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k-1}} = C_{k+1} - C_k + 2(n-k-1)C_{k-1}$.

Proof. For (1), it is easy to see that for any $k \geq 1$, the maximum number of $MMP(0,0,k,0)$-matches occurs in a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ only when $\sigma_1 \ldots \sigma_k \in S_k(132)$ and $\sigma_{k+1} \ldots \sigma_n$ is strictly increasing. Thus $Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k}} = C_k$ for $n \geq k+1$.

For (2), suppose that $k \geq 3$ and $a_{n,k} = Q_{n,132}^{(0,0,k,0)}(x)|_{x^{n-k-1}}$ where $n \geq k+1$. Then suppose that $\sigma = \sigma_1 \ldots \sigma_{n+1} \in S_{n+1}(132)$ is such that $\mathrm{mmp}^{(0,0,2,0)}(\sigma) = n-k$. Then if $\sigma_{n+1} = n+1$, we must have $\mathrm{mmp}^{(0,0,2,0)}(\sigma_1 \ldots \sigma_n) = n-k-1$ so that we have $a_{n,k}$ choices for $\sigma_1 \ldots \sigma_n$. If $\sigma_1 = n$, then $\mathrm{mmp}^{(0,0,2,0)}(\sigma_2 \ldots \sigma_{n+1}) = n-k$ so that we have C_{k-1} choices for $\sigma_2 \ldots \sigma_{n+1}$. If $\sigma_n = n$, then $\sigma_{n+1} = 1$ and $\mathrm{mmp}^{(0,0,2,0)}(\sigma_1 \ldots \sigma_n) = n-k$ so that we have C_{k-1} choices for $\sigma_1 \ldots \sigma_n$. If $\sigma_i = n$ where $2 \leq i \leq k$, then $\sigma_1 \ldots \sigma_k$ cannot contribute to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$ so that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = \mathrm{mmp}^{(0,0,k,0)}(\sigma_{k+1} \ldots \sigma_{n+1}) \leq n-i-k < n-k-1$. If $\sigma_i = n$ where $n-k+1 \leq n-1$, then $\sigma_{i+1} \ldots \sigma_{n+1}$ cannot contribute to $\mathrm{mmp}^{(0,0,k,0)}(\sigma)$ so that $\mathrm{mmp}^{(0,0,k,0)}(\sigma) = \mathrm{mmp}^{(0,0,k,0)}(\sigma_1 \ldots \sigma_i) \leq n-i-k < n-k-1$. Finally if $\sigma_i = n$ where $k+1 \leq i \leq n-k-2$, then

$$\begin{aligned}\mathrm{mmp}^{(0,0,k,0)}(\sigma) &= \mathrm{mmp}^{(0,0,k,0)}(\sigma_1 \ldots \sigma_i) + \mathrm{mmp}^{(0,0,k,0)}(\sigma_{i+1} \ldots \sigma_{n+1}) \\ &\leq i-k+(n+1-i-k) = n+1-2k < n-k-1.\end{aligned}$$

Thus it follows that for $n \geq k+1$, $a_{n,k}$ satisfies the recursion

$$a_{n+1,k} = a_{n,k} + 2C_{k-1}. \tag{22}$$

In general, if $n = k+1$, then there are $C_{k+1} - C_k$ permutations in $S_n(132)$ avoiding $MMP(0,0,k,0)$, namely, those that do not have $\sigma_{k+1} = k+1$. Using this as the base case, we may solve recursion (22) to obtain that $a_{n,k} = C_{k+1} - C_k + 2(n-k-1)C_{k-1}$. ☐

Again, we can easily use Mathematica to compute some examples of these generating functions.

$$\begin{aligned}&Q_{132}^{(0,0,1,0)}(t,x) = 1 + t + (1+x)t^2 + \left(1+3x+x^2\right)t^3 + \left(1+6x+6x^2+x^3\right)t^4 + \\ &\left(1+10x+20x^2+10x^3+x^4\right)t^5 + \left(1+15x+50x^2+50x^3+15x^4+x^5\right)t^6 + \\ &\left(1+21x+105x^2+175x^3+105x^4+21x^5+x^6\right)t^7 + \\ &\left(1+28x+196x^2+490x^3+490x^4+196x^5+28x^6+x^7\right)t^8 + \\ &\left(1+36x+336x^2+1176x^3+1764x^4+1176x^5+336x^6+36x^7+x^8\right)t^9 + \cdots .\end{aligned}$$

It is easy to explain several of the coefficients of $Q_{n,132}^{(0,0,1,0)}(x)$. That is, the following hold.

Theorem 9. *1. $Q_{n,132}^{(0,0,1,0)}(0) = 1$ for $n \geq 1$,*

2. $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-1}} = 1$ for $n \geq 2$,

3. $Q_{n,132}^{(0,0,1,0)}(x)|_{x} = \binom{n}{2}$ *for* $n \geq 2$, *and*

4. $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$ *for* $n \geq 3$.

Proof. It is easy to see that $n(n-1)\dots 1$ is the only permutation $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(0,0,1,0)}(\sigma) = 0$. Thus $Q_{n,132}^{(0,0,1,0)}(0) = 1$ for all $n \geq 1$. Similarly, for $n \geq 2$, $\sigma = 12\dots(n-1)n$ is the only permutation in $S_n(132)$ with $\mathrm{mmp}^{(0,0,1,0)}(\sigma) = n-1$ so that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-1}} = 1$ for $n \geq 2$.

To prove (3), let $\sigma^{(i,j)} = n(n-1)\dots(j+1)(j-1)\dots ij(i-1)\dots 1$ for any $1 \leq i < j \leq n$. It is easy to see that $\mathrm{mmp}^{(0,0,1,0)}(\sigma^{(i,j)}) = 1$ and that these are the only permutations σ in $S_n(132)$ such that $\mathrm{mmp}^{(0,0,1,0)}(\sigma) = 1$. Thus $Q_{n,132}^{(0,0,1,0)}(x)|_{x} = \binom{n}{2}$ for $n \geq 2$.

For (4), we prove by induction that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$ for $n \geq 3$. Clearly the theorem holds for $n = 3, 4$. Now suppose that $n \geq 5$ and $\sigma \in S_n(132)$ and $\mathrm{mmp}^{(0,0,1,0)}(\sigma) = n-2$. Then if $\sigma_n = n$, it must be the case that $\sigma_1 \dots \sigma_{n-1} \in S_{n-1}$ and $\mathrm{mmp}^{(0,0,1,0)}(\sigma_1 \dots \sigma_{n-1}) = n-3$ so by induction we have $\binom{n-1}{2}$ choices for $\sigma_1 \dots \sigma_{n-1}$. If $\sigma_i = n$, then it must be the case that $\sigma = (n-k+1)\dots(n-1)n12\dots(n-k)$ so that there are $n-1$ such permutations where $\sigma_n \neq n$. $\square$

More generally, one can observe that x^j and x^{n-j-1} have the same coefficient in $Q_{n,132}^{(0,0,1,0)}(x)$. This will be proved later; see the discussion of (42).

$$\begin{aligned}
&Q_{132}^{(0,0,2,0)}(t,x) = 1 + t + 2t^2 + (3+2x)t^3 + \left(5 + 7x + 2x^2\right)t^4 + \\
&\left(8 + 21x + 11x^2 + 2x^3\right)t^5 + \left(13 + 53x + 49x^2 + 15x^3 + 2x^4)\right)t^6 + \\
&\left(21 + 124x + 174x^2 + 89x^3 + 19x^4 + 2x^5\right)t^7 + \\
&\left(34 + 273x + 546x^2 + 411x^3 + 141x^4 + 23x^5 + 2x^6\right)t^8 + \\
&\left(55 + 577x + 1557x^2 + 1635x^3 + 804x^4 + 205x^5 + 27x^6 + 2x^7\right)t^9 + \cdots.
\end{aligned}$$

We then have the following proposition.

Proposition 1. *1.* $Q_{n,132}^{(0,0,2,0}(0) = F_n$ *where* F_n *is the* n*-th Fibonacci number and*

2. $Q_{n,132}^{(0,0,2,0)}(x)|_{x^{n-3}} = 3 + 4(n-3)$.

Proof. In this case, we know that $Q_{132}^{(0,0,2,0)}(t,0) = \frac{1}{1-t(C_0+C_1t)} = \frac{1}{1-t-t^2}$, so the sequence $\{Q_{n,132}^{(0,0,2,0)}(0)\}_{n\geq 0}$ is the Fibonacci numbers. This result is known [5], since the avoidance of $MMP(0,0,2,0)$ is equivalent to the avoidance of the patterns 123 and 213 simultaneously, so in this case we are dealing with the multi-avoidance of the classical patterns 132, 123, and 213.

The fact that $Q_{n,132}^{(0,0,2,0)}(x)|_{x^{n-3}} = 3 + 4(n-3)$ is a special case of Theorem 8. $\square$

The sequence $(Q_{n,132}^{(0,0,2,0)}(x)|_{x})_{n\geq 3}$ which is $2, 7, 21, 53, 124, 273, 577, \dots$ does not appear in the OEIS.

$$Q_{132}^{(0,0,3,0)}(t,x) = 1+t+2t^2+5t^3+(9+5x)t^4+\left(18+19x+5x^2\right)t^5+$$
$$\left(37+61x+29x^2+5x^3\right)t^6+\left(73+188x+124x^2+39x^3+5x^4\right)t^7+$$
$$(146+523x+500x^2+207x^3+49x^4+5x^5)t^8+$$
$$(293+1387x+1795x^2+1013x^3+310x^4+59x^5+5x^6)t^9+\cdots.$$

In this case, the sequence $\{Q_{n,132}^{(0,0,3,0)}(0)\}_{n\geq 0}$ is A077947 in the OEIS which also counts the number of sequences of codewords of total length n from the code $C=\{0,10,110,111\}$. The basic idea of a combinatorial explanation of this fact is not that difficult to present. Indeed, a permutation avoiding the patterns 132 and $MMP(0,0,3,0)$ is such that to the left of n, the largest element, one can either have no elements, one element $(n-1)$, two elements in increasing order $(n-2)(n-1)$, or two elements in decreasing order $(n-1)(n-2)$. We can then recursively build the codeword corresponding to the permutation beginning with, say, 0, 10, 110 and 111, respectively, corresponding to the four cases; one then applies the same map to the subpermutation to the right of n.

The sequence $(Q_{n,132}^{(0,0,3,0)}(x)|_x)_{n\geq 4}$ which is

$$5, 19, 61, 188, 532, 1387, \ldots$$

does not appear in the OEIS.

$$Q_{132}^{(0,0,4,0)}(t,x) = 1+t+2t^2+5t^3+14t^4+(28+14x)t^5+(62+56x+14x^2)t^6+$$
$$(143+188x+84x^2+14x^3)t^7+(331+603x+307x^2+112x^3+14x^4)t^8+$$
$$(738+1907x+1455x^2+608x^3+140x^4+14x^5)t^9+\cdots.$$

Here neither the sequences $(Q_{n,132}^{(0,0,4,0)}(0))_{n\geq 1}$ nor the sequence $(Q_{n,132}^{(0,0,4,0)}(x)|_x)_{n\geq 5}$ appear in the OEIS.

5 $Q_{n,132}^{(0,k,0,0)}(x) = Q_{n,132}^{(0,0,0,k)}(x)$

Note it follows from Lemma 1 that $Q_{n,132}^{(0,k,0,0)}(x) = Q_{n,132}^{(0,0,0,k)}(x)$ for all $k\geq 1$. Thus in this section, we shall only consider the generating functions $Q_{132}^{(0,k,0,0)}(t,x)$.

First we consider the case $k=1$. It is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,1,0,0)}(x)$ to $Q_{n,132}^{(0,1,0,0)}(x)$, since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,1,0,0)$ in σ. Similarly our choices for $B_i(\sigma)$ will contribute a factor of $C_{n-i}x^{n-i}$ to $Q_{n,132}^{(0,1,0,0)}(x)$, since the presence of n to the left of these elements guarantees that they all match the pattern $MMP(0,1,0,0)$ in σ. Note that n does not match the pattern $MMP(0,1,0,0)$ in σ. It follows that

$$Q_{n,132}^{(0,1,0,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(0,1,0,0)}(x) C_{n-i} x^{n-i}. \tag{23}$$

Multiplying both sides of (23) by t^n and summing for $n \geq 1$ will show that

$$-1 + Q_{132}^{(0,1,0,0)}(t,x) = tQ_{132}^{(0,1,0,0)}(t,x)\ C(tx). \tag{24}$$

Thus

$$Q_{132}^{(0,1,0,0)}(t,x) = \frac{1}{1 - tC(tx)}$$

which is the same as the generating function for $Q_{132}^{(1,0,0,0)}(t,x)$.

Next suppose that $k > 1$. Again, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,k,0,0)}(x)$ to $Q_{n,132}^{(0,k,0,0)}(x)$, since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,k,0,0)$ in σ. Now if $i \geq k$, then our choices for $B_i(\sigma)$ will contribute a factor of $C_{n-i}x^{n-i}$ to $Q_{n,132}^{(0,k,0,0)}(x)$, since the presence of n and the elements of $A_i(\sigma)$ guarantee that the elements of $B_i(\sigma)$ all match the pattern $MMP(0,k,0,0)$ in σ. However, if $i < k$, then our choices for $B_i(\sigma)$ will contribute a factor of $Q_{n,132}^{(0,k-i,0,0)}(x)$ to $Q_{n,132}^{(0,k,0,0)}(x)$ since the presence of n and the elements of $A_i(\sigma)$ to the left of n element guarantees that the elements of $B_i(\sigma)$ match the pattern $MMP(0,k,0,0)$ in σ if and only if they match the pattern $MMP(0,k-i,0,0)$ in $B_i(\sigma)$. Note that n does not match the pattern $MMP(0,k,0,0)$ for any $k \geq 1$. It follows that

$$\begin{aligned} Q_{n,132}^{(0,k,0,0)}(x) &= \sum_{i=1}^{k-1} Q_{i-1,132}^{(0,k,0,0)}(x) Q_{n-i,132}^{(0,k-i,0,0)}(x) + \sum_{i=k}^{n} Q_{i-1,132}^{(0,k,0,0)}(x) C_{n-i}x^{n-i} \\ &= \sum_{i=1}^{k-1} C_{i-1} Q_{n-i,132}^{(0,k-i,0,0)}(x) + \sum_{i=k}^{n} Q_{i-1,132}^{(0,k,0,0)}(x) C_{n-i}x^{n-i}. \end{aligned} \tag{25}$$

Here the last equation follows from the fact that $Q_{i-1,132}^{(0,k,0,0)}(x) = C_{i-1}$ if $i \leq k-1$. Multiplying both sides of (25) by t^n and summing for $n \geq 1$ will show that

$$\begin{aligned} &-1 + Q_{132}^{(0,k,0,0)}(t,x) \\ &= t\sum_{i=1}^{k-1} C_{i-1}t^{i-1} Q_{132}^{(0,k-i,0,0)}(t,x) + tC(tx)(Q_{132}^{(0,k,0,0)}(t,x) - (C_0 + C_1 t + \cdots + C_{k-2}t^{k-2})). \end{aligned}$$

Thus we have the following theorem.

Theorem 10.

$$Q_{132}^{(0,1,0,0)}(t,x) = \frac{1}{1 - tC(tx)}. \tag{26}$$

For $k > 1$,

$$Q_{132}^{(0,k,0,0)}(t,x) = \frac{1 + t\sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t,x) - C(tx))}{1 - tC(tx)} \tag{27}$$

and

$$Q_{132}^{(0,k,0,0)}(t,0) = \frac{1 + t\sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t,0) - 1)}{1 - t}. \tag{28}$$

Then one can compute that

$$
\begin{aligned}
Q_{132}^{(0,1,0,0)}(t,0) &= \frac{1}{(1-t)}; \\
Q_{132}^{(0,2,0,0)}(t,0) &= \frac{1-t+t^2}{(1-t)^2}; \\
Q_{132}^{(0,3,0,0)}(t,0) &= \frac{1-2t+2t^2+t^3-t^4}{(1-t)^3}; \\
Q_{132}^{(0,4,0,0)}(t,0) &= \frac{1-3t+4t^2-t^3+3t^4-5t^5+2t^6}{(1-t)^4}, \text{ and} \\
Q_{132}^{(0,5,0,0)}(t,0) &= \frac{1-4t+7t^2-5t^3+4t^4+6t^5-21t^6+18t^7-5t^8}{(1-t)^5}.
\end{aligned}
$$

We can explain the highest coefficient of x in $Q_n^{(0,k,0,0)}(x)$ for any $k \geq 1$.

Proposition 2. *For all $k \geq 1$, the highest power of x that occurs in $Q_n^{(0,k,0,0)}(x)$ is x^{n-k} and $Q_n^{(0,k,0,0)}(x)|_{x^{n-k}} = C_k C_{n-k}$.*

Proof. It is easy to see that to obtain the largest number of matches of $MMP(0,k,0,0)$ for a permutation $\sigma \in S_n(132)$, we need only arrange the largest k elements $n, n-1, \ldots, n-k+1$ such that they avoid 132, followed by the elements $1, \ldots, n-k$ under the same condition. Thus the highest power of x that occurs in $Q_n^{(0,k,0,0)}(x)$ is x^{n-k} and its coefficient is $C_k C_{n-k}$. $\square$

Again we can use Mathematica or Maple to compute the first few terms of the generating function $Q_{132}^{(0,k,0,0)}(t,x)$ for small k. Since $Q_{132}^{(0,1,0,0)}(t,x) = Q_{132}^{(1,0,0,0)}(t,x)$, we will not list that generating function again.

$$
\begin{aligned}
&Q_{132}^{(0,2,0,0)}(t,x) = 1+t+2t^2+(3+2x)t^3+\left(4+6x+4x^2\right)t^4+ \\
&\left(5+12x+15x^2+10x^3\right)t^5+\left(6+20x+36x^2+42x^3+28x^4\right)t^6+ \\
&\left(7+30x+70x^2+112x^3+126x^4+84x^5\right)t^7+ \\
&\left(8+42x+120x^2+240x^3+360x^4+396x^5+264x^6\right)t^8+ \\
&\left(9+56x+189x^2+450x^3+825x^4+1188x^5+1287x^6+858x^7\right)t^9+\cdots.
\end{aligned}
$$

The only permutations $\sigma \in S_n(132)$ such that $\mathrm{mmp}^{(0,2,0,0)}(\sigma) = 0$ are the identity permutation plus all the adjacent transpositions $(i, i+1)$ which explains why $Q_n^{(0,2,0,0)}(0) = n$ for all $n \geq 1$. This is a known result [5] since avoiding $MMP(0,2,0,0)$ is equivalent to avoiding simultaneously the classical patterns 321 and 231. Hence in this case, we are dealing with the simultaneous avoidance of the patterns 132, 321 and 231. The sequence $(Q_n^{(0,2,0,0)}(x)|_{x^{n-3}})_{n\geq 3}$, which is

$$3, 6, 15, 42, 126, 396, 1287, \ldots,$$

appears to be sequence A120589 in the OEIS which has no listed combinatorial interpretation.

$$
\begin{aligned}
&Q_{132}^{(0,3,0,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (9+5x)t^4 + \left(14 + 18x + 10x^2)\right) t^5 + \\
&\left(20 + 42x + 45x^2 + 25x^3\right) t^6 + \left(27 + 80x + 126x^2 + 126x^3 + 70x^4\right) t^7 + \\
&\left(35 + 135x + 280x^2 + 392x^3 + 378x^4 + 210x^5\right) t^8 + \\
&\left(44 + 210x + 540x^2 + 960x^3 + 1260x^4 + 1088x^5 + 660x^6\right) t^9 + \cdots .
\end{aligned}
$$

$$
\begin{aligned}
&Q_{132}^{(0,4,0,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (28+14x)t^5 + \left(48 + 56x + 28x^2\right) t^6 + \\
&\left(75 + 144x + 140x^2 + 70x^3\right) t^7 + \left(110 + 300x + 432x^2 + 392x^3 + 196x^4\right) t^8 + \\
&\left(154 + 550x + 1050x^2 + 1344x^3 + 1176x^4 + 588x^5\right) t^9 + \cdots .
\end{aligned}
$$

The sequences $\{Q_n^{(0,3,0,0)}(0)\}_{n\geq 1}$, $(Q_n^{(0,2,0,0)}(x)|_x)_{n\geq 4}$, $(Q_n^{(0,3,0,0)}(x)|_{x^{n-4}})_{n\geq 4}$, $\{Q_n^{(0,4,0,0)}(0)\}_{n\geq 1}$, and $\{Q_n^{(0,4,0,0)}(x)|_x\}_{n\geq 5}$ do not appear in the OEIS.

6 The function $Q_{n,132}^{(k,0,\emptyset,0)}(x)$

Note that the pattern $MMP(k,0,\emptyset,0)$ is a generalization of the number of left-to-right minima statistic (which corresponds to the case $k=0$).

First we compute the generating function for $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ which corresponds to the elements that are both left-to-right minima and right-to-left maxima. Consider the permutations $\sigma \in S_n(132)$ where $\sigma_1 = n$. Clearly such permutations contribute $xQ_{n-1,132}^{(\emptyset,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$. For $i > 1$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ since the presence of n to the right of these elements ensures that no point in $A_i(\sigma)$ matches the pattern $MMP(\emptyset,0,\emptyset,0)$. Similarly, our choices for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(\emptyset,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(\emptyset,0,\emptyset,0)$ in σ. Thus

$$
Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x) = xQ_{n-1,132}^{(\emptyset,0,\emptyset,0)}(x) + \sum_{i=2}^{n} C_{i-1} Q_{n-i,132}^{(\emptyset,0,\emptyset,0)}(x). \tag{29}
$$

Multiplying both sides of (29) by t^n and summing over all $n \geq 1$, we obtain that

$$
-1 + Q_{132}^{(\emptyset,0,\emptyset,0)}(t,x) = txQ_{132}^{(\emptyset,0,\emptyset,0)}(t,x) + tQ_{132}^{(\emptyset,0,\emptyset,0)}(t,x)\ (C(t)-1). \tag{30}
$$

Thus we have the following theorem.

Theorem 11.

$$Q_{132}^{(\emptyset,0,\emptyset,0)}(t,x) = \frac{1}{1-tx+t-tC(t)} \tag{31}$$

and

$$Q_{132}^{(\emptyset,0,\emptyset,0)}(t,0) = \frac{1}{1+t-tC(t)}. \tag{32}$$

One can compute that

$$\begin{aligned}
&Q_{132}^{(\emptyset,0,\emptyset,0)}(t,x) = 1 + xt + \left(1+x^2\right)t^2 + \left(2+2x+x^3\right)t^3 + \left(6+4x+3x^2+x^4\right)t^4 + \\
&\left(18+13x+6x^2+4x^3+x^5\right)t^5 + \left(57+40x+21x^2+8x^3+5x^4+x^6\right)t^6 + \\
&\left(186+130x+66x^2+30x^3+10x^4+6x^5+x^7\right)t^7 + \\
&\left(622+432x+220x^2+96x^3+40x^4+12x^5+7x^6+x^8\right)t^8 + \\
&\left(2120+1466x+744x^2+328x^3+130x^4+51x^5+14x^6+8x^7+x^9\right)t^9 + \cdots.
\end{aligned}$$

Clearly the highest degree term in $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ is x^n which comes from the permutation $n(n-1)\ldots 21$. Similarly, the coefficient of x^{n-2} in $Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)$ is $n-1$ which comes from the permutations $n(n-1)\ldots(i+3)(i+2)i(i+1)(i-1)(i-2)\ldots 21$ for $i=1,\ldots,n-1$.

The sequence $(Q_{n,132}^{(\emptyset,0,\emptyset,0)}(0))_{n\geq 1}$ is the Fine numbers (A000957 in the OEIS). The Fine numbers $(\mathbb{F}_n)_{n\geq 0}$ can be defined by the generating function

$$\mathbb{F}(t) = \sum_{n\geq 0}\mathbb{F}_n t^n = \frac{1-\sqrt{1-4t}}{3t-\sqrt{1-4t}}.$$

It is straightforward to verify that

$$\frac{1-\sqrt{1-4t}}{3t-\sqrt{1-4t}}\cdot\frac{1+\sqrt{1-4t}}{1+\sqrt{1-4t}} = \frac{1}{1+t-tC(t)}.$$

$\mathbb{F}_n$ counts the number of 2-Motzkin paths with no level steps at height 0; see [2, 3]. $\mathbb{F}_n$ also counts the number of ordered rooted trees with n edges that have root of even degree. The sequence $(Q_{n,132}^{(\emptyset,0,\emptyset,0)}(x)|_x)_{n\geq 1}$ is A065601 in the OEIS, which counts the number of Dyck paths of length 2n with exactly one hill.

Problem 4. Find simple bijective proofs for the last two facts.

Next we compute the generating function for $Q_{n,132}^{(0,0,\emptyset,0)}(x)$. First consider the permutations $\sigma \in S_n^{(1)}(132)$. Clearly such permutations contribute $xQ_{n-1,132}^{(0,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(0,0,\emptyset,0)}(x)$. For $i>1$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(0,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(0,0,\emptyset,0)}(x)$ since neither n nor any of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(0,0,\emptyset,0)$ in σ. Similarly, our choices for $B_i(\sigma)$ will contribute a factor of

$Q_{n-i,132}^{(0,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(0,0,\emptyset,0)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(0,0,\emptyset,0)$ in σ. Thus

$$Q_{n,132}^{(0,0,\emptyset,0)}(x) = xQ_{n-1,132}^{(0,0,\emptyset,0)}(x) + \sum_{i=2}^{n} Q_{i-1,132}^{(0,0,\emptyset,0)}(x)Q_{n-i,132}^{(0,0,\emptyset,0)}(x). \tag{33}$$

Multiplying both sides of (33) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(0,0,\emptyset,0)}(t,x) = txQ_{132}^{(0,0,\emptyset,0)}(t,x) + tQ_{132}^{(0,0,\emptyset,0)}(t,x)\ (Q_{132}^{(0,0,\emptyset,0)}(t,x) - 1), \tag{34}$$

so that

$$0 = 1 + Q_{132}^{(0,0,\emptyset,0)}(t,x)(tx - t - 1) + t(Q_{132}^{(0,0,\emptyset,0)}(t,x))^2.$$

Thus

$$Q_{132}^{(0,0,\emptyset,0)}(t,x) = \frac{(1+t-tx) - \sqrt{(1+t-tx)^2 - 4t}}{2t}.$$

Next we compute a recursion for $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ where $k \geq 1$. It is clear that n can never match the pattern $MMP(k,0,\emptyset,0)$ for $k \geq 1$ in any $\sigma \in S_n(132)$. For $i \geq 1$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of $Q_{i-1,132}^{(k-1,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ since none of the elements to the right of n have any effect on whether an element in $A_i(\sigma)$ matches the pattern $MMP(k,0,\emptyset,0)$ and the presence of n ensures that an element in $A_i(\sigma)$ matches $MMP(k,0,\emptyset,0)$ in σ if and only if it matches $MMP(k-1,0,\emptyset,0)$ in $A_i(\sigma)$. Similarly, our choices for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(k,0,\emptyset,0)}(x)$ to $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(k,0,\emptyset,0)$. Thus

$$Q_{n,132}^{(k,0,\emptyset,0)}(x) = \sum_{i=1}^{n} Q_{i-1,132}^{(k-1,0,\emptyset,0)}(x)Q_{n-i,132}^{(k,0,\emptyset,0)}(x). \tag{35}$$

Multiplying both sides of (35) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q_{132}^{(0,0,\emptyset,0)}(t,x) = tQ_{132}^{(k-1,0,\emptyset,0)}(t,x)\ Q_{132}^{(k,0,\emptyset,0)}(t,x). \tag{36}$$

Thus we have the following theorem.

Theorem 12.

$$Q_{132}^{(0,0,\emptyset,0)}(t,x) = \frac{(1+t-tx) - \sqrt{(1+t-tx)^2 - 4t}}{2t}. \tag{37}$$

For $k \geq 1$,

$$Q_{132}^{(k,0,\emptyset,0)}(t,x) = \frac{1}{1 - tQ_{132}^{(k-1,0,\emptyset,0)}(t,x)}. \tag{38}$$

Thus

$$Q_{132}^{(0,0,\emptyset,0)}(t,0) = 1$$

and for $k \geq 1$,

$$Q_{132}^{(k,0,\emptyset,0)}(t,0) = \frac{1}{1 - tQ_{132}^{(k-1,0,\emptyset,0)}(t,0)}. \tag{39}$$

Next we consider the constant term and the coefficient of of x in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ for $k \geq 1$.

Proposition 3. *For all $k \geq 1$,*

$$Q_{132}^{(k,0,\emptyset,0)}(t,0) = Q_{132}^{(k,0,0,0)}(t,0).$$

Proof. Note that $Q_{132}^{(1,0,\emptyset,0)}(t,0) = \frac{1}{1-t} = Q_{132}^{(1,0,0,0)}(t,0)$. If we compare the recursions (39) and (8), we see that we have that $Q_{132}^{(k,0,\emptyset,0)}(t,0) = Q_{132}^{(k,0,0,0)}(t,0)$ for all $k \geq 1$. This fact is easy to see directly. That is, suppose that $\sigma \in S_n(132)$ has a $MMP(k,0,0,0)$-match. Then it is easy to see that if i is the smallest t such that σ_t matches $MMP(k,0,0,0)$ in σ, then there can be no $j < i$ such that $\sigma_j < \sigma_i$ because otherwise, σ_j would match $MMP(k,0,0,0)$. That is, σ_i is also a $MMP(k,0,\emptyset,0)$-match. Thus if σ has a $MMP(k,0,0,0)$-match, then it also has a $MMP(k,0,\emptyset,0)$-match. The converse of this statement is trivial. Hence the number of $\sigma \in S_n(132)$ with no $MMP(k,0,0,0)$-matches equals the number of $\sigma \in S_n(132)$ with no $MMP(k,0,\emptyset,0)$-matches. ☐

The recursion (38) has the same form as the recursion (6). Thus we can use the same method of proof that we did to establish the recursion (12) to prove that

$$Q^{(k,0,\emptyset,0)}(t,x)|_x = Q^{(k-1,0,\emptyset,0)}(t,x)|_x \frac{t\frac{d}{dt}Q^{(k,0,\emptyset,0)}(t,0)}{\frac{d}{dt}tQ^{(k-1,0,\emptyset,0)}(t,0)}. \tag{40}$$

For example, we know that

$$Q^{(1,0,\emptyset,0)}(t,x)|_x = Q^{(0,0,1,0)}(t,x)|_x = \sum_{n\geq 2}\binom{n}{2}t^n = \frac{t^2}{(1-t)^3}. \tag{41}$$

Since $Q^{(k,0,\emptyset,0)}(t,0) = Q^{(k,0,0,0)}(t,0)$ for all $k \geq 1$, one can use (40) and Mathematica to show that

$$\begin{aligned}
Q^{(2,0,\emptyset,0)}(t,x)|_x &= \frac{t^3}{(1-t)(1-2t)^2}, \\
Q^{(3,0,\emptyset,0)}(t,x)|_x &= \frac{t^4}{(1-t)(1-3t+t^2)^2}, \\
Q^{(4,0,\emptyset,0)}(t,x)|_x &= \frac{t^5}{(1-t)^3(1-3t)^2}, \text{ and} \\
Q^{(5,0,\emptyset,0)}(t,x)|_x &= \frac{t^6}{(1-t)(1-5t+6t^2-t^3)^2}.
\end{aligned}$$

We also have the following proposition concerning the coefficient of the highest power of x in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$.

Proposition 4. *For all $k \geq 1$, the highest power of x appearing in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ is x^{n-k} and for all $n \geq k$, $Q_{n,132}^{(k,0,\emptyset,0)}(x)|_{x^{n-k}} = 1$.*

Proof. It is easy to see that for any $k \geq 1$, the permutation $\sigma \in S_n(132)$ with the maximal number of $MMP(k,0,\emptyset,0)$-matches for $n \geq k+1$, will be of the form $(n-k)(n-k-1)\ldots 21(n-k+1)(n-k+2)\ldots n$. Thus the highest coefficient x^s that will appear in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ will be x^{n-k}, with coefficient 1. $\square$

Using Theorem 12, one can compute that

$$Q_{132}^{(0,0,\emptyset,0)}(t,x) = 1 + xt + x(1+x)t^2 + x\left(1+3x+x^2\right)t^3 + x\left(1+6x+6x^2+x^3\right)t^4 + \left(1+10x+20x^2+10x^3+x^4\right)t^5 + x\left(1+15x+50x^2+50x^3+15x^4+x^5\right)t^6 + x\left(1+21x+105x^2+175x^3+105x^4+21x^5+x^6\right)t^7 + x\left(1+28x+196x^2+490x^3+490x^4+196x^5+28x^6+x^7\right)t^8 + x\left(1+36x+336x^2+1176x^3+1764x^4+1176x^5+336x^6+36x^7+x^8\right)t^9 + \cdots.$$

If we compare $Q_{132}^{(0,0,\emptyset,0)}(t,x)$ to $Q_{132}^{(0,0,1,0)}(t,x)$, we see that for $n \geq 1$,

$$Q_{n,132}^{(0,0,\emptyset,0)}(x) = xQ_{n,132}^{(0,0,1,0)}(x). \tag{42}$$

Note the $Q_{n,132}^{(0,0,\emptyset,0)}(x)$ has an obvious symmetry property. That is, the following holds.

Theorem 13. *For all $n \geq 1$,*

$$x^{n+1}Q_{n,132}^{(0,0,\emptyset,0)}\left(\frac{1}{x}\right) = Q_{n,132}^{(0,0,\emptyset,0)}(x).$$

Proof. Since the distribution of occurrences of the pattern $MMP(0,0,\emptyset,0)$ is the same as the distribution of the statistic the number of left-to-right minima, and the distribution of occurrences of $MMP(0,0,1,0)$ is the distribution of the statistic the number of non-left-to-right minima, this shows that the statistics the number of left-to-right minima and 1 + the number of non-left-to-right minima are equidistributed on 132-avoiding permutations. Actually, this proves a more general claim, namely, that on $S_n(132)$, the joint distribution of the pair $(MMP(0,0,\emptyset,0)-1, MMP(0,0,1,0))$ is the same as the distribution of $(MMP(0,0,1,0), MMP(0,0,\emptyset,0)-1)$, which often is not the case, but is here because the sum $MMP(0,0,\emptyset,0) + MMP(0,0,1,0)$ applied to a permutation equals the length of the permutation. That is, if we let

$$R_n(x,y) = \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(0,0,\emptyset,0)}(\sigma)} y^{\mathrm{mmp}^{(0,0,1,0)}(\sigma)}, \tag{43}$$

then this shows that $yR_n(x,y)$ is symmetric in x and y for all n.

We shall sketch a combinatorial proof of this fact. First we construct a bijection T from $S_n(132)$ onto $S_n(123)$ that will make the fact that $yR_n(x,y)$ is symmetric apparent. If $\sigma = \sigma_1 \ldots \sigma_k \in S_k$ and $\tau = \tau_1 \ldots \tau_\ell \in S_\ell$, then we let

$$\sigma \oplus \tau = \sigma_1 \ldots \sigma_k (k+\tau_1) \ldots (k+\tau_\ell)$$

and

$$\sigma \ominus \tau = (\ell + \sigma_1) \dots (\ell + \sigma_k) \tau_1 \dots \tau_\ell.$$

Then $\bigcup_n S_n(132)$ is recursively generated by starting with the permutation 1 and closing under the operations of $\sigma \ominus \tau$ and $\sigma \oplus 1$. Then we can define a recursive bijection $T : \bigcup_n S_n(132) \to \bigcup_n S_n(123)$ by letting $T(1) = 1$, $T(\sigma \ominus \tau) = T(\sigma) \ominus T(\tau)$, and $T(\sigma \oplus 1) = X(T(\sigma))$, where $X(\sigma)$ is constructed from σ as follows.

Take the permutation $\sigma \in S_n(123)$ and fix the positions and values of the left-to-right minima. Append one position to the end of σ, and renumber the non-left-to-right minima in decreasing order. For example, if $\sigma = 4762531$, then 4, 2, and 1 are the left-to-right minima. After fixing those positions and values and appending one position, the permutation looks like $4xx2xx1x$. Then we fill in the x's with 8, 7, 6, 5, 3, in that order, to obtain 48726513. The map X is essentially based on the *Simion-Schmidt bijection* described in [5].

It is straightforward to prove by induction that if $T(\sigma) = \tau$, then σ_j matches the pattern $MMP(0, 0, \emptyset, 0)$ in σ if and only if τ_j matches the pattern $MMP(0, 0, \emptyset, 0)$ in τ. That is, the map T preserves left-to-right minima. Note that if σ_j does not match the pattern $MMP(0, 0, \emptyset, 0)$ in σ, then it must match the pattern $MMP(0, 0, 1, 0)$ in σ. Thus it follows that

$$\begin{aligned} R_n(x, y) &= \sum_{\sigma \in S_n(132)} x^{\mathrm{mmp}^{(0,0,\emptyset,0)}(\sigma)} y^{\mathrm{mmp}^{(0,0,1,0)}(\sigma)} \\ &= \sum_{\sigma \in S_n(123)} x^{\text{LRmin}(\sigma)} y^{\text{non-LRmin}(\sigma)} \end{aligned}$$

where LRmin(σ) is the number of left-to-right minima of σ and non-LRmin$(\sigma) = n -$ LRmin(σ).

Next observe that specifying the left-to-right minima of a permutation $\sigma \in S_n(123)$ completely determines σ. That is, if $\sigma_1 > \sigma_{i_2} > \cdots > \sigma_{i_k}$ are the left-to-right minima of σ, where $1 < i_2 < \cdots < i_k \leq n$, then the remaining elements must be placed in decreasing order, as in the map X, since any pair that are not decreasing will form a 123-pattern with a previous left-to-right minimum. This means that $X : S_n(123) \to S_{n+1}(123)$ is one-to-one, and since LRmin$(X(\sigma)) =$ LRmin(σ) and non-LRmin$(X(\sigma)) = 1 +$ non-LRmin(σ), it follows that

$$yR_n(x, y) = \sum_{\sigma \in S_n(123)} x^{\text{LRmin}(X(\sigma))} y^{\text{non-LRmin}(X(\sigma))}.$$

But it is easy to see that for any permutation $X(\sigma)$, reversing and then complementing $X(\sigma)$, which rotates the graph of $X(\sigma)$ by 180° around its center, produces a permutation of the form $X(\tau)$ for some $\tau \in S_n(123)$ such that LRmin$(X(\sigma)) =$ non-LRmin$(X(\tau))$ and non-LRmin$(X(\sigma)) =$ LRmin$(X(\tau))$. Thus

$$\sum_{\sigma \in S_n(123)} x^{\text{LRmin}(X(\sigma))} y^{\text{non-LRmin}(X(\sigma))}$$

is symmetric in x and y. Hence, $yR_n(x,y)$ is symmetric in x and y. Thus, if r and c are the reverse and complement maps, respectively, then $Y : S_n(132) \to S_n(132)$ given by $Y(\sigma) = T^{-1}X^{-1}rcXT\sigma$ is a bijection that swaps the statistics $\mathrm{mmp}(0,0,\emptyset,0)-1$ and $\mathrm{mmp}(0,0,1,0)$. $\square$

One can compute that

$$
\begin{aligned}
&Q_{132}^{(1,0,\emptyset,0)}(t,x) = 1 + t + (1+x)t^2 + \left(1+3x+x^2\right)t^3 + \\
&\left(1+6x+6x^2+x^3\right)t^4 + \left(1+10x+20x^2+10x^3+x^4\right)t^5 + \\
&\left(1+15x+50x^2+50x^3+15x^4+x^5\right)t^6 + \\
&\left(1+21x+105x^2+175x^3+105x^4+21x^5+x^6\right)t^7 + \\
&\left(1+28x+196x^2+490x^3+490x^4+196x^5+28x^6+x^7\right)t^8 + \\
&\left(1+36x+336x^2+1176x^3+1764x^4+1176x^5+336x^6+36x^7+x^8\right)t^9 + \cdots .
\end{aligned}
$$

One can observe that $Q_{132}^{(1,0,\emptyset,0)}(t,x) = Q_{132}^{(0,0,1,0)}(t,x)$. We provide here a combinatorial proof of this fact. Actually, we will prove a stronger statement that we record as the following theorem.

Theorem 14. *The two pairs of statistics $(MMP(1,0,\emptyset,0), MMP(0,0,1,0))$ and $(MMP(0,0,1,0), MMP(1,0,\emptyset,0))$ are equidistributed on $S_n(132)$. That is, there are as many permutations in $S_n(132)$ with k occurrences of the pattern $MMP(1,0,\emptyset,0)$ and ℓ occurrences of the pattern $MMP(0,0,1,0)$ as those with k occurrences of the pattern $MMP(0,0,1,0)$ and ℓ occurrences of the pattern $MMP(1,0,\emptyset,0)$.*

Proof. We will construct a map φ on $\cup_n S_n(132)$, recursively interchanging occurrences of the involved patterns. The base case, $n=1$, obviously holds: $\varphi(1) := 1$ and neither of the patterns occurs in 1.

Assume that the claim holds for 132-avoiding permutations of lengths less than n, and consider a permutation $\pi \in S_n(132)$ such that $\pi = \pi_1 n \pi_2$ and each letter in π_1, if any, is larger than any letter in π_2. Consider two cases.

Case 1. π_1 is empty. In this case, we can define $\varphi(\pi) := n\varphi(\pi_2)$. Since n is neither an occurrence of $MMP(1,0,\emptyset,0)$ nor an occurrence of $MMP(0,0,1,0)$. We get the desired property by the induction hypothesis.

Case 2. π_1 is not empty. Note that n is an occurrence of the pattern $MMP(0,0,1,0)$, and because of n, each left-to-right minimum in π_1 is actually an occurrence of the pattern $MMP(1,0,\emptyset,0)$. Further, each non-left-to-right minimum in π_1 is obviously an occurrence of the pattern $MMP(0,0,1,0)$. We now let $\varphi(\pi) := (Y(\mathrm{red}(\pi_1)) \oplus 1) \ominus \varphi(\pi_2)$ where Y, $\oplus$, and $\ominus$ are defined while proving combinatorially (42), which deals with the equidistribution of the statistics $MMP(0,0,\emptyset,0)-1$ and $MMP(0,0,1,0)$. Indeed, $\varphi(\pi_2)$ will interchange the occurrences of the patterns by the induction hypothesis. Also, assuming that $\pi_1 n$ has k occurrences of the pattern $MMP(1,0,\emptyset,0)$ and ℓ occurrences of the pattern $MMP(0,0,1,0)$, $Y(\mathrm{red}(\pi_1)) \oplus 1$ will have ℓ occurrences of the pattern $MMP(1,0,\emptyset,0)$

and k occurrences of the pattern $MMP(0,0,1,0)$ (because n will stay an occurrence of the pattern $MMP(0,0,1,0)$).

$\square$

One can compute that

$$Q_{132}^{(2,0,\emptyset,0)}(t,x) = 1+t+2t^2+(4+x)t^3+\left(8+5x+x^2\right)t^4+\left(16+17x+8x^2+x^3\right)t^5+ \left(32+49x+38x^2+12x^3+x^4\right)t^6+\left(64+129x+141x^2+77x^3+17x^4+x^5\right)t^7+ \left(128+321x+453x^2+361x^3+143x^4+23x^5+x^6\right)t^8+ \left(256+769x+1326x^2+1399x^3+834x^4+247x^5+30x^6+x^7\right)t^9+\cdots.$$

The sequence $(Q_{n,132}^{(2,0,\emptyset,0)}(x)|_x)_{n\geq 4}$ is sequence A000337 in the OEIS, whose n-th term is $(n-1)2^n+1$. Thus $Q_{n,132}^{(2,0,\emptyset,0)}(x)|_x = (n-4)2^{n-3}+1$ for $n \geq 4$. One further can compute the following functions.

$$Q_{132}^{(3,0,\emptyset,0)}(t,x) = 1+t+2t^2+5t^3+(13+x)t^4+\left(34+7x+x^2\right)t^5+ \left(89+32x+10x^2+x^3\right)t^6+\left(233+122x+59x^2+14x^3+x^4\right)t^7+ \left(610+422x+272x^2+106x^3+19x^4+x^5\right)t^8+ \left(1597+1376x+1090x^2+591x^3+182x^4+25x^5+x^6\right)t^9+\cdots.$$

$$Q_{132}^{(4,0,\emptyset,0)}(t,x) = 1+t+2t^2+5t^3+14t^4+(41+x)t^5+\left(122+9x+x^2\right)t^6+ \left(365+51x+12x^2+x^3\right)t^7+\left(1094+235x+84x^2+16x^3+x^4\right)t^8+ \left(3281+966x+454x^2+139x^3+21x^4+x^5\right)t^9+\cdots.$$

$$Q_{132}^{(5,0,\emptyset,0)}(t,x) = 1+t+2t^2+5t^3+14t^4+42t^5+(131+x)t^6+\left(417+11x+x^2\right)t^7+ \left(1341+74x+14x^2+x^3\right)t^8+\left(4334+396x+113x^2+18x^3+x^4\right)t^9+\cdots.$$

The second highest power of x that occurs in $Q_{n,132}^{(k,0,\emptyset,0)}(x)$ is x^{n-k-1}. Our next result will show that $Q_{n,132}^{(k,0,\emptyset,0)}(x)|_{x^{n-k-1}}$ has a regular pattern for large enough n. That is, we have the following theorem.

Theorem 15. *For $n \geq 3$ and $k \geq 1$,*

$$Q_{n+k-1,132}^{(k,0,\emptyset,0)}|_{x^{n-2}} = 2(k-1)+\binom{n}{2}. \tag{44}$$

Proof. Note that $Q^{(1,0,\emptyset,0)}(t,x) = Q^{(0,0,1,0)}(t,x)$ and by Theorem 9, we have that $Q_{n,132}^{(0,0,1,0)}(x)|_{x^{n-2}} = \binom{n}{2}$. Thus the theorem holds for $k=1$.

By induction, assume that $Q^{(k,0,\emptyset,0)}_{n+k-1,132}|_{x^{n-2}} = 2(k-1)+\binom{n}{2}$. We know by (35) that

$$Q^{(k+1,0,\emptyset,0)}_{n+k,132}(x) = \sum_{i=1}^{n+k} Q^{(k,0,\emptyset,0)}_{i-1,132}(x) Q^{(k+1,0,\emptyset,0)}_{n+k-i,132}(x). \tag{45}$$

Note that for $2 \leq i \leq n-k-2$, the highest coefficient of x in $Q^{(k+1,0,\emptyset,0)}_{n+k-i,132}(x)$ is $x^{n+k-i-(k+1)} = x^{n-i-1}$. However the highest coefficient of x in $Q^{(k,0,\emptyset,0)}_{i-1,132}(x)$ is x^{i-2} so that the only terms on the RHS of (45) that can contribute to the coefficient of x^{n-2} is $i=1$, $i=n-k-1$, and $i=n-k$. By Proposition 4, we know that

$$Q^{(k+1,0,\emptyset,0)}_{n+k-1,132}(x)|_{x^{n-2}} = 1 = Q^{(k,0,\emptyset,0)}_{n+k-2,132}(x)|_{x^{n-2}}$$

so that the $i=1$ and $i=n-k-1$ terms in (45) contribute 2 to $Q^{(k+1,0,\emptyset,0)}_{n+k,132}(x)|_{x^{n-2}}$. Now the $i=n+k$ term in (45) contributes

$$Q^{(k,0,\emptyset,0)}_{n+k-1,132}(x)|_{x^{n-2}} = 2(k-1)+\binom{n}{2}$$

to $Q^{(k+1,0,\emptyset,0)}_{n+k,132}(x)|_{x^{n-2}}$. Thus

$$Q^{(k+1,0,\emptyset,0)}_{n+k,132}(x)|_{x^{n-2}} = 2k+\binom{n}{2}.$$

□

The sequences $(Q^{(3,0,\emptyset,0)}_{n,132}(x)|_x)_{n\geq 4}$, $(Q^{(4,0,\emptyset,0)}_{n,132}(x)|_x)_{n\geq 5}$, and $(Q^{(5,0,\emptyset,0)}_{n,132}(x)|_x)_{n\geq 5}$ do not appear in the OEIS.

7 The function $Q^{(\emptyset,0,k,0)}_{n,132}(x)$

First we compute the generating function for $Q^{(\emptyset,0,0,0)}_{n,132}(x)$. Observe that n will always match the pattern $MMP(\emptyset,0,0,0)$ in any $\sigma \in S_n$. For $i \geq 1$, it is easy to see that as we sum over all the permutations σ in $S^{(i)}_n(132)$, our choices for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q^{(\emptyset,0,0,0)}_{n,132}(x)$ since the presence of n to the right of an element in $A_i(\sigma)$ ensures that it does match the pattern $MMP(\emptyset,0,0,0)$ in σ. Similarly our choices for $B_i(\sigma)$ will contribute a factor of $Q^{(\emptyset,0,0,0)}_{n-i,132}(x)$ to $Q^{(\emptyset,0,0,0)}_{n,132}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(\emptyset,0,0,0)$. Thus

$$Q^{(\emptyset,0,0,0)}_{n,132}(x) = x\sum_{i=1}^{n} C_{i-1} Q^{(\emptyset,0,0,0)}_{n-i,132}(x). \tag{46}$$

Multiplying both sides of (46) by t^n and summing over all $n \geq 1$, we obtain that

$$-1 + Q^{(\emptyset,0,0,0)}_{132}(t,x) = txC(t)\; Q^{(\emptyset,0,0,0)}_{132}(t,x), \tag{47}$$

so that

$$Q_{132}^{(\emptyset,0,0,0)}(t,x) = \frac{1}{1-txC(t)}.$$

Next suppose that $k \geq 1$. In this case n in $\sigma \in S_n^{(i)}(132)$ will match the pattern $MMP(\emptyset,0,k,0)$ in σ if and only if $i > k$. For $i \geq 1$, it is easy to see that as we sum over all the permutations σ in $S_n^{(i)}(132)$, our choices for $A_i(\sigma)$ will contribute a factor of C_{i-1} to $Q_{n,132}^{(\emptyset,0,k,0)}(x)$ since the presence of n to the right ensures that none of these elements will match the pattern $MMP(\emptyset,0,k,0)$. Similarly, our choices for $B_i(\sigma)$ will contribute a factor of $Q_{n-i,132}^{(\emptyset,0,k,0)}(x)$ to $Q_{n,132}^{(\emptyset,0,k,0)}(x)$ since neither n nor any of the elements to the left of n have any effect on whether an element in $B_i(\sigma)$ matches the pattern $MMP(\emptyset,0,k,0)$. Thus

$$Q_{n,132}^{(\emptyset,0,k,0)}(x) = \sum_{i=1}^{k} C_{i-1}Q_{n-i,132}^{(\emptyset,0,k,0)}(x) + x\sum_{i=k+1}^{n} C_{i-1}Q_{n-i,132}^{(\emptyset,0,k,0)}(x). \tag{48}$$

Multiplying both sides of (48) by t^n and summing over all $n \geq 1$, we obtain that

$$-1+Q_{132}^{(\emptyset,0,k,0)}(t,x) = t(\sum_{j=0}^{k-1} C_j t^j)Q_{132}^{(\emptyset,0,k,0)}(t,x) + xtQ_{132}^{(\emptyset,0,k,0)}\ (C(t)-\sum_{j=0}^{k-1} C_j t^j). \tag{49}$$

Thus we have the following theorem.

Theorem 16.

$$Q_{132}^{(\emptyset,0,0,0)}(t,x) = \frac{1}{1-txC(t)}. \tag{50}$$

For $k \geq 1$*,*

$$Q_{132}^{(\emptyset,0,k,0)}(t,x) = \frac{1}{1--txC(t)-t(1-x)(\sum_{j=0}^{k-1} C_j t^j)} \tag{51}$$

and

$$Q_{132}^{(\emptyset,0,k,0)}(t,0) = \frac{1}{1-t(\sum_{j=0}^{k-1} C_j t^j)}. \tag{52}$$

Proposition 5. $Q_{132}^{(\emptyset,0,k,0)}(t,0) = Q^{(0,0,k,0)}(t,0)$ *for all* $k \geq 1$*.*

Proof. The proposition follows immediately from Theorems 7 and 16. That is, we have

$$Q_{132}^{(0,0,k,0)}(t,0) = \frac{1}{1-t(\sum_{j=0}^{k-1} C_j t^j)} = Q_{132}^{(\emptyset,0,k,0)}(t,0).$$

This fact is easy to see directly. That is, suppose that $\sigma = \sigma_1 \ldots \sigma_n \in S_n(132)$ and σ contains a $MMP(0,0,k,0)$-match. Then it is easy to see that if i is the largest such that σ_i matches $MMP(0,0,k,0)$, then there can be no $j > i$ such that $\sigma_j > \sigma_i$ because otherwise, σ_j would match $MMP(0,0,k,0)$. Thus if σ has a $MMP(0,0,k,0)$-match, then it also has a $MMP(\emptyset,0,k,0)$-match. Again, the converse is trivial. Hence the number of $\sigma \in S_n(132)$ with no $MMP(0,0,k,0)$-matches equals the number of $\sigma \in S_n(132)$ with no $MMP(\emptyset,0,k,0)$-matches. $\square$

One can compute

$$Q_{132}^{(\emptyset,0,0,0)}(t,x) = 1 + xt + \left(x+x^2\right)t^2 + \left(2x+2x^2+x^3\right)t^3 + \left(5x+5x^2+3x^3+x^4\right)t^4 + \left(14x+14x^2+9x^3+4x^4+x^5\right)t^5 + \left(42x+42x^2+28x^3+14x^4+5x^5+x^6\right)t^6 + \left(132x+132x^2+90x^3+48x^4+20x^5+6x^6+x^7\right)t^7 + \left(429x+429x^2+297x^3+165x^4+75x^5+27x^6+7x^7+x^8\right)t^8 + \left(1430x+1430x^2+1001x^3+572x^4+275x^5+110x^6+35x^7+8x^8+x^9\right)t^9 + \cdots.$$

Recall that $Q_{132}^{(1,0,0,0)}(t,x) = \frac{1}{1-tC(tx)}$ so that $Q_{132}^{(1,0,0,0)}(tx,\frac{1}{x}) = Q_{132}^{(\emptyset,0,0,0)}(t,x)$. This can easily be explained by the fact that every σ_i, $1 \le i \le n$, matches either $MMP(1,0,0,0)$ or $MMP(\emptyset,0,0,0)$.

One can then compute the following.

$$Q_{132}^{(\emptyset,0,1,0)}(t,x) = 1 + t + (1+x)t^2 + (1+4x)t^3 + \left(1+12x+x^2\right)t^4 + \left(1+34x+7x^2\right)t^5 + \left(1+98x+32x^2+x^3\right)t^6 + \left(1+294x+124x^2+10x^3\right)t^7 + \left(1+919x+448x^2+61x^3+x^4\right)t^8 + \left(1+2974x+1576x^2+298x^3+13x^4\right)t^9 + \cdots.$$

In this case, it is easy to see that the only $\sigma \in S_n(132)$ that avoids the pattern $MMP(\emptyset,0,1,0)$ is the strictly decreasing permutation. Thus $Q_{n,132}^{(\emptyset,0,1,0)}(0) = 1$ for all $n \ge 1$.

It is also easy to see that the permutation that maximizes the number of matches of $MMP(\emptyset,0,1,0)$ in $S_{2n}(132)$ is $(2n-1)(2n)(2n-3)(2n-2)\dots12$ which explains why the highest power of x in $Q_{2n,132}^{(\emptyset,0,1,0)}(x)$ is x^n which has coefficient 1. More generally, one can see that the permutations that maximize the number of matches of $MMP(\emptyset,0,k-1,0)$ in $S_{kn}(132)$ are the permutations of the form

$$\tau^{(n)}(kn)\tau^{(n-1)}(k(n-1))\tau^{(n-2)}(k(n-2))\dots\tau^{(1)}k,$$

where for each $i = 1,\dots,n$, $\tau^{(i)}$ is a permutation of $(i-1)k+1,\dots,(i-1)k+k-1$ which avoids 132. It follows that the highest power of x occurring in $Q_{kn,132}^{(\emptyset,0,k,0)}(x)$ is x^n which occurs with a coefficient of C_{k-1}^n.

It is also not difficult to see that the highest power of x in $Q_{2n+1,132}^{(\emptyset,0,1,0)}(x)$ is x^n which has the coefficient $3n+1$. That is, we can get a coefficient of x^n in a $\sigma \in S_n(132)$ by taking

$$\begin{aligned}&(2n+1)(2n-1)(2n)(2n-3)(2n-2)\dots12,\\&(2n-1)(2n)(2n+1)(2n-3)(2n-2)\dots12, \text{ or}\\&(2n)(2n-1)(2n+1)(2n-3)(2n-2)\dots12,\end{aligned}$$

or starting with $(2n)(2n+1)\tau$ where $\tau \in S_{2n-1}(132)$ which has $n-1$ occurrences of $MMP(\emptyset,0,1,0)$.

The sequence $(Q_{n,132}^{(\emptyset,0,1,0)}(x)|_x)_{n\ge2}$ seems to be A014043 in the OEIS which has the generating function $\frac{1-2z\sqrt{1-4z}}{2z^2(1-z)^2}$.

$$Q_{132}^{(\emptyset,0,2,0)}(t,x) = 1 + t + 2t^2 + (3+2x)t^3 + (5+9x)t^4 + (8+34x)t^5 + \left(13+115x+4x^2\right)t^6 + \left(21+376x+32x^2\right)t^7 + \left(34+1219x+177x^2\right)t^8 + \left(55+3980x+819x^2+8x^3\right)t^9 + \cdots.$$

The sequence $(Q_{n,132}^{(\emptyset,0,2,0)}(0))_{n\geq 2}$ is the Fibonacci numbers. We can give a combinatorial explanation for this fact as well. That is, the permutations in $S_n(132)$ that avoid the pattern $MMP(\emptyset,0,2,0)$ are of the form $n\alpha$, where α is a permutation in $S_{n-1}(132)$ that avoids $MMP(\emptyset,0,2,0)$, or of the form $(n-1)n\beta$, where β is a permutation in $S_{n-2}(132)$ that avoids $MMP(\emptyset,0,2,0)$. It follows that

$$Q_{n,132}^{(\emptyset,0,2,0)}(0) = Q_{n-1,132}^{(\emptyset,0,2,0)}(0) + Q_{n-2,132}^{(\emptyset,0,2,0)}(0).$$

The sequence $(Q_{n,132}^{(\emptyset,0,2,0)}(x)|_x)_{n\geq 3}$ does not appear in the OEIS.

$$Q_{132}^{(\emptyset,0,3,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + (9+5x)t^4 + (18+24x)t^5 + (37+95x)t^6 + (73+356x)t^7 + \left(146+1259x+25x^2\right)t^8 + \left(293+4354x+215x^2\right)t^9 + \cdots.$$

The sequence $(Q_{n,132}^{(\emptyset,0,3,0)}(0))_{n\geq 0}$ is sequence A077947 in the OEIS, which has the generating function $\frac{1}{1-x-x^2-2x^3}$. However, the sequence $(Q_{n,132}^{(\emptyset,0,3,0)}(x)|_x)_{n\geq 4}$ does not appear in the OEIS.

$$Q_{132}^{(\emptyset,0,4,0)}(t,x) = 1 + t + 2t^2 + 5t^3 + 14t^4 + (28+14x)t^5 + (62+70x)t^6 + (143+286x)t^7 + (331+1099x)t^8 + (738+4124x)t^9 + \cdots.$$

The sequence $(Q_{n,132}^{(\emptyset,0,4,0)}(0))_{n\geq 0}$ does not appear in the OEIS.

References

[1] P. Brändén and A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, *Elect. J. Comb.* **18(2)** (2011), #P5, 14pp.

[2] E. Deutsch and L.W. Shapiro, A survey of the Fine numbers, *Discrete Math.* **241** (2001), 241–265.

[3] E. Deutsch and L.W. Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, *Discrete Math.* **256** (2002), 655–670.

[4] Í. Hilmarsson, I. Jónsdóttir, S. Sigurdardottir and S. Vidarsdóttir, Wilf classification of mesh patterns of short length, in preparation.

[5] S. Kitaev, Patterns in permutations and words, *Monographs in Theoretical Computer Science* (with a foreword by Jeffrey B. Remmel), Springer-Verlag, ISBN 978-3-642-17332-5, 2011.

[6] S. Kitaev and J. Remmel, Marked mesh patterns, preprint.

[7] S. Kitaev, J. Remmel and M. Tiefenbruck, Marked mesh patterns in 132-avoiding permutations II, in preparation.

[8] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://www.research.att.com/~njas/sequences/`.

[9] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, (1999).

[10] H. Úlfarsson, A unification of permutation patterns related to Schubert varieties, arXiv:1002.4361 (2011).