

FRAME PATTERNS IN n -CYCLES

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ABSTRACT. In this paper, we study the distribution of the number of occurrences of the simplest frame pattern, called the μ pattern, in n -cycles. Given an n -cycle C , we say that a pair $\langle i, j \rangle$ matches the μ pattern if $i < j$ and as we traverse around C in a clockwise direction starting at i and ending at j , we never encounter a k with $i < k < j$. We say that $\langle i, j \rangle$ is a nontrivial μ -match if $i + 1 < j$. Also, an n -cycle C is incontractible if there is no i such that $i + 1$ immediately follows i in C .

We show that the number of incontractible n -cycles in the symmetric group S_n is D_{n-1} , where D_n is the number of derangements in S_n . Further, we prove that the number of n -cycles in S_n with exactly k μ -matches can be expressed as a linear combination of binomial coefficients of the form $\binom{n-1}{i}$ where $i \leq 2k + 1$. We also show that the generating function $NTI_{n,\mu}(q)$ of q raised to the number of nontrivial μ -matches in C over all incontractible n -cycles in S_n is a new q -analogue of D_{n-1} , which is different from the q -analogues of the derangement numbers that have been studied by Garsia and Remmel and by Wachs. We show that there is a rather surprising connection between the charge statistic on permutations due to Lascoux and Schützenberger and our polynomials in that the coefficient of the smallest power of q in $NTI_{2k+1,\mu}(q)$ is the number of permutations in S_{2k+1} whose charge path is a Dyck path. Finally, we show that $NTI_{n,\mu}(q) \big|_q \binom{n-1}{2-k}$ and $NT_{n,\mu}(q) \big|_q \binom{n-1}{2-k}$ are the number of partitions of k for sufficiently large n .

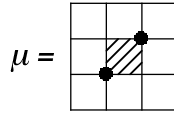
1. INTRODUCTION

Mesh patterns were introduced in [2] by Brändén and Claesson, and they were studied in a series of papers (e.g. see [6] by Kitaev and Liese, and references therein). A particular class of mesh patterns is *boxed patterns* introduced in [1] by Avgustinovich et al., who later suggested to call this type of patterns *frame patterns*. The simplest frame pattern which is called the μ pattern is defined as follows. Let S_n denote the set of all permutations of $\{1, \dots, n\}$. Given $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$, we say that a pair $\langle \sigma_i, \sigma_j \rangle$ is an occurrence of the μ pattern in σ if $i < j$, $\sigma_i < \sigma_j$, and there is no $i < k < j$ such that $\sigma_i < \sigma_k < \sigma_j$ (the last condition is indicated by the shaded area in Figure 2). The μ pattern is shown in Figure 1 using the notation in [2].

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FIGURE 1. The mesh pattern μ .

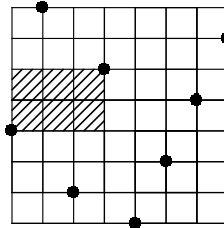
Similarly, we say that the pair $\langle \sigma_i, \sigma_j \rangle$ is an occurrence of the μ' pattern in σ if $i < j$, $\sigma_i > \sigma_j$, and there is no $i < k < j$ such that $\sigma_i > \sigma_k > \sigma_j$. For example, if $\sigma = 4\ 8\ 2\ 6\ 1\ 3\ 5\ 7$, then the occurrences of μ in σ are

$$\langle 4, 8 \rangle, \langle 4, 6 \rangle, \langle 4, 5 \rangle, \langle 2, 6 \rangle, \langle 2, 3 \rangle, \langle 6, 7 \rangle, \langle 1, 3 \rangle, \langle 3, 5 \rangle, \langle 5, 7 \rangle$$

and the occurrences of μ' in σ are

$$\langle 4, 2 \rangle, \langle 4, 3 \rangle, \langle 8, 2 \rangle, \langle 8, 6 \rangle, \langle 8, 7 \rangle, \langle 2, 1 \rangle, \langle 6, 1 \rangle, \langle 6, 3 \rangle, \langle 6, 5 \rangle.$$

We let $N_\mu(\sigma)$ (resp., $N_{\mu'}(\sigma)$) denote the number of occurrences of the μ (resp., μ') in σ . The *reverse* of $\sigma = \sigma_1 \dots \sigma_n \in S_n$, σ^r , is the permutation $\sigma_n \sigma_{n-1} \dots \sigma_1$, and the *complement* of σ , σ^c , is the permutation $(n+1-\sigma_1)(n+1-\sigma_2) \dots (n+1-\sigma_n)$. It is easy to see that $N_\mu(\sigma) = N_{\mu'}(\sigma^r) = N_{\mu'}(\sigma^c)$ and thus, since the reverse and complement are trivial bijections from S_n to itself, studying the distribution of μ -matches in S_n is equivalent to studying the distribution of μ' -matches in S_n .

FIGURE 2. The graph of the permutation 48261357 with the occurrence $\langle 4, 6 \rangle$ highlighted.

If we graph a given permutation σ as dots on a grid as in Figure 2, then one can see that each occurrence of μ is a pair of increasing dots such that there are no dots within the rectangle created by the original dots. The occurrence $\langle 4, 6 \rangle$ is highlighted in Figure 2; in particular, there are no dots within the shaded rectangle.

Jones and Remmel studied the distribution of cycle-occurrences of classical consecutive patterns in [4]. See [5] for a comprehensive introduction to the theory of permutation patterns. In this paper, we shall study the distribution of the *cycle-occurrences* of μ in the cycle structure of permutations in the symmetric group S_n . That is, suppose that we are given a k -cycle C in a permutation σ in the symmetric group S_n . Then we will always write $C = (c_0, \dots, c_{k-1})$ where c_0 is the smallest element of the cycle. We will always draw such a cycle with c_0 at the top and assume that we traverse around the cycle in a clockwise

direction. For example, if $C = (1, 4, 6, 2, 7, 5, 8, 3)$, then we would picture C as in Figure 3. We say that c_s is *cyclically between* c_i and c_j in C , if starting at c_i , we encounter c_s before we encounter c_j as we traverse around the cycle in a clockwise direction. Alternatively, we can identify C with the permutation $c_0c_1 \dots c_{k-1}$. In this notation, c_s is cyclically between c_i and c_j if either $i < s < j$, or $j < i < s$, or $s < j < i$. For example, in the cycle $C = (1, 4, 6, 2, 7, 5, 8, 3)$, 8 is cyclically between 2 and 3. Then we say that the pair $\langle c_i, c_j \rangle$ is a *cycle-occurrence of μ in C* if $c_i < c_j$ and there is no c_s such that $c_i < c_s < c_j$ and c_s is cyclically between c_i and c_j . Similarly, we say that the pair $\langle c_i, c_j \rangle$ is a cycle-occurrence of μ' in C , if $c_i > c_j$ and there is no c_s such that $c_i > c_s > c_j$ and c_s is cyclically between c_i and c_j . As is the case with permutations, the study of the number of cycle-occurrences of μ in the cycle structures of permutations is equivalent to the study of the number of cycle-occurrences of μ in the cycle structures of permutations. That is, given the cycle structure C_1, \dots, C_k of a permutation $\sigma \in S_n$, the cycle complement of σ , $\sigma^{\text{CYC-C}}$, is the permutation whose cycle structure arises from the cycle structure of σ by replacing each number i by $n + 1 - i$. For example, if σ consists of the cycle $(1, 4, 2, 6), (3, 8)(5, 9, 7)$, then $\sigma^{\text{CYC-C}}$ consists of the cycles $(9, 6, 8, 4), (7, 2), (5, 1, 3)$. It is then easy to see that for all $\sigma \in S_n$, σ has k cycle-occurrences of μ if and only if $\sigma^{\text{CYC-C}}$ has k cycle-occurrences of μ' .

Let \mathcal{C}_n be the set of n -cycles in S_n . If $C = (1, c_1, \dots, c_{n-1}) \in \mathcal{C}_n$, then it is clear that $\langle i, i + 1 \rangle$ is always a cycle-occurrence of μ in C . We shall call such cycle-occurrences *trivial occurrences of μ* or *trivial μ -matches* in C and all other cycle-occurrences of μ in C will be called *nontrivial occurrences of μ* or *nontrivial μ -matches* in C . We let $N_\mu(C)$ denote the number of occurrences of μ in C and $NT_\mu(C)$ denote the number of nontrivial occurrences of μ in C . For example, if $C = (1, 4, 6, 2, 7, 5, 8, 3)$, the nontrivial occurrences of μ in C are the pairs $\langle 1, 4 \rangle, \langle 2, 7 \rangle, \langle 2, 5 \rangle, \langle 4, 6 \rangle$, and $\langle 5, 8 \rangle$, so that $NT_\mu(C) = 5$. Clearly, if $C = (1, c_1, \dots, c_{n-1})$ is an n -cycle in \mathcal{C}_n , then $N_\mu(C) = (n - 1) + NT_\mu(C)$, since for all $1 \leq i \leq n - 1$, $\langle i, i + 1 \rangle$ will be a trivial occurrence of μ in C . If C is a 1-cycle, then $NT_\mu(C) = N_\mu(C) = 0$.

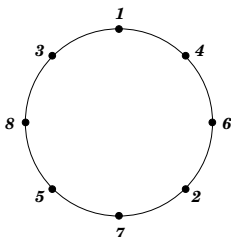


FIGURE 3. The cycle $C = (1, 4, 6, 2, 7, 5, 8, 3)$.

If $\sigma_1 \dots \sigma_n$ is any sequence of distinct integers, we let $\text{red}(\sigma)$ denote the permutation of S_n that is obtained by replacing the i th largest element of $\{\sigma_1, \dots, \sigma_n\}$ by i for $i = 1, \dots, n$. For example, $\text{red}(3\ 7\ 10\ 5\ 2) = 2\ 4\ 5\ 3\ 1$. Similarly, if $C = (c_0, \dots, c_{k-1})$ is a k -cycle in some permutation $\sigma \in S_n$, we let $\text{red}(C)$ be the k -cycle in S_k which is obtained by replacing the i th largest element of C by i . For example, if $C = (2, 6, 7, 3, 9)$, then

$\text{red}(C) = (1, 3, 4, 2, 5)$. In such a situation, we let $NT_\mu(C) = NT_\mu(\text{red}(C))$. Finally, if σ consists of cycles $C^{(1)}, \dots, C^{(\ell)}$, then we let $CNT_\mu(\sigma) = \sum_{i=1}^{\ell} NT_\mu(C^{(i)})$.

We note that if one wishes to study the generating function

$$(1) \quad CNT_\mu(q, x, t) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} q^{CNT_\mu(\sigma)} x^{\text{cyc}(\sigma)},$$

where $\text{cyc}(\sigma)$ denotes the number of cycles in σ , then it is enough to study the generating function

$$(2) \quad NT_\mu(q, t) = \sum_{n \geq 1} \frac{t^n}{n!} NT_{n,\mu}(q)$$

where $NT_{n,\mu}(q) = \sum_{C \in \mathcal{C}_n} q^{NT_\mu(C)}$. That is, it easily follows from the exponential formula that if w is a weight function on n -cycles $C \in \mathcal{C}_n$ and for any permutation σ whose cycle structure is C_1, \dots, C_k , we define $w(\sigma) = \prod_{i=1}^k w(\text{red}(C_i))$, then

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} w(\sigma) x^{\text{cyc}(\sigma)} = e^{x \sum_{n \geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{C}_n} w(C)}.$$

Hence

$$(3) \quad CNT_\mu(q, x, t) = e^{x NT_\mu(q, t)}.$$

The main focus of this paper is to study the polynomials $NT_{n,\mu}(q)$. In Table 1 we provide polynomials $NT_{n,\mu}(q)$ for $n \leq 10$ that were calculated using Mathematica.

Given a polynomial $f(x) = \sum_{i=0}^n c_i x^i$, we let $f(x)|_{x^i} = c_i$ denote the coefficient of x^i in $f(x)$. It is easy to explain several of the coefficients that appear in $NT_{n,\mu}(q)$. For example, for $n \geq 2$, $NT_{n,\mu}(q)|_{q^0} = 1$ because $(1, 2, \dots, n)$ is the only n -cycle that has no nontrivial occurrences of μ . Similarly, $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}}} = 1$ since the only n -cycle with $\binom{n-1}{2}$ nontrivial occurrences of μ is $(1, n, n-1, n-2, \dots, 2)$. We computed that the sequence $(NT_{n,\mu}(q)|_{q^1})_{n \geq 3}$ starts out $1, 3, 6, 10, 15, 21, \dots$ which leads to a conjecture that $NT_{n,\mu}(q)|_{q^1} = \binom{n-1}{2}$ for $n \geq 3$. Similarly, we computed that the sequence $(NT_{n,\mu}(q)|_{q^2})_{n \geq 4}$ starts out $1, 6, 20, 50, 105, 196, \dots$ which is the initial terms of sequence A002415 in the *On-Line Encyclopedia of Integer Sequences (OEIS)* whose n th term is given by the formula $\frac{n^2(n^2-1)}{12}$. We will verify this by proving that for $n \geq 5$, $NT_{n,\mu}(q)|_{q^2} = \binom{n-1}{3} + 2\binom{n-1}{4}$.

In fact, we shall show that for any fixed $k \geq 0$, there are constants $c_0, c_1, \dots, c_{2k+1}$ such that $NT_{n,\mu}(q)|_{q^k} = \sum_{i=1}^{2k+1} c_i \binom{n-1}{i}$ for all $n \geq 2k+2$. To prove this result, we will need to study what we call incontractible n -cycles which are n -cycles C for which there are no integers i such that $i+1$ immediately follows i in C . We let \mathcal{IC}_n denote the set of incontractible n -cycles in \mathcal{C}_n . We will show that $|\mathcal{IC}_n|$ is D_{n-1} where D_n is the number of derangements of S_n , i.e. the number of $\sigma \in S_n$ such that σ has no fixed points. We will also study the polynomials

$$NTI_{n,\mu}(q) = \sum_{C \in \mathcal{IC}_n} q^{NT_\mu(C)}.$$

TABLE 1. Polynomials $NT_{n,\mu}(q)$.

$NT_{1,\mu}(q)$	1
$NT_{2,\mu}(q)$	1
$NT_{3,\mu}(q)$	$1 + q$
$NT_{4,\mu}(q)$	$1 + 3q + q^2 + q^3$
$NT_{5,\mu}(q)$	$1 + 6q + 6q^2 + 7q^3 + 2q^4 + q^5 + q^6$
$NT_{6,\mu}(q)$	$1 + 10q + 20q^2 + 31q^3 + 23q^4 + 15q^5 + 13q^6 + 3q^7 + 2q^8 + q^9 + q^{10}$
$NT_{7,\mu}(q)$	$1 + 15q + 50q^2 + 106q^3 + 135q^4 + 126q^5 + 119q^6 + 66q^7 + 46q^8 + 25q^9 + 19q^{10} + 5q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15}$
$NT_{8,\mu}(q)$	$1 + 21q + 105q^2 + 301q^3 + 561q^4 + 736q^5 + 850q^6 + 726q^7 + 603q^8 + 418q^9 + 299q^{10} + 174q^{11} + 101q^{12} + 65q^{13} + 33q^{14} + 27q^{15} + 7q^{16} + 5q^{17} + 3q^{18} + 2q^{19} + q^{20} + q^{21}$
$NT_{9,\mu}(q)$	$1 + 28q + 196q^2 + 742q^3 + 1870q^4 + 3311q^5 + 4820q^6 + 5541q^7 + 5675q^8 + 5007q^9 + 4055q^{10} + 3093q^{11} + 2116q^{12} + 1461q^{13} + 888q^{14} + 646q^{15} + 338q^{16} + 217q^{17} + 126q^{18} + 80q^{19} + 44q^{20} + 35q^{21} + 11q^{22} + 7q^{23} + 5q^{24} + 3q^{25} + 2q^{26} + q^{27} + q^{28}$
$NT_{10,\mu}(q)$	$1 + 36q + 336q^2 + 1638q^3 + 5328q^4 + 12253q^5 + 22392q^6 + 32864q^7 + 41488q^8 + 45433q^9 + 44119q^{10} + 40008q^{11} + 32781q^{12} + 25689q^{13} + 18551q^{14} + 13710q^{15} + 9137q^{16} + 6179q^{17} + 3971q^{18} + 2568q^{19} + 1640q^{20} + 1098q^{21} + 640q^{22} + 374q^{23} + 251q^{24} + 148q^{25} + 100q^{26} + 56q^{27} + 46q^{28} + 15q^{29} + 11q^{30} + 7q^{31} + 5q^{32} + 3q^{33} + 2q^{34} + q^{35} + q^{36}$

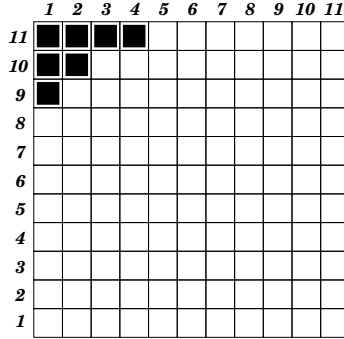
We have computed Table 2 for the polynomials $NTI_{n,\mu}(q)$ using Mathematica.

We shall show that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ are the number of partitions of k for sufficiently large n . We show this by plotting the *non- μ -matches* of an n -cycle $C \in \mathcal{C}_n$, i.e. the pairs of integers $\langle i, j \rangle$ with $i < j$ which are not occurrences of the μ pattern in C , on an $n \times n$ grid and shading a cell in the j th row and i th column if and only if $\langle i, j \rangle$ is a non- μ -match of C . For example, the non- μ -matches of the cycle $C = (1, 8, 7, 6, 5, 11, 4, 3, 10, 2, 9)$ are $\langle 1, 9 \rangle, \langle 1, 10 \rangle, \langle 1, 11 \rangle, \langle 2, 10 \rangle, \langle 2, 11 \rangle, \langle 3, 11 \rangle$ and $\langle 4, 11 \rangle$. These pairs can be plotted on the 11×11 grid as shown in Figure 4. The shaded plot in Figure 4 is of the form of the *Ferrers diagram* of the integer partition $\lambda = (4, 2, 1)$. We will show that if $C \in \mathcal{C}_n$ and there are fewer than $n - 2$ non-matches in C , then shaded squares in the plot of the non- μ -matches in C will be of the form of a Ferrers diagram of an integer partition. Moreover, we shall show that if $C \in \mathcal{C}_n$ has fewer than $n - 2$ non-matches, then C must be incontractible. Hence, it follows that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ are the number of partitions of k for sufficiently large n .

We will show that $NTI_{n,\mu}(q)|_{q^k} = 0$ for $n \geq 2k + 2$ and that the lowest power of q which appears in either $NTI_{2k,\mu}(q)$ or $NTI_{2k+1,\mu}(q)$ is q^k . We computed that the sequence $(NTI_{2k+1,\mu}(q)|_{q^k})_{k \geq 3}$ starts out $1, 2, 5, 14, \dots$ which led us to conjecture that

TABLE 2. Polynomials $NTI_{n,\mu}(q)$.

$NTI_{1,\mu}(q)$	1
$NTI_{2,\mu}(q)$	0
$NTI_{3,\mu}(q)$	q
$NTI_{4,\mu}(q)$	$q^2 + q^3$
$NTI_{5,\mu}(q)$	$2q^2 + 3q^3 + 2q^4 + q^5 + q^6$
$NTI_{6,\mu}(q)$	$6q^3 + 13q^4 + 10q^5 + 8q^6 + 3q^7 + 2q^8 + q^9 + q^{10}$
$NTI_{7,\mu}(q)$	$5q^3 + 27q^4 + 51q^5 + 56q^6 + 48q^7 + 34q^8 + 19q^9 + 13q^{10}$ $+ 5q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15}$
$NTI_{8,\mu}(q)$	$29q^4 + 134q^5 + 255q^6 + 327q^7 + 323q^8 + 264q^9 + 187q^{10} + 139q^{11} +$ $80q^{12} + 51q^{13} + 26q^{14} + 20q^{15} + 7q^{16} + 5q^{17} + 3q^{18} + 2q^{19} + q^{20} + q^{21}$
$NTI_{9,\mu}(q)$	$14q^4 + 181q^5 + 694q^6 + 1413q^7 + 2027q^8 + 2307q^9 + 2139q^{10} + 1841q^{11} +$ $1392q^{12} + 997q^{13} + 652q^{14} + 458q^{15} + 282q^{16} + 177q^{17} + 102q^{18} + 64q^{19} +$ $36q^{20} + 27q^{21} + 11q^{22} + 7q^{23} + 5q^{24} + 3q^{25} + 2q^{26} + q^{27} + q^{28}$
$NTI_{10,\mu}(q)$	$130q^5 + 1128q^6 + 3965q^7 + 8509q^8 + 13444q^9 + 16918q^{10} +$ $18015q^{11} + 17121q^{12} + 14712q^{13} + 11663q^{14} + 8784q^{15} + 6347q^{16} +$ $4406q^{17} + 2945q^{18} + 1920q^{19} + 1280q^{20} + 819q^{21} + 541q^{22} +$ $311q^{23} + 206q^{24} + 121q^{25} + 82q^{26} + 47q^{27} + 37q^{28} + 15q^{29} +$ $11q^{30} + 7q^{31} + 5q^{32} + 3q^{33} + 2q^{34} + q^{35} + q^{36}$

FIGURE 4. The non-matches of $(1, 8, 7, 6, 5, 11, 4, 3, 10, 2, 9)$ plotted on an 11×11 grid

$NTI_{2k+1,\mu}(q)|_{q^k} = C_k = \frac{1}{n+1} \binom{2n}{n}$ where C_k is the k th *Catalan number*. We shall prove this conjecture and our proof of this conjecture led to a surprising connection between the *charge statistic on permutations* as defined by Lascoux and Schützenberger [7] and our problem. That is, given a permutation $\sigma = \sigma_1 \dots \sigma_n$, one defines the index, $\text{ind}_\sigma(\sigma_i)$, of σ_i in σ as follows. First, $\text{ind}_\sigma(1) = 0$. Then, inductively, $\text{ind}_\sigma(i+1) = \text{ind}_\sigma(i)$ if $i+1$ appears to the right of i in σ and $\text{ind}_\sigma(i+1) = 1 + \text{ind}_\sigma(i)$ if $i+1$ appears to the left of i in σ . For example, if $\sigma = 1\ 4\ 8\ 5\ 9\ 6\ 2\ 7\ 3$ and we use subscripts to indicate the index of i in σ , then

we see that the indices associated with σ are

$$\sigma = 1_0 \ 4_1 \ 8_2 \ 5_1 \ 9_2 \ 6_1 \ 2_0 \ 7_1 \ 3_0.$$

The charge of σ , $\text{ch}(\sigma)$, is equal to $\sum_{i=1}^n \text{ind}_\sigma(i)$. Thus, for example, if $\sigma = 1 \ 4 \ 8 \ 5 \ 9 \ 6 \ 2 \ 7 \ 3$, then $\text{ch}(\sigma) = 8$. We will associate a path which we call the *charge path* of σ whose vertices are elements $(i, \text{ind}_\sigma(\sigma_i))$ and whose edges are $\{(i, \text{ind}_\sigma(\sigma_i)), (i + 1, \text{ind}_\sigma(\sigma_{i+1}))\}$ for $i = 1, \dots, n$. For example, if $\sigma = 1 \ 4 \ 8 \ 5 \ 9 \ 6 \ 2 \ 7 \ 3$ as above, then charge path of σ , which we denote by $\text{chpath}(\sigma)$, is pictured in Figure 5. In our particular example, the charge graph of σ is a *Dyck path* (a lattice path with steps $(1,1)$ and $(1,-1)$ from the origin $(0,0)$ to $(2n,0)$ that never goes below the x -axis) and if $C = (1, 4, 8, 5, 9, 6, 2, 7, 3)$ is the n -cycle induced by σ , then $NT_\mu(C) = 4$. This is no accident. That is, we shall show that if $C = (1, \sigma_2, \dots, \sigma_{2n+1})$ is $2n + 1$ -cycle in \mathcal{IC}_{2n+1} , then $NT_\mu(C) = n$ if and only if the charge path of $\sigma = 1\sigma_2 \dots \sigma_{2n+1}$ is a Dyck path of length $2n$.

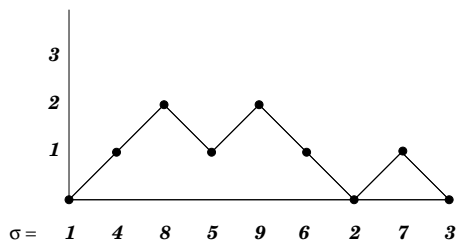


FIGURE 5. The charge path of $\sigma = 1 \ 4 \ 8 \ 5 \ 9 \ 6 \ 2 \ 7 \ 3$.

The outline of this paper is as follows. In Section 2, we shall introduce the notion of the contraction of an n -cycle and use it to show that for all $n \geq 1$, there are positive integers c_1, \dots, c_{2k+1} such that $NT_{n,\mu}(q)|_{q^k} = \sum_{s=\lfloor \frac{3+\sqrt{1+8k}}{2} \rfloor}^{2k+1} c_s \binom{n-1}{s-1}$. In Section 3, we shall study incontractible cycles and the polynomials $NTI_{n,\mu}(q)$. We shall show that for all $n \geq 1$, $NTI_{2n+1,\mu}(q)|_{q^n} = C_n$ where C_n is n th Catalan number. In Section 4, we study the non- μ -matches in n -cycles of \mathcal{C}_n and use them to show that $NTI_{n,\mu}(q)|_{q \binom{n-1}{2}-k}$ and $NT_{n,\mu}(q)|_{\binom{n-1}{2}-k}$ are the number of partitions of k for sufficiently large n . In Section 5, we will state our conclusions and some open problems.

2. THE CONTRACTION OF A CYCLE.

Define a *bond* of an n -cycle $C \in \mathcal{C}_n$ to be a pair of consecutive integers $(a, a + 1)$ such that a is immediately followed by $a + 1$ in the cycle C . For example, the bonds of the cycle $(1, 4, 5, 6, 2, 7, 8, 3)$ are the pairs $(4, 5)$, $(5, 6)$, and $(7, 8)$. If $C = (c_1, c_2, \dots, c_n)$ is an n -cycle where $c_1 = 1$, define the sets R_1, R_2, \dots, R_k recursively as follows. First, let $1 = c_1 \in R_1$. Then inductively, if $c_i \in R_j$, then $c_{i+1} \in R_j$ if $c_{i+1} = c_i + 1$, and $c_{i+1} \in R_{j+1}$ otherwise. For example, for the cycle $C = (1, 2, 4, 6, 7, 8, 3, 5)$ these sets are $R_1 = \{1, 2\}, R_2 = \{4\}, R_3 = \{6, 7, 8\}, R_4 = \{3\}, R_5 = \{5\}$. We will call these sets

the *consecutive runs* of C . Define the *contraction* of a cycle C with consecutive runs R_1, R_2, \dots, R_k to be

$$\text{cont}(C) = \text{red}(\max(R_1), \max(R_2), \dots, \max(R_k)),$$

where $\max(R_i)$ denotes the maximum element in a run R_i . For example, the contraction of $C = (1, 2, 4, 6, 7, 8, 3, 5)$ is

$$\text{cont}(1, 2, 4, 6, 7, 8, 3, 5) = \text{red}(2, 4, 8, 3, 5) = (1, 3, 5, 2, 4).$$

We say that C *contracts* to A if $\text{cont}(C) = A$. Clearly, a cycle C is *incontractible* if $\text{cont}(C) = C$.

We claim that the number of non-trivial μ matches does not change as we pass from C to $\text{cont}(C)$. That is, we have the following theorem.

Theorem 1. *For any n -cycle $C \in \mathcal{C}_n$,*

$$NT_\mu(C) = NT_\mu(\text{cont}(C)).$$

Proof. We proceed by induction on n . The theorem is clearly true for $n = 1$. Now, suppose that we are given an n -cycle $C \in \mathcal{C}_n$ and R_1, \dots, R_k are the consecutive runs. If $k = n$, then, clearly, C is *incontractible*. Otherwise, suppose that s is the least i such that $|R_i| \geq 2$. Let $R_s = \{i, i+1, \dots, j\}$. Then it is easy to see that in C , there are no pairs $\langle a, t \rangle$ which are nontrivial occurrences of μ in C for $i \leq a \leq j-1$ since $a+1$ immediately follows a in C . Similarly, there are no pairs $\langle s, b \rangle$ which are nontrivial occurrences of μ in C for $i+1 \leq b \leq j$ since $b-1$ immediately precedes b in C . Now, suppose that C' arises from C by removing $i+1, \dots, j$ and replacing each $s > j$ by $s - (j-i)$. Then it is easy to see that

- (1) if $s < t \leq i$, $\langle s, t \rangle$ is a nontrivial occurrence of μ in C if and only if $\langle s, t \rangle$ is a nontrivial occurrence of μ in C' ,
- (2) if $s \leq i < j < t$, then $\langle s, t \rangle$ is a nontrivial occurrence of μ in C if and only if $\langle s, t - (j-i) \rangle$ is a nontrivial occurrence of μ in C' , and
- (3) if $j \leq s < t$, then $\langle s, t \rangle$ is a nontrivial occurrence of μ in C if and only if $\langle s - (j-i), t - (j-i) \rangle$ is a nontrivial occurrence of μ in C' .

It follows that $NT_\mu(C) = NT_\mu(C')$. Moreover, it is easy to see that $\text{cont}(C) = \text{cont}(C')$. By induction $NT_\mu(C') = NT_\mu(\text{cont}(C'))$. Hence, $NT_\mu(C) = NT_\mu(C') = NT_\mu(\text{cont}(C')) = NT_\mu(\text{cont}(C))$. \square

It is easy to count the number of n -cycles $C \in \mathcal{C}_n$ for which $\text{cont}(C) = A$. That is, we have the following theorem.

Theorem 2. *Let $A \in \mathcal{IC}_\ell$ be an *incontractible* cycle of length ℓ . The number of n -cycles $C \in \mathcal{C}_n$ such that $\text{cont}(C) = A$ is $\binom{n-1}{\ell-1}$.*

Proof. If $\text{cont}(C) = A$, then there are ℓ consecutive runs of C , namely R_1, \dots, R_ℓ such that $\text{red}(\max(R_1), \dots, \max(R_\ell)) = A$. Since the maximum of one of the consecutive runs must be n , there are $\binom{n-1}{\ell-1}$ ways to choose the rest of the maxima. This will determine the consecutive runs and then we order them so that $\text{red}(\max(R_1), \dots, \max(R_\ell)) = A$.

For example, suppose $A = (1, 3, 5, 2, 4)$ and $n = 8$. The 8-cycle that contracts to A corresponding to the choice $\{1, 3, 4, 7\}$ out of the $\binom{7}{4}$ choices is obtained by letting the maxima of the consecutive runs be $\{1, 3, 4, 7, 8\}$. Therefore, the consecutive runs are $\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}, \{8\}$. Then we arrange them so that $\text{red}(\max(R_1), \dots, \max(R_\ell)) = A$ and you get $\{1\}, \{4\}, \{8\}, \{2, 3\}, \{5, 6, 7\}$, and so the cycle is $(1, 4, 8, 2, 3, 5, 6, 7)$. \square

Theorems 1 and 2 imply the following theorem.

Theorem 3. *For any $k \geq 0$ and $n \geq 1$,*

$$(4) \quad NT_{n,\mu}(q)|_{q^k} = \sum_{s=1}^n \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}.$$

Proof. It is easy to see that (4) follows by partitioning the n -cycles $C \in \mathcal{C}_n$ such that $NT_\mu(C) = k$ by their contractions. \square

We shall prove in the next section that the smallest power of q that occurs in either $NTI_{2n,\mu}(q)$ or $NTI_{2n+1,\mu}(q)$ is q^n . Hence it follows that

$$\sum_{s=1}^n \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}.$$

Furthermore, the maximum number of nontrivial occurrences of μ that one can have for a cycle of length s is $\binom{s-1}{2}$, (see page 4). This means that for a cycle of length s to have k nontrivial occurrences, we must have that $\frac{(s-1)(s-2)}{2} \geq k$ or, equivalently, $s^2 - 3s + 2 - 2k \geq 0$. It follows that it must be the case that $s \geq \frac{3 + \sqrt{1+8k}}{2}$. Hence,

$$(5) \quad \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1} = \sum_{s=\lfloor \frac{3 + \sqrt{1+8k}}{2} \rfloor}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}.$$

Thus we have the following theorem.

Theorem 4. *For any $k \geq 0$ and $n \geq 1$,*

$$(6) \quad NT_{n,\mu}(q)|_{q^k} = \sum_{s=\lfloor \frac{3 + \sqrt{1+8k}}{2} \rfloor}^{2k+1} c_s \binom{n-1}{s-1}$$

where $c_s = NTI_{s,\mu}(q)|_{q^k}$.

It follows from our tables for $NTI_{n,\mu}(q)$ and the fact that $NTI_{11,\mu}(q)|_{q^5} = C_5 = 42$ which we will prove in Corollary 10 that for all $n \geq 1$

$$\begin{aligned}
NT_{n,\mu}(q)|_q &= \binom{n-1}{2}, \\
NT_{n,\mu}(q)|_{q^2} &= \binom{n-1}{3} + 2\binom{n-1}{4}, \\
NT_{n,\mu}(q)|_{q^3} &= \binom{n-1}{3} + 3\binom{n-1}{4} + 6\binom{n-1}{5} + 5\binom{n-1}{6}, \\
NT_{n,\mu}(q)|_{q^4} &= 2\binom{n-1}{4} + 13\binom{n-1}{5} + 27\binom{n-1}{6} + 29\binom{n-1}{7} + 14\binom{n-1}{8}, \text{ and} \\
NT_{n,\mu}(q)|_{q^5} &= \binom{n-1}{4} + 10\binom{n-1}{5} + 51\binom{n-1}{6} + 134\binom{n-1}{7} + \\
&\quad 181\binom{n-1}{8} + 130\binom{n-1}{9} + 42\binom{n-1}{10}.
\end{aligned}$$

3. INCONTRACTIBLE n -CYCLES

Let IC_n denote the number of incontractible n -cycles in \mathcal{C}_n . Clearly, there is only 1 incontractible 3-cycle, namely, $(1, 3, 2)$ and there are only 2 incontractible 4-cycles, namely, $(1, 4, 3, 2)$ and $(1, 3, 2, 4)$. Our next theorem shows the numbers IC_n satisfies the recursion of the number of derangements.

Theorem 5. *For all $n \geq 5$,*

$$(7) \quad IC_n = (n-2)IC_{n-1} + (n-2)IC_{n-2}.$$

Proof. Let $[n] = \{1, \dots, n\}$. Suppose that $n \geq 5$ and $C = (1, c_2, \dots, c_n)$ is an n -cycle in \mathcal{IC}_n . Let $C \upharpoonright_{[n-1]} = (1, c'_2, \dots, c'_{n-1})$ be the cycle obtained from C by removing n from C . For example if $C = (1, 3, 6, 2, 4, 5)$ then $C \upharpoonright_{[5]} = (1, 3, 2, 4, 5)$. We then have two cases depending on whether $C \upharpoonright_{[n-1]}$ is incontractible or not.

Case 1. $C \upharpoonright_{[n-1]} \in \mathcal{IC}_{n-1}$.

In this situation, it is easy to see that if $D = C \upharpoonright_{[n-1]}$, there are exactly $n-2$ cycles $C' \in \mathcal{IC}_n$ such that $D = C' \upharpoonright_{[n-1]}$. These cycles are the result of inserting n immediately after i in D for $i = 1, \dots, n-2$. That is, let $D^{(i)}$ be the result of inserting n immediately after i in the cycle structure of D where $1 \leq i \leq n-2$. Then it is easy to see that $D^{(i)} \in \mathcal{IC}_n$ and $D = D^{(i)} \upharpoonright_{[n-1]}$. For example, if $D = (1, 3, 5, 4, 2)$ is the 5-cycle pictured at the top of Figure 6, then $D^{(1)}, \dots, D^{(4)}$ are pictured on the second row of Figure 6, reading from left to right. Thus, there are $(n-2) IC_{n-1}$ n -cycles $C \in \mathcal{IC}_n$ such that $C \upharpoonright_{[n-1]}$ is an element of \mathcal{IC}_{n-1} . (Note that $D^{(n-1)}$ is not incontractible because it is obtained by inserting n directly after $n-1$.)

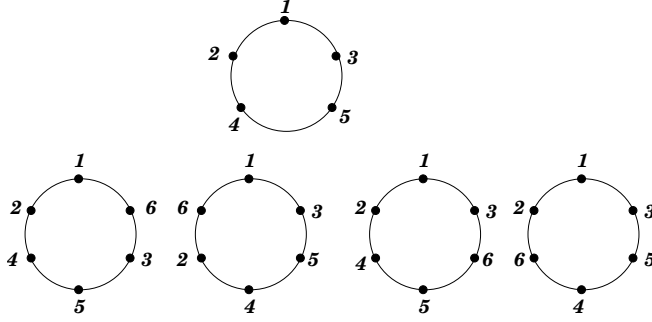


FIGURE 6. The four elements of \mathcal{IC}_6 that arise by inserting 6 into $C = (1, 3, 5, 4, 2)$.

We note that if $D = (1 = d_1, \dots, d_{n-1})$ is an $(n - 1)$ -cycle in \mathcal{IC}_{n-1} , then

$$NT_\mu(D^{(i)}) \geq 1 + NT_\mu(D) \text{ for } i = 1, \dots, n - 2.$$

That is, if $\langle d_s, d_t \rangle$ is a nontrivial occurrence of μ in D , then the insertion of n does not effect whether $\langle d_s, d_t \rangle$ is a nontrivial occurrence of μ in $D^{(i)}$. Moreover, we are always guaranteed that $\langle i, n \rangle$ is a nontrivial occurrence of μ in $D^{(i)}$ since $i \leq n - 2$. It is possible that $NT_\mu(D^{(i)}) - NT_\mu(D)$ is greater than or equal to 1. For example, in Figure 6, it is easy to see that if $D = (1, 3, 5, 4, 2)$, then there is only two pairs which are nontrivial occurrences of μ in C , namely $\langle 1, 3 \rangle$ and $\langle 3, 5 \rangle$, while in $D^{(1)}$, the insertion of 6 after 1 created three new nontrivial occurrences of μ in $D^{(1)}$ namely, the pairs $\langle 1, 6 \rangle$, $\langle 2, 6 \rangle$, and $\langle 4, 6 \rangle$. Thus, $NT_\mu(D^{(1)}) - NT_\mu(D) = 5 - 2 = 3$. On the other hand, it is easy to see that if $D \in \mathcal{IC}_{n-1}$, then

$$(8) \quad NT_\mu(D^{(n-2)}) = 1 + NT_\mu(D).$$

That is, if we insert n immediately after $n - 2$ in D , then for $i = 1, \dots, n - 3$, $\langle i, n \rangle$ cannot be a nontrivial occurrence of μ in $D^{(n-2)}$ because $n - 2$ will be between i and n in D . Thus, we will create exactly one more pair which is a nontrivial occurrence of μ in $D^{(n-2)}$ that was not in D , namely, $\langle n - 2, n \rangle$.

Case 2 $C \upharpoonright_{[n-1]} \notin \mathcal{IC}_n$.

In this case, it must be that in C , n is the only element between i and $i + 1$ in C for some i so that in $C \upharpoonright_{[n-1]}$, $i + 1$ immediately follows i . Clearly, i is the only j in $C \upharpoonright_{[n-1]}$ such that $j + 1$ immediately follows j . Hence, we can construct an element of $D \in \mathcal{IC}_{n-2}$ from $C \upharpoonright_{[n-1]}$ by removing $i + 1$ and then replacing j by $j - 1$ for $i + 1 < j \leq n - 1$. Vice versa, given $D \in \mathcal{IC}_{n-2}$ and $1 \leq i \leq n - 2$, let $D^{[i]}$ be the cycle that results from D by first replacing elements $j \geq i + 1$, by $j + 1$, then replacing i by a pair i immediately followed by $i + 1$, and finally inserting n between i and $i + 1$. This process is pictured in Figure 7 for the cycle $D = (1, 3, 2, 4)$. In such a situation, we shall say that $D^{[i]}$ arises by expanding D at i .

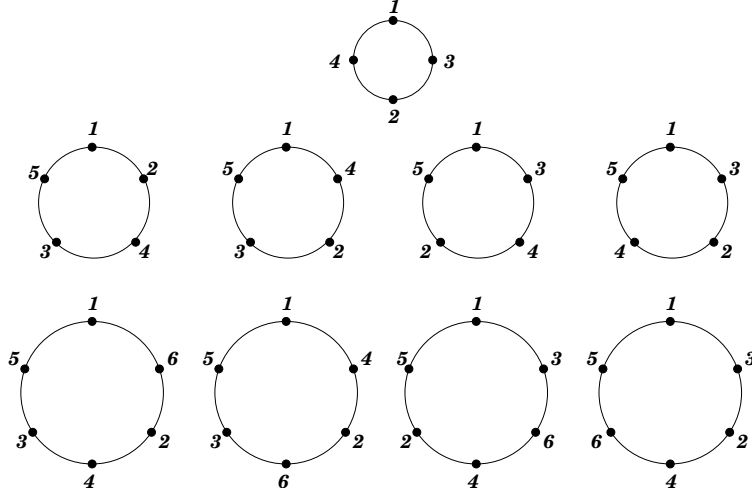


FIGURE 7. The four elements of \mathcal{IC}_6 that arise by expanding $D = (1, 3, 2, 4)$.

We should note that if $D = (1 = d_1, \dots, d_{n-2})$ is an $(n-2)$ -cycle in \mathcal{IC}_{n-2} , then

$$NT_\mu(D^{[i]}) \geq 1 + NT_\mu(D) \text{ for } i = 1, \dots, n-2.$$

In the first step of the expansion of D at i , we replace each $j \geq i+1$ by $j+1$ and then replace i by a consecutive pair i followed by $i+1$ to get a cycle D_i . It is clear that this keeps the number of nontrivial occurrences of μ the same. That is, it is easy to see that there will be no pair $\langle i, t \rangle$ which is a nontrivial occurrence of μ in D_i since i is immediately followed by $i+1$ in D_i . Moreover, it is easy to check that

- (1) if $s < t \leq i$, then $\langle s, t \rangle$ is a nontrivial occurrence of μ in D if and only if $\langle s, t \rangle$ is a nontrivial occurrence of μ in D_i ,
- (2) if $s < i < t$, then $\langle s, t \rangle$ is a nontrivial occurrence of μ in D if and only if $\langle s, t+1 \rangle$ is a nontrivial occurrence of μ in D_i .
- (3) $\langle i, t \rangle$ is a nontrivial occurrence of μ in D if and only if $\langle i+1, t+1 \rangle$ is a nontrivial occurrence of μ in D_i , and
- (4) if $i < s < t$, then $\langle s, t \rangle$ is a nontrivial occurrence of μ in D if and only if $\langle s+1, t+1 \rangle$ is a nontrivial occurrence of μ in D_i .

Then for every pair $\langle r, s \rangle$ which is a nontrivial occurrence of μ in D_i , there exists a nontrivial occurrence of μ in $D^{[i]}$ after inserting n between i and $i+1$. Finally we will create at least one new nontrivial occurrence of μ in D , namely $\langle i, n \rangle$.

Again it is possible that $NT_\mu(D^{[i]}) - NT_\mu(D)$ is greater than or equal to 1. For example, in Figure 7, it is easy to see that if $D = (1, 3, 5, 4, 2)$, there are only two pairs that are nontrivial occurrences of μ in D , namely $\langle 1, 3 \rangle$ and $\langle 2, 4 \rangle$, which correspond to the pairs $\langle 1, 4 \rangle$ and $\langle 3, 5 \rangle$ that are nontrivial occurrences of μ in $D^{[2]}$. However, the insertion of 6 between 2 and 3 in D_2 created two new nontrivial occurrences of μ in $D^{[2]}$, namely, $\langle 2, 6 \rangle$ and $\langle 4, 6 \rangle$. Thus, $NT_\mu(D^{[2]}) - NT_\mu(D) = 4 - 2 = 2$. On the other hand, it is easy to see

that if $D \in \mathcal{IC}_{n-2}$, then

$$(9) \quad NT_\mu(D^{[n-2]}) = 1 + NT_\mu(D).$$

That is, if we insert n immediately after $n-2$ in D_{n-2} , then for $i = 1, \dots, n-3$, $\langle i, n \rangle$ cannot be a nontrivial occurrences of μ in $D^{[n-2]}$ because $n-2$ will be between i and n in $D^{[n-2]}$. Thus, we will create exactly one occurrence of μ in $D^{[n-2]}$ that was not in D_{n-2} , namely, $\langle n-2, n \rangle$. □

We have two important corollaries to our proof of Theorem 5.

Corollary 6. *For all $n \geq 2$, $IC_n = D_{n-1}$ where D_n is the number of derangements of S_n .*

Proof. It is well-known that $D_1 = 0 = IC_2$, $D_2 = 1 = IC_3$, and that for $n \geq 2$, $D_{n+1} = nD_{n-1} + nD_{n-2}$. Thus, the corollary follows by recursion. □

Corollary 7. *For all $n \geq 2$, the lowest power of q appearing in $NTI_{2n,\mu}(q)$ and $NTI_{2n+1,\mu}(q)$ is q^n .*

Proof. We have shown by direct calculation that the lowest power of q appearing in $NTI_{4,\mu}(q)$ and $NTI_{5,\mu}(q)$ is q^2 . Thus, the corollary holds for $n = 2$. Now, assume the corollary holds for $n \geq 2$. Then we shall show the corollary holds for $n+1$. We have shown that each $(2n+2)$ -cycle C in \mathcal{IC}_{2n+2} is either of the form $D^{(i)}$ for some $D \in \mathcal{IC}_{2n+1}$ and $1 \leq i \leq 2n$ in which case $NT_\mu(D^{(i)}) \geq 1 + NT_\mu(D)$, or of the form $E^{[i]}$ for some $E \in \mathcal{IC}_{2n}$ and $1 \leq i \leq 2n$ in which case $NT_\mu(E^{[i]}) \geq 1 + NT_\mu(E)$. But by induction, we know that $NT_\mu(D) \geq n$ and $NT_\mu(E) \geq n$ so that $NT_\mu(C) \geq n+1$. Thus, the smallest possible power of q that can appear in $NTI_{2n+2,\mu}(q)$ is $n+1$. On the other hand, we can assume by induction that there is a $D \in \mathcal{IC}_{2n+1}$ such that $NT_\mu(D) = n$ in which case we know that $NT_\mu(D^{(n-2)}) = 1 + NT_\mu(D) = n+1$. Hence, the coefficient of q^{n+1} in $NTI_{2n+2,\mu}(q)$ is non-zero.

We have shown that each $(2n+3)$ -cycle C in \mathcal{IC}_{2n+3} is either of the form $D^{(i)}$ for some $D \in \mathcal{IC}_{2n+2}$ and $1 \leq i \leq 2n+1$ in which case $NT_\mu(D^{(i)}) \geq 1 + NT_\mu(D)$, or of the form $E^{[i]}$ for some $E \in \mathcal{IC}_{2n+1}$ and $1 \leq i \leq 2n+1$ in which case $NT_\mu(E^{[i]}) \geq 1 + NT_\mu(E)$. But by induction, we know that $NT_\mu(D) \geq n+1$ and $NT_\mu(E) \geq n$ so that $NT_\mu(C) \geq n+1$. Thus, the smallest possible power of q that can appear in $NTI_{2n+3,\mu}(q)$ is $n+1$. On the other hand, we can assume by induction that there is a $D \in \mathcal{IC}_{2n+1}$ such that $NT_\mu(D) = n$ in which case we know that $NT_\mu(D^{[n-2]}) = 1 + NT_\mu(D) = n+1$. Hence, the coefficient of q^{n+1} in $NTI_{2n+3,\mu}(q)$ is non-zero. □

We have not been able to find a recursion for the polynomials $NTI_{n,\mu}(q)$. The problem with the recursion implicit in the proof of Theorem 5 is that for n -cycles $D \in \mathcal{IC}_n$, the contributions of $\sum_{i \geq 1} q^{NT_\mu(D^{(i)})}$ are not uniform. For example, if $D = (1, 3, 5, 4, 2)$, then $NT_\mu(D) = 2$, $NT_\mu(D^{(1)}) = 5$, $NT_\mu(D^{(2)}) = 4$, $NT_\mu(D^{(3)}) = 4$, and $NT_\mu(D^{(4)}) = 3$ so that $\sum_{i=1}^4 q^{NT_\mu(D^{(i)})} = (q + 2q + q^3)q^{NT_\mu(D)}$. However, $D = (1, 5, 4, 3, 2)$, then

$NT_\mu(D) = 6$, $NT_\mu(D^{(1)}) = 10$, $NT_\mu(D^{(2)}) = 9$, $NT_\mu(D^{(3)}) = 8$, and $NT_\mu(D^{(4)}) = 7$ so that $\sum_{i=1}^4 q^{NT_\mu(D^{(i)})} = (q + q^2 + q^3 + q^4)q^{NT_\mu(D)}$. A similar phenomenon occurs for $\sum_{i \geq 1} NT_\mu(D^{[i]})$. For example, it is easy to see from Figure 6 that if $D = (1, 3, 2, 4)$, then $NT_\mu(D) = 2$, $NT_\mu(D^{[1]}) = 3$, $NT_\mu(D^{[2]}) = 4$, $NT_\mu(D^{[3]}) = 3$, and $NT_\mu(D^{[4]}) = 3$, so that $\sum_{i=1}^4 q^{NT_\mu(D^{[i]})} = (3q + q^2)q^{NT_\mu(D)}$. However, as one can see from Figure 8 below that if $D = (1, 4, 3, 2)$, then $NT_\mu(D) = 3$, $NT_\mu(D^{[1]}) = 6$, $NT_\mu(D^{[2]}) = 5$, $NT_\mu(D^{[3]}) = 4$, and $NT_\mu(D^{[4]}) = 4$, so that $\sum_{i=1}^4 q^{NT_\mu(D^{[i]})} = (2q + q^2 + q^3)q^{NT_\mu(D)}$.

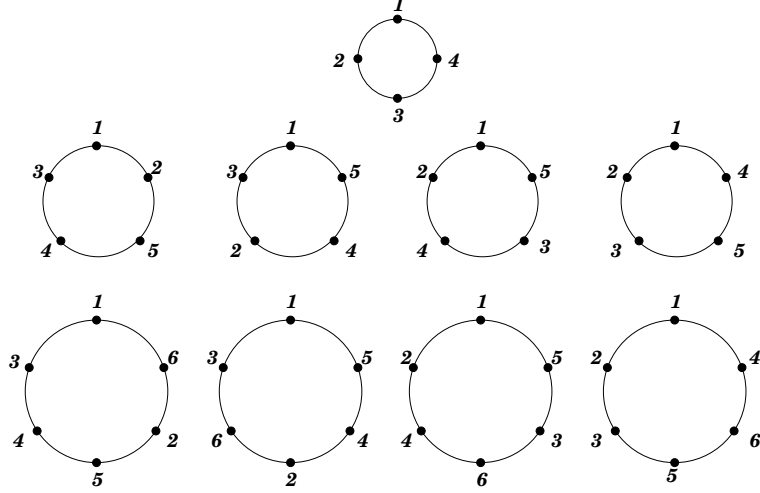


FIGURE 8. The four elements of \mathcal{IC}_6 that arise by expanding $D = (1, 4, 3, 2)$.

Our next goal is to show that $NTI_{2n+1, \mu}(q)|_{q^n} = C_n$ where C_n is the n th Catalan number.

Theorem 8. *Suppose that $\sigma = \sigma_1 \dots \sigma_{2n+1} \in S_{2n+1}$ and $\text{chpath}(\sigma) = P$ is a Dyck path of length $2n$. Then the n -cycle $C_\sigma = (\sigma_1, \dots, \sigma_{2n+1})$ is incontractible and $NT_\mu(C) = n$.*

Proof. Since the path P starts at $(0, 0)$, it follows that $\text{ind}_\sigma(\sigma_1) = 0$. This can only happen if $\sigma_1 = 1$. Thus, $C_\sigma = (1, \sigma_2, \dots, \sigma_{2n+1})$ has the standard form of an n -cycle. Note that the only way $i+1$ immediately follows i in C_σ is if there is a j such that $\sigma_j = i$ and $\sigma_{j+1} = i+1$. But then $\text{ind}_\sigma(\sigma_j) = \text{ind}_\sigma(\sigma_{j+1})$ which would imply that the charge path of σ has a level which means that there is an edge drawn horizontally. Since we are assuming that the charge path of C_σ is a Dyck path, there can be no such i and, hence, C_σ is incontractible.

Next we claim that if the i th step of P is an up-step, then $\langle \sigma_i, \sigma_{i+1} \rangle$ is a nontrivial occurrence of μ in C_σ . That is, if the i th step of P is an up-step, then $\text{ind}_\sigma(\sigma_i) = k$ and $\text{ind}_\sigma(\sigma_{i+1}) = k+1$ for some $k \geq 0$. Since P is a Dyck path, there must be some $j > i+1$ such that $\text{ind}_\sigma(\sigma_j) = k$ because we must pass through level k from the point $(i+1, k+1)$ in $\text{chpath}(\sigma)$ to get back to the point $(2n, 0)$ which is the end point of the path. Let i_j denote the smallest $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$. It follows from the definition of the

function ind_σ that it must be the case that $\sigma_i = \ell$ and $\sigma_{i_j} = \ell + 1$. Moreover, it must be the case that $\sigma_{i+1} > \ell + 1$ so that $\langle \sigma_i, \sigma_{i+1} \rangle$ is a nontrivial occurrence of μ in C_σ . We claim that there can be no $t \neq \sigma_{i+1}$ such that $\langle \sigma_i, t \rangle$ is a nontrivial occurrence of μ in C_σ . That is, since P is a Dyck path, we know that all the vertices on P between (i, k) and (i_j, k) lie on levels strictly greater than k . It is easy to see from our inductive definition of ind_σ that it must be the case that σ_{i+1} is the least element of $\sigma_{i+1}, \sigma_{i+2}, \dots, \sigma_{i_j-1}$ and hence the fact that σ_{i+1} immediately follows σ_i in C_σ implies that none of the pairs $\langle \sigma_i, \sigma_s \rangle$ is a nontrivial occurrence of μ in C_σ for $i + 1 < s < i_j$. But then as we traverse around the cycle C_σ , the fact that $\sigma_i = \ell$ and $\sigma_{i_j} = \ell + 1$ implies that none of the pairs $\langle \sigma_i, \sigma_s \rangle$ where either $i_j < s \leq 2n + 1$ or $1 \leq s < i$ are nontrivial occurrences of μ in C_σ . Thus, if the i th step of P is an up-step, then $\langle \sigma_i, t \rangle$ is an occurrence of μ in C_σ for exactly one t , namely $t = \sigma_{i+1}$.

Next suppose that the i th step of P is a down-step. Then, we claim that there is no t such that $\langle \sigma_i, t \rangle$ is an occurrence of μ in C_σ . We then have two cases.

Case 1. $\text{ind}_\sigma(\sigma_i) = k$ and there is a $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$.

In this case, let i_j be the least $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$. It then follows that if $\sigma_i = \ell$, then $\sigma_{i_j} = \ell + 1$. Because P is a Dyck path, it must be the case that $\text{ind}_\sigma(\sigma_s) < k$ for all $i < s < i_j$ so that $\sigma_s < \ell$ for all $i < s < i_j$. But this means that as we traverse around the cycle C_σ , the first element that we encounter after $\sigma_i = \ell$ which is bigger than ℓ is $\sigma_{i_j} = \ell + 1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_i, t \rangle$ is a nontrivial occurrence of μ in C_σ .

Case 2. $\text{ind}_\sigma(\sigma_i) = k$ and there is no $j > i$ such that $\text{ind}_\sigma(\sigma_j) = k$.

In this case since P is a Dyck path, all the elements σ_s such that $s > i$ must have $\text{ind}_\sigma(\sigma_s) < k$ and hence $\sigma_s < \sigma_i$ since, in a charge graph, all the elements whose index in σ is less than k are smaller than all the elements whose index is equal to k in σ . We now have two subcases.

Subcase 2.1. There is an $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

Let s_j be the smallest $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$. Since σ_i was the rightmost element whose index relative to σ equals k , it follows that if $\sigma_i = \ell$, then $\sigma_{s_j} = \ell + 1$ since it is the leftmost element whose index relative to σ equals $k + 1$. Moreover, for all $t < s_j$, either σ_t has index less than k in σ or σ_t has index k in σ . In either case, our definition of ind_σ ensures that $\sigma_t < \ell$. But this means that as we traverse around the cycle C_σ , the first element that we encounter after $\sigma_i = \ell$ which is bigger than ℓ is $\sigma_{s_j} = \ell + 1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_i, t \rangle$ is a nontrivial occurrence of μ in C_σ .

Subcase 2.2. There is no $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

In this case, σ_i has the highest possible index in σ and it is the rightmost element whose index in σ is k which means that $\sigma_i = 2n + 1$. Hence, there is no $t > \sigma_i$ such that $\langle \sigma_i, t \rangle$ is an occurrence of μ in C_σ .

The only other possibility is that we have not considered is that $i = 2n + 1$. Since P is a Dyck path, we know that $\text{ind}_\sigma(\sigma_{2n+1}) = 0$, $\text{ind}_\sigma(\sigma_1) = 0$, and $\text{ind}_\sigma(\sigma_2) = 1$. But then it follows that if $\sigma_{2n+1} = \ell$, then $\sigma_1 < \ell$ and $\sigma_2 = \ell + 1$. But this means that as we traverse around the cycle C_σ , the first element that we encounter after $\sigma_{2n+1} = \ell$ which is bigger than ℓ is $\sigma_2 = \ell + 1$ and hence there is no $t > \sigma_i$ such that $\langle \sigma_{2n+1}, t \rangle$ is a nontrivial occurrence of μ in C_σ .

Thus, we have shown that the nontrivial occurrences of μ in C_σ correspond to the up-steps of $\text{chpath}(\sigma)$ and hence $NT_\mu(C_\sigma) = n$. \square

Our next theorem will show that if $C = (1, \sigma_2, \dots, \sigma_{2n+1})$ is a $(2n + 1)$ -cycle in \mathcal{IC}_{2n+1} where $NT_\mu(C) = n$, then the charge path of $\sigma_C = 1\sigma_2 \dots \sigma_{2n+1}$ must be a Dyck path.

Theorem 9. *Let $C = (1, \sigma_2, \dots, \sigma_{2n+1}) \in \mathcal{IC}_{2n+1}$ and $\sigma_C = 1\sigma_2 \dots \sigma_{2n+1}$. Then, if $NT_\mu(C) = n$, the charge path of σ_C must be a Dyck path.*

Proof. Our proof proceeds by induction on n . For $n = 1$, the only element of \mathcal{IC}_3 is $C = (1, 3, 2)$ and it is easy to see that $\text{chpath}(132)$ is a Dyck path.

Now assume by induction that if $D = (1, \sigma_2, \dots, \sigma_{2n+1})$ is a $(2n + 1)$ -cycle in \mathcal{IC}_{2n+1} such that $NT_\mu(D) = n$, then the charge path of $\sigma_D = 1\sigma_2 \dots \sigma_{2n+1}$ is a Dyck path. Now suppose that $C = (1, \sigma_2, \dots, \sigma_{2n+3})$ is a $(2n + 3)$ -cycle in \mathcal{IC}_{2n+3} such that $NT_\mu(C) = n + 1$. Let $\sigma_C = \sigma = 1\sigma_2 \dots \sigma_{2n+3}$. Then, we know by our proof of Theorem 5 that either $C = D^{(j)}$ for some $D \in \mathcal{IC}_{2n+2}$ or $C = D^{[j]}$ for some $D \in \mathcal{IC}_{2n+1}$. We claim that C cannot be of the form $D^{(j)}$ for some $D \in \mathcal{IC}_{2n+2}$. That is, we know that for all $D \in \mathcal{IC}_{2n+2}$ we have $NT_\mu(D) \geq n + 1$ and, hence, $NT_\mu(D^{(j)}) \geq 1 + NT_\mu(D) \geq n + 2$. Thus, there must be some $D \in \mathcal{IC}_{2n+1}$ such that $C = D^{[j]}$ for some j . But then we know that $P_D = \text{chpath}(D)$ is a Dyck path. Let $D = (1, \tau_2, \dots, \tau_{2n+1})$ and $\sigma_D = \tau = \tau_1 \dots \tau_{2n+1}$. Note that it follows from our arguments in Theorem 5 that $NT_\mu(D^{[j]}) - NT_\mu(D)$ is equal to the number of pairs $\langle s, 2n + 3 \rangle$ which are nontrivial occurrences of μ in $D^{[j]}$. It is also the case that the charge path of σ is easily constructed from the charge path of τ . That is, suppose that $\tau_i = j$. Then $1, \dots, j$ are in the same order in σ and τ so that $\text{ind}_\sigma(s) = \text{ind}_\tau(s)$ for $1 \leq s \leq j$. In going from τ to σ , we first increased the value of any $t > j$ by one and then inserted $j + 1$ immediately after j to get a permutation α . Thus $\text{ind}_\alpha(j) = \text{ind}_\alpha(j + 1)$. Moreover, $j + 1, \dots, 2n + 2$ are in the same relative order in α as $j, \dots, 2n + 1$ in τ so that for $j + 1 \leq t \leq 2n + 1$, $\text{ind}_\tau(t) = \text{ind}_\alpha(t + 1)$. Finally, inserting $2n + 3$ in α between j and $j + 1$ has no effect on the indices assigned to $1, \dots, 2n + 2$. Thus for $1 \leq s \leq j$, $\text{ind}_\sigma(j) = \text{ind}_\tau(j)$ and for $j + 1 \leq t \leq 2n + 1$, $\text{ind}_\sigma(t + 1) = \text{ind}_\tau(t)$. This means that one can construct the charge path of σ by essentially starting with the charge graph of τ , then replacing the vertex $(i, \text{ind}_\tau(\tau_i))$ by a horizontal edge, and finally replacing that horizontal edge by a pair of edges $\{(i, \text{ind}_\sigma(\sigma_i)), (i + 1, \text{ind}_\sigma(2n + 3))\}$ and $\{(i + 1, \text{ind}_\sigma(2n + 3)), ((i + 2), \text{ind}_\sigma(\sigma_{i+2}))\}$. In particular, this means the charge paths from 1 up to i are identical for both σ and τ and the charge path from $i + 2$ to $2n + 3$ in σ is identical to the charge path from i to $2n + 1$ in τ .

We now consider several cases depending on which i is such that $\tau_i = j$ and the value $\text{ind}_\tau(\tau_i)$ in the Dyck path P_D .

Case 1. $j = \tau_i$ where $\text{ind}_\tau(\tau_i) < \text{ind}_\tau(\tau_{i+1})$.

In this case, we will have $\sigma_i = \tau_i$, $\sigma_{i+1} = 2n + 3$, $\sigma_{i+2} = 1 + \tau_i$, and $\sigma_{i+3} = 1 + \tau_{i+1}$. Since P_D is a Dyck path, there will be some k such that $\text{ind}_\tau(\tau_i) = k$, and $\text{ind}_\tau(\tau_{i+1}) = k + 1$ so that we will be in the situation pictured in Figure 9. Because $\text{ind}_\sigma(\sigma_{i+3}) = k + 1$, it must be the case that $\text{ind}_\sigma(2n + 3) > k + 1$. It is easy to see that $\langle \sigma_i, 2n + 3 \rangle$ is an occurrence of μ in C . We then have two subcases.

Subcase 1A. There is an $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

In this case, let s_i be the largest s such that $s < i$ and $\text{ind}_\sigma(\sigma_s) = k + 1$. Because P_D is a Dyck path, it must be the case that for all $s_i < t < i$, $\text{ind}_\sigma(\sigma_{s_i}) \leq k$ because on the path from $(s, k + 1)$ to (i, k) , we cannot have an $s < t < i$ such that $\text{ind}_\sigma(\sigma_t) > k + 1$ without having some $t < u < i$ such that $\text{ind}_\sigma(\sigma_u) = k + 1$ which would violate our choice of s_i . But this means that $\sigma_t < \sigma_i$ for all $s_i < t < \sigma_i$ and hence, $\langle \sigma_{s_i}, 2n + 3 \rangle$ is a nontrivial occurrence of μ in C . Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

Subcase 1B. There is no $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

In this case, let s_i be the largest s such that $i < s$ and $\text{ind}_\sigma(\sigma_s) = k$. Because P_D is a Dyck path such an s must exist since $\text{ind}_\sigma(\sigma_{i+3}) = k + 1$ and a Dyck path that reaches level $k + 1$ must subsequently descend to level k . But this means that for all $t < i$, $\text{ind}_\sigma(\sigma_t) \leq k$ and hence, $\sigma_t \leq \sigma_i$. Moreover, for all $r > s_i$, $\text{ind}_\sigma(\sigma_r) < k$ and hence, $\sigma_r \leq \sigma_i$. But it then follows that since $\sigma_{s_i} > \sigma_i$, $\langle \sigma_{s_i}, 2n + 3 \rangle$ is a nontrivial occurrence of μ in C . Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

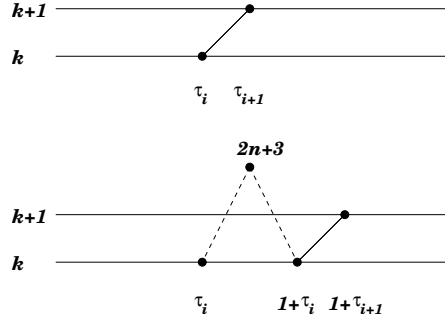


FIGURE 9. The charge path of $D^{[\tau_{2n+1}]}$ in Case 1.

Case 2. $j = \tau_{2n+1}$.

In this case, we will have $\sigma_{2n+1} = \tau_{2n+1}$, $\sigma_{2n+2} = 2n + 3$ and $\sigma_{2n+3} = \tau_{2n+1} + 1$. Thus, the charge path of $D^{[\tau_{2n+1}]}$ looks like one of the two situations pictured in Figure 10. That is, we have two subcases.

Subcase 2A. $\text{ind}_\sigma(\sigma_{2n+2}) = 1$.

In this case, $\text{chpath}(\sigma)$ will be a Dyck path. This case can only happen if $\text{ind}_\tau(\tau_i) \leq 1$ for

all $1 \leq i \leq 2n + 1$.

Subcase 2B. $\text{ind}_\sigma(\sigma_{2n+2}) \geq 2$.

In this case, $\text{chpath}(\sigma)$ will not be a Dyck path. This case can only happen if there is an s such that $\text{ind}_\tau(\tau_s) = \text{ind}_\sigma(\tau_i + s) \geq 2$. But then since $\text{ind}_\sigma(\sigma_{2n}) = 1$, it must be the case that $\sigma_{2n} < \tau_s + 1 < 2n + 3$. Hence both $\langle \sigma_{2n+1}, 2n + 3 \rangle$ and $\langle \sigma_{2n}, 2n + 3 \rangle$ are nontrivial occurrences of μ in C so that $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

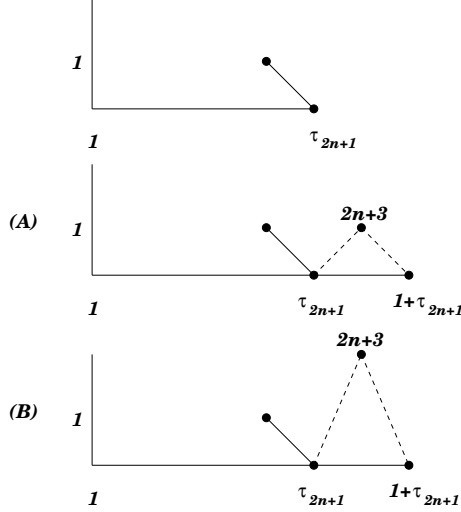


FIGURE 10. The charge path of $D^{[\tau_{2n+1}]}$ in Case 2.

Case 3. $j = \tau_i$ where $\text{ind}_\tau(\tau_{i-1}) > \text{ind}_\tau(\tau_i) > \text{ind}_\tau(\tau_{i+1})$.

In this case, we will have $\sigma_{i-1} = 1 + \tau_{i-1}$, $\sigma_i = \tau_i$, $\sigma_{i+1} = 2n + 3$, $\sigma_{i+2} = 1 + \tau_i$, and $\sigma_{i+3} = \tau_{i+1}$. There are now two cases.

Case 3A. $\tau_{i-1} < 2n + 1$.

In this case, we will have the situation pictured in Figure 11. That is, since $\sigma_{i-1} = 1 + \tau_{i-1} < 2n + 2$, it must be the case the either $2n + 2$ is to the left of σ_{i-1} in which case $\text{ind}_\sigma(2n + 2) > k + 1$ or $2n + 2$ is the right of σ_{i+3} in which case $\text{ind}_\sigma(2n + 2) \geq k + 1$ and $2n + 3$ is to the left of $2n + 2$ in σ . In either case, it must be that $\text{ind}_\sigma(2n + 3) > k + 1$. It is then easy to see that both $\langle \sigma_{i-1}, 2n + 3 \rangle$ and $\langle \sigma_i, 2n + 3 \rangle$ are nontrivial occurrences of μ in C so that $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

Case 3B. $\tau_{i-1} = 2n + 1$.

In this case, we will have the situation pictured in Figure 12. In this situation, the charge path of $D^{[\tau_i]}$ will be a Dyck path.

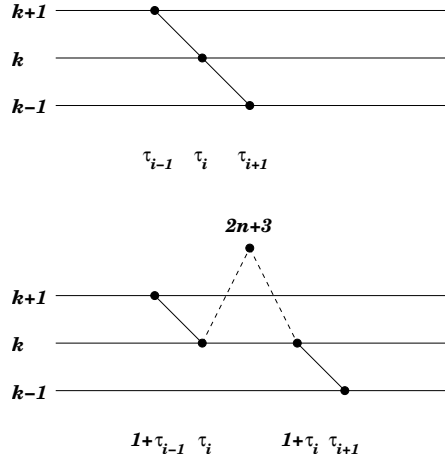


FIGURE 11. The charge path of $D^{[\tau_i]}$ in Case 3A.

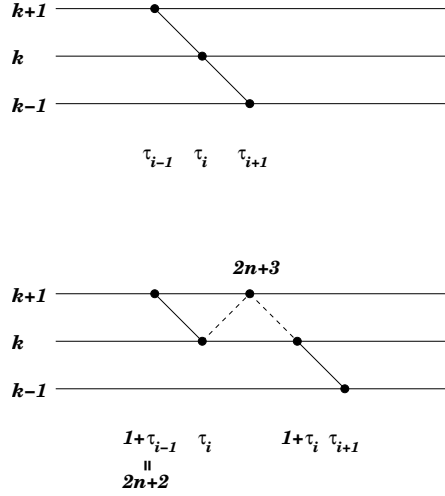


FIGURE 12. The charge path of $D^{[\tau_i]}$ in Case 3B.

Case 4. $j = \tau_i$ where $\text{ind}_\tau(\tau_{i-1}) < \text{ind}_\tau(\tau_i) > \text{ind}_\tau(\tau_{i+1})$.

In this case, we will have $\sigma_{i-1} = \tau_{i-1}$, $\sigma_i = \tau_i$, $\sigma_{i+1} = 2n+3$, $\sigma_{i+2} = 1+\tau_i$, and $\sigma_{i+3} = \tau_{i+1}$. There are now two subcases.

Subcase 4A. $\text{ind}_\sigma(2n+3) > k+1$.

In this case, we have the situation pictured in Figure 13. It will always be the case that $\langle \sigma_i, 2n+3 \rangle$ is a nontrivial occurrence of μ in C . We then have two more subcases.

Subcase 4A1. There is an $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

In this case, let s_i be the largest s such that $s < i$ and $\text{ind}_\sigma(\sigma_s) = k + 1$. Because P_D is a Dyck path, we can argue as in Case 1A that it must be the case that for all $s_i < t < i$, $\text{ind}_\sigma(\sigma_{s_t}) \leq k$. But this means that $\sigma_t < \sigma_i$ for all $s_i < t < \sigma_i$ and hence, $\langle \sigma_{s_i}, 2n + 3 \rangle$ is also a nontrivial occurrence of μ in C . Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

Subcase 4A2. There is no $s < i$ such that $\text{ind}_\sigma(\sigma_s) = k + 1$.

In this case, let s_i be the largest s such that $i < s$ and $\text{ind}_\sigma(\sigma_s) = k$. Note that s_i exists since $\text{ind}_\sigma(\sigma_{i+2}) = k$. But this means that for all $t < i$, $\text{ind}_\sigma(\sigma_t) \leq k$ and, hence, $\sigma_t < \sigma_i$. Moreover, for all $r > s_i$, $\text{ind}_\sigma(\sigma_r) < k - 1$ and hence, $\sigma_r < \sigma_i$. But it then follows that since $\sigma_{s_i} > \sigma_i$, $\langle \sigma_{s_i}, 2n + 3 \rangle$ is also a nontrivial occurrence of μ in C . Thus, $NT_\mu(C) \geq NT_\mu(D) + 2 = n + 2$.

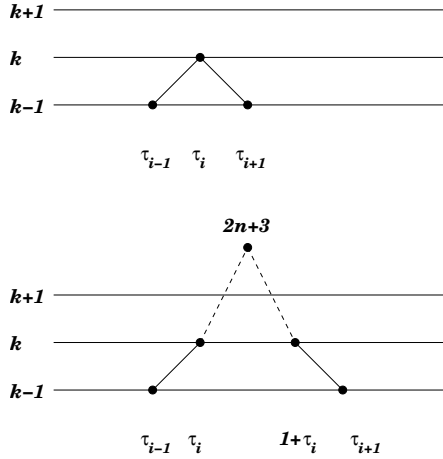


FIGURE 13. The charge path of $D^{[\tau_i]}$ in Case 4A.

Subcase 4B. $\text{ind}_\sigma(2n + 3) = k + 1$.

In this case, we have the situation pictured in Figure 14 and, hence, $\text{chpath}(\sigma)$ will be a Dyck path.

Thus, we have shown that if $D \in \mathcal{IC}_{2n+1}$ is such that $NT_\mu(D) = n$ and $D^{[i]} = C$, then either $\text{chpath}(\sigma)$ is a Dyck path or $NT_\mu(D^{[i]}) \geq n + 2$. Hence, if $NT_\mu(C) = n + 1$, then $\text{chpath}(\sigma)$ is a Dyck path. \square

Theorems 8 and 9 yield the following corollary.

Corollary 10. $NTI_{2n+1,\mu}(q)|_{q^n} = C_n$ where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n th Catalan number.

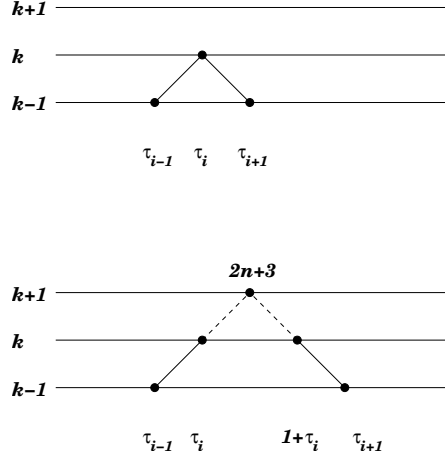


FIGURE 14. The charge path of $D^{[\tau_i]}$ in Case 4B.

4. PLOTS AND INTEGER PARTITIONS

In this section we will show that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ and $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ are equal to the number of partitions of k for sufficiently large n by plotting the *non- μ -matches* on a grid and mapping them to integer partitions.

Given an n -cycle $C \in \mathcal{C}_n$, we say that a pair $\langle i, j \rangle$ with $i < j$ is a non- μ -match of C if $\langle i, j \rangle$ is not an occurrence of μ in C . In other words, $\langle i, j \rangle$ is a non- μ -match if there exists an integer x such that $i < x < j$ and x is cyclically between i and j in C . Furthermore, let $\mathcal{NM}_\mu(C)$ be the set of non- μ -matches in C and let $NM_\mu(C) = |\mathcal{NM}_\mu(C)|$. Let

$$(10) \quad NM_{n,\mu}(q) = \sum_{C \in \mathcal{C}_n} q^{NM_\mu(C)}.$$

Note that for any $C \in \mathcal{C}_n$, $NM_\mu(C) + NT_\mu(C) = \binom{n-1}{2}$ so that $NM_{n,\mu}(q)|_{q^k} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$. Table 3 shows the polynomials $NM_{n,\mu}(q)$ for $1 \leq n \leq 10$. Our data suggests the following theorem.

Theorem 11. *For $k < n - 2$, $NM_{n,\mu}(q)|_{q^k} = a(k)$ where $a(k)$ is the number of integer partitions of k .*

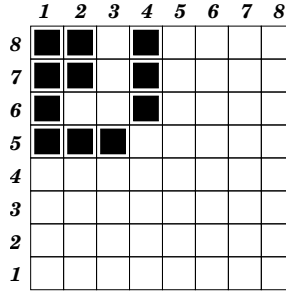
To prove Theorem 11, we need to define what we call the plot of non- μ -matches in $C \in \mathcal{C}_n$. We consider an $n \times n$ grid where the rows are labeled with $1, 2, \dots, n$, reading from bottom to top, and the columns are labeled with $1, 2, \dots, n$, reading from left to right. The cell (i, j) is the cell which lies in the i th row and j th column. Then, given a set S of ordered pairs $\langle i, j \rangle$ with $1 \leq i \leq n$, we let $plot_n(S)$ denote the diagram that arises by shading a cell (j, i) on the $n \times n$ grid if and only if $\langle i, j \rangle \in S$. Given an n -cycle $C \in \mathcal{C}_n$, we let $NMplot(C) = plot_n(\mathcal{NM}_\mu(C))$. For example, if $C = (1, 4, 5, 3, 8, 7, 2, 6)$, then

$$\mathcal{NM}_\mu(C) = \{\langle 1, 5 \rangle, \langle 1, 6 \rangle, \langle 1, 7 \rangle, \langle 1, 8 \rangle, \langle 2, 5 \rangle, \langle 2, 7 \rangle, \langle 2, 8 \rangle, \langle 3, 5 \rangle, \langle 4, 6 \rangle, \langle 4, 7 \rangle, \langle 4, 8 \rangle\}$$

TABLE 3. Polynomials $NM_{n,\mu}(q)$

$n = 1$	1
2	1
3	$1 + q$
4	$1 + q + 3q^2 + q^3$
5	$1 + q + 2q^2 + 7q^3 + 6q^4 + 6q^5 + q^6$
6	$1 + q + 2q^2 + 3q^3 + 13q^4 + 15q^5 + 23q^6 + 31q^7 + 20q^8 + 10q^9 + q^{10}$
7	$1 + q + 2q^2 + 3q^3 + 5q^4 + 19q^5 + 25q^6 + 46q^7 + 66q^8 + 119q^9 + 126q^{10} + 135q^{11} + 106q^{12} + 50q^{13} + 15q^{14} + q^{15}$
8	$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 27q^6 + 33q^7 + 65q^8 + 101q^9 + 174q^{10} + 299q^{11} + 418q^{12} + 603q^{13} + 726q^{14} + 850q^{15} + 736q^{16} + 561q^{17} + 301q^{18} + 105q^{19} + 21q^{20} + q^{21}$
9	$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 35q^7 + 44q^8 + 80q^9 + 126q^{10} + 217q^{11} + 338q^{12} + 646q^{13} + 888q^{14} + 1461q^{15} + 2116q^{16} + 3093q^{17} + 4055q^{18} + 5007q^{19} + 5675q^{20} + 5541q^{21} + 4820q^{22} + 3311q^{23} + 1870q^{24} + 742q^{25} + 196q^{26} + 28q^{27} + q^{28}$
10	$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 46q^8 + 56q^9 + 100q^{10} + 148q^{11} + 251q^{12} + 374q^{13} + 640q^{14} + 1098q^{15} + 1640q^{16} + 2568q^{17} + 3971q^{18} + 6179q^{19} + 9137q^{20} + 13710q^{21} + 18551q^{22} + 25689q^{23} + 32781q^{24} + 40008q^{25} + 44119q^{26} + 45433q^{27} + 41488q^{28} + 32864q^{29} + 22392q^{30} + 12253q^{31} + 5328q^{32} + 1638q^{33} + 336q^{34} + 36q^{35} + q^{36}$

and $NMplot(C)$ is pictured in Figure 15.

FIGURE 15. $NMplot((1, 4, 5, 3, 8, 7, 2, 6))$.

First, we observe that if $C \in \mathcal{C}_n$, then $\mathcal{NM}_\mu(C)$ completely determines C . That is, we have the following theorem.

Theorem 12. *If S is a set of ordered pairs such that $\mathcal{NM}_\mu(C) = S$ for some n -cycle $C \in \mathcal{C}_n$, then C is the only n -cycle such that $\mathcal{NM}_\mu(C) = S$.*

Proof. Our proof proceeds by induction on n . Clearly, the theorem holds for $n = 1$ and $n = 2$. Now, assume that the theorem holds for n . Let $C' \in \mathcal{C}_{n+1}$ and $S = \mathcal{NM}_\mu(C')$. Let $C = (1, c_2, \dots, c_n)$ be the n -cycle that is obtained from C' by removing $n + 1$. Then it is easy to see that $\mathcal{NM}_\mu(C) = S - \{\langle i, n + 1 \rangle : \langle i, n + 1 \rangle \in S\}$. Hence, C is the unique n -cycle in \mathcal{C}_n such that $\mathcal{NM}_\mu(C) = S - \{\langle i, n + 1 \rangle : \langle i, n + 1 \rangle \in S\}$. Let $C^{(i)}$ denote the cycle that results by inserting $n + 1$ immediately after i in C . Since C is unique, it follows that C' must equal $C^{(i)}$ for some $1 \leq i \leq n$. However, it is easy to see that if $1 \leq i < n$, then $\langle j, n + 1 \rangle \in \mathcal{NM}_\mu(C^{(i)})$ for $1 \leq j < i$ and $\langle i, n + 1 \rangle \notin \mathcal{NM}_\mu(C^{(i)})$. If $i = n$, then $\langle j, n + 1 \rangle \in \mathcal{NM}_\mu(C^{(i)})$ for $1 \leq j < n$. Thus, it follows that $\mathcal{NM}_\mu(C^{(1)}), \dots, \mathcal{NM}_\mu(C^{(n)})$ are pairwise distinct. Hence, there is exactly one cycle C' such that $\mathcal{NM}_\mu(C') = S$. \square

An integer sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a *partition of n* if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\sum_{i=1}^\ell \lambda_i = n$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a partition of n , we will write $\lambda \vdash n$, and let $|\lambda| = n$ denote the size of λ and $\ell(\lambda) = \ell$ denote the length of λ . The *Ferrers diagram* of a partition λ on the $m \times m$ grid, denoted $FD_m(\lambda)$, is the diagram that results by shading the squares $(1, m), (2, m), \dots, (\lambda_1, m)$ in the top row, shading the squares $(1, m - 1), (2, m - 1), \dots, (\lambda_2, m - 1)$ in the second row from the top, and, in general, shading the squares $(1, m - i + 1), (2, m - i + 1), \dots, (\lambda_i, m - i + 1)$ in the i th row from the top. For example, the Ferrers diagram of $\lambda = (5, 4, 3, 1, 1)$ on a 8×8 grid is pictured in Figure 16.

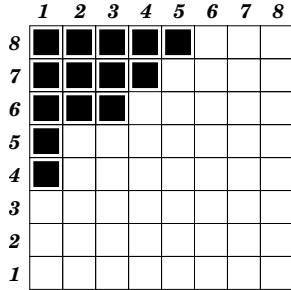


FIGURE 16. $FD_8(5, 4, 3, 1, 1)$

It is easy to see that the following three properties characterize the Ferrers diagrams of partitions λ in an $n \times n$ grid.

- (1) If λ is not empty, then the square $(n, 1)$ is shaded in $FD_m(\lambda)$.
- (2) If $x \neq 1$ and the square (y, x) is shaded in $FD_m(\lambda)$, then the square $(y, x - 1)$ is shaded in $FD_m(\lambda)$.
- (3) If $y \neq n$ and the square (y, x) is shaded in $FD_m(\lambda)$, then the square $(y + 1, x)$ is shaded in $FD_m(\lambda)$.

This means that if S is a set of pairs (i, j) such that $plot_n(S)$ is a Ferrers diagram, then it must be the case that

- (1) If S is not empty, then the square $(1, n) \in S$.
- (2) If $x \neq 1$ and the square $(x, y) \in S$, then $(x - 1, y) \in S$.

(3) If $y \neq n$ and the square $(x, y) \in S$, then $(x, y + 1) \in S$.

Our next lemma and corollary will show that if $C \in \mathcal{C}_n$ and $NM_\mu(C) < n - 2$, then $\mathcal{NM}_\mu(C)$ has the same three properties.

Lemma 13. *Assume that $C \in \mathcal{C}_n$.*

- (1) *If $\langle 1, n \rangle \notin \mathcal{NM}_\mu(C)$ and $NM_\mu(C) \neq 0$, then $NM_\mu(C) \geq n - 2$.*
- (2) *If $x \neq 1$, $\langle x, y \rangle \in \mathcal{NM}_\mu(C)$, and $\langle x - 1, y \rangle \notin \mathcal{NM}_\mu(C)$, then $NM_\mu(C) \geq n - 2$.*
- (3) *If $y \neq n$, $\langle x, y \rangle \in \mathcal{NM}_\mu(C)$, and $\langle x, y + 1 \rangle \notin \mathcal{NM}_\mu(C)$, then $NM_\mu(C) \geq n - 2$.*

Proof. Let $C = (1, c_1, \dots, c_{n-1}) \in \mathcal{C}_n$ be an n -cycle where $n \geq 4$.

For part (1), suppose that $\langle 1, n \rangle \notin \mathcal{NM}_\mu(C)$ and $NM_\mu(C) \neq 0$. Then, there are no integers that are cyclically in between 1 and n in C so that $c_1 = n$. Since $NM_\mu(C) \neq 0$, $C \neq (1, n, n - 1, \dots, 3, 2)$. Thus, C must be of the form

$$C = (1, n, n - 1, \dots, n - a, b, \underbrace{\dots}_A b + 1, \underbrace{\dots}_B)$$

where $b + 1 < n - a$. That is, the sequence $c_1 > c_2 > \dots > c_{a+1}$ consists of the decreasing interval from n to $n - a$ and then $b \leq n - a - 2$. It follows that the pairs $\langle 1, b + 1 \rangle, \langle b, n \rangle, \dots, \langle b, n - a \rangle$ are all *non- μ -matches* in C which accounts for $a + 2$ non- μ -matches in C . Let A be the set of integers cyclically between b and $b + 1$ in C and let B be the set of integers cyclically between $b + 1$ and 1 in C . Note that

- (1) if $x \in A$ and $x < b$, then $\langle x, n \rangle \in \mathcal{NM}_\mu(C)$,
- (2) if $x \in A$ and $x > b$, then $\langle 1, x \rangle \in \mathcal{NM}_\mu(C)$,
- (3) if $x \in B$ and $x < b$, then $\langle x, b + 1 \rangle \in \mathcal{NM}_\mu(C)$, and
- (4) if $x \in B$ and $x > b$, then $\langle 1, x \rangle \in \mathcal{NM}_\mu(C)$.

It follows that each element in A and B is part of at least one non- μ -match in C so that we know that $NM_\mu(C) \geq a + 2 + |A| + |B|$. Thus, since $|A| + |B| = n - a - 4$, we have $NM_\mu(C) \geq n - 2$.

For part (2), suppose that $x \neq 1$, $\langle x, y \rangle \in \mathcal{NM}_\mu(C)$, and $\langle x - 1, y \rangle \notin \mathcal{NM}_\mu(C)$. Then x cannot be cyclically between $x - 1$ and y in C . Also, there exists an integer z with $x < z < y$ such that z is cyclically between x and y in C , but z is not cyclically between $x - 1$ and y in C . It follows that C is of the form

$$C = (\underbrace{1, \dots, x}_{A_3}, \underbrace{\dots, z, \dots}_{A_1}, x - 1, \underbrace{\dots}_{A_2}, y, \underbrace{\dots}_{A_3})$$

where A_1 is the set of integers cyclically between x and $x - 1$ in C that are not equal to z , A_2 is the set of integers cyclically between $x - 1$ and y in C , and A_3 is the set of integers between y and x in C . Note that $\langle x, y \rangle$ and $\langle x - 1, z \rangle$ are in $\mathcal{NM}_\mu(C)$. Moreover,

- (1) if $a \in A_1$ and $a < x - 1$, then $\langle a, y \rangle \in \mathcal{NM}_\mu(C)$,
- (2) if $a \in A_1$ and $a > x$, then $\langle x - 1, a \rangle \in \mathcal{NM}_\mu(C)$,
- (3) if $a \in A_2$ and $a < x - 1$, then $\langle a, z \rangle \in \mathcal{NM}_\mu(C)$,
- (4) there are no $a \in A_2$ with $x - 1 < a < y$ since $\langle x - 1, y \rangle \notin \mathcal{NM}_\mu(C)$,

- (5) if $a \in A_2$ and $a > y$, then $\langle 1, a \rangle \in \mathcal{NM}_\mu(C)$,
- (6) if $a \in A_3$ and $a < z$, then $\langle a, y \rangle \in \mathcal{NM}_\mu(C)$, and
- (7) if $a \in A_3$ and $a > z$, then $\langle x, a \rangle \in \mathcal{NM}_\mu(C)$.

Therefore, any integer $a \in A_1 \cup A_2 \cup A_3$ is part of a distinct non- μ -match in C . Since $|A_1 \cup A_2 \cup A_3| = n - 4$, it follows that $NM_\mu(C) \geq 2 + n - 4 = n - 2$.

For part (3), suppose that $y \neq n$, $\langle x, y \rangle \in \mathcal{NM}_\mu(C)$, and $\langle x, y + 1 \rangle \notin \mathcal{NM}_\mu(C)$. Then y cannot be cyclically between x and $y + 1$ in C . Also, there exists an integer z with $x < z < y$ such that z is cyclically between x and y in C , but z is not cyclically between x and $y + 1$ in C . Thus, C must be of the form

$$C = (\underbrace{1, \dots, x}_{B_3}, \underbrace{\dots, y + 1}_{B_1}, \underbrace{\dots, z, \dots, y}_{B_2}, \underbrace{\dots}_{B_3})$$

where B_1 is the set of integers cyclically between x and $y + 1$ in C , B_2 is the set of integers cyclically between $y + 1$ and y in C that are not equal to z , and B_3 is the set of integers cyclically between y and x in C . Note that $\langle x, y \rangle$ and $\langle z, y + 1 \rangle$ are in $\mathcal{NM}_\mu(C)$. Moreover,

- (1) if $b \in B_1$ and $b < x$, then $\langle b, y \rangle \in \mathcal{NM}_\mu(C)$,
- (2) there is no $b \in B_1$ with $x < b < y + 1$ since $\langle x, y + 1 \rangle \notin \mathcal{NM}_\mu(C)$,
- (3) if $b \in B_1$ and $b > y + 1$, then $\langle z, b \rangle \in \mathcal{NM}_\mu(C)$,
- (4) if $b \in B_2$ and $b < y$, then $\langle b, y + 1 \rangle \in \mathcal{NM}_\mu(C)$,
- (5) if $b \in B_2$ and $b > y + 1$, then $\langle x, b \rangle \in \mathcal{NM}_\mu(C)$,
- (6) if $b \in B_3$ and $b < z$, then $\langle b, y \rangle \in \mathcal{NM}_\mu(C)$, and
- (7) if $b \in B_3$ and $b > z$ then $\langle x, b \rangle \in \mathcal{NM}_\mu(C)$.

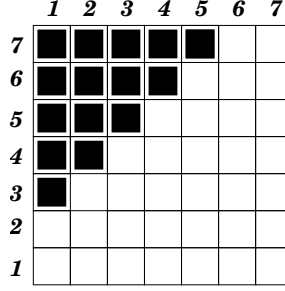
Therefore, any integer $b \in B_1 \cup B_2 \cup B_3$ is part of a distinct non- μ -match in C . Since $|B_1 \cup B_2 \cup B_3| = n - 4$, it follows that $NM_\mu(C) \geq 2 + n - 4 = n - 2$. \square

Corollary 14. *If $NM_\mu(C) < n - 2$, then $NMplot(C)$ is a Ferrers diagram.*

Proof. This corollary follows directly from Lemma 13. That is, suppose that $C \in \mathcal{C}_n$ and $NM_\mu(C) < n - 2$. First, if $NM_\mu(C) = 0$, then $C = (1, n, n - 1, \dots, 2)$ in which case $FD_n(C)$ has no shaded squares which correspond to the empty partition. Thus, assume that $1 \leq NM_\mu(C) < n - 2$. Then, it follows from part (1) of Lemma 13 that $\langle 1, n \rangle \in \mathcal{NM}_\mu(C)$. Next it follows from part (2) of Lemma 13 that if $x \neq 1$ and $\langle x, y \rangle \in \mathcal{NM}_\mu(C)$, then $\langle x - 1, y \rangle \in \mathcal{NM}_\mu(C)$. Finally, it follows from part (3) of Lemma 13 that if $y \neq n$ and $\langle x, y \rangle \in \mathcal{NM}_\mu(C)$, then $\langle x, y + 1 \rangle \in \mathcal{NM}_\mu(C)$. Hence, the shaded cells $NMplot(C)$ must be a Ferrers diagram of a partition λ of $NM_\mu(C)$. \square

If $n \geq 3$, we let T_n be the plot of the Ferrers diagram in the $n \times n$ grid corresponding to the partition $\lambda = (n - 2, n - 3, \dots, 1)$. Thus, for example, T_7 is pictured in Figure 17. One can see that the sets that T_n have the property that $|\{\lambda : \lambda \text{ is a partition and } FD_n(\lambda) \subseteq T_n\}| = C_n$ where C_n is the n th Catalan number since the lower boundaries of the plots of $FD(\lambda)$ for such λ correspond to Dyck paths.

Our next theorem will show that for any $n \geq 3$ and any partition $\lambda \subseteq T_n$, we can construct an n -cycle $C \in \mathcal{C}_n$ such that $NMplot(C) = FD_n(\lambda)$.

FIGURE 17. The Ferrers diagram T_7 on the 7×7 grid.

Theorem 15. *Suppose that $n \geq 3$ and λ is a partition such that $\lambda \subseteq T_n$. Then, there is an n -cycle $C \in \mathcal{C}_n$ such that $NMplot(C) = FD_n(\lambda)$.*

Proof. We proceed by induction on n . For $n = 3$, it is easy to see that if $C^{(1)} = (1, 3, 2)$, the $NMplot(C^{(1)})$ is empty and if $C^{(2)} = (1, 2, 3)$, then $NMplot(C^{(2)}) = T_3$. Note that $C^{(1)}$ and $C^{(2)}$ have the property that if the largest part of the corresponding partition is of size i , then 3 immediately follows $i + 1$ in the cycle. Thus, our theorem holds for $n = 3$.

Now, suppose that $n > 3$ and $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition which is contained in T_n . It follows that $\lambda^- = (\lambda_2, \dots, \lambda_k)$ is contained in T_{n-1} . Assume by induction that there is an $(n-1)$ -cycle C' such that $NMplot(C') = FD_{n-1}(\lambda^-)$ and $n-1$ immediately follows $\lambda_2 + 1$ in C' . Thus, in C' , the pairs $\langle \lambda_2 + 1, n-1 \rangle, \langle \lambda_2 + 2, n-1 \rangle, \dots, \langle n-3, n-1 \rangle$ must match μ in C' . This means that if $\lambda_2 + 1 \leq n-3$, then $n-3$ must lie between $n-2$ and $n-1$ in C' . Next if $\lambda_2 + 1 \leq n-4$, then $n-4$ must lie between $n-3$ and $n-1$ in C' . In general, if $\lambda_2 + 1 \leq n-k$, then $n-k$ must lie between $n-k+1$ and $n-1$ in C' . Thus, C' must be of the following form

$$C' = (\underbrace{\dots}_{A_{n-1}} n-2 \underbrace{\dots}_{A_{n-2}} n-3 \underbrace{\dots}_{A_{n-3}} n-4 \underbrace{\dots}_{A_{n-4}} \dots (\lambda_2 + 2) \underbrace{\dots}_{A_{\lambda_2+2}} (\lambda_2 + 1), (n-1) \underbrace{\dots}_{A_{n-1}}).$$

where A_i is the set of elements cyclically between $n-i$ and $n-i-1$ in C' for $i = 1, \dots, n - \lambda_2 - 2$. Now let C be the cycle that results from C' by inserting n immediately after $\lambda_1 + 1$ in C' . Inserting n into C' does not effect on whether pairs $\langle i, j \rangle$ with $1 \leq i < j \leq n-1$ are μ -matches in C . That is, for such pairs $\langle i, j \rangle \in \mathcal{NM}_\mu(C)$ if and only if $\langle i, j \rangle \in \mathcal{NM}_\mu(C')$. Thus, the diagram of $NMplot(C')$ and $NMplot(C)$ are the same up to row $n-1$. Now, in row n , we know that the cells $(n, 1), \dots, (n, \lambda_1)$ are shaded since the fact that n immediately follows $\lambda_1 + 1$ in C means that $\langle i, n \rangle \in \mathcal{NM}_\mu(C)$ for $i = 1, \dots, \lambda_1$. However, it is easy to see from the form of C' above that $\langle n-2, n \rangle, \langle n-3, n \rangle, \dots, \langle \lambda_1 + 1, n \rangle$ are μ -matches in C since $n-2, n-3, \dots, \lambda_1 + 1$ appear in decreasing order as we traverse clockwise around the cycle C . Thus, $NMplot(C) = FD_n(\lambda)$. \square

Note the proof of Theorem 15 gives a simple algorithm to construct an n -cycle C_λ such that $NMplot(C_\lambda) = FD_n(\lambda)$ for any $\lambda \subseteq T_n$. For example, suppose that $n = 12$ and $\lambda = (3, 3, 2, 1)$ so that the Ferrers diagram in the 12×12 grid is pictured in Figure 18.

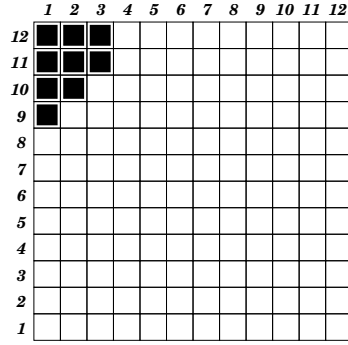


FIGURE 18. $sh_{12}(3, 3, 2, 1)$.

Since there are no non- μ -matches in the first eight rows, we must start with the cycle $C_8 = (1, 8, 7, 6, 5, 4, 3, 2)$. Then, our proof of Theorem 15 tells us that we should build up the cycle structure by first creating a cycle $C_9 \in \mathcal{C}_9$ by inserting 9 immediately after 2 in C_8 , since the number of non- μ -matches in row 9 is 1. Then we create a cycle $C_{10} \in \mathcal{C}_{10}$ by inserting 10 immediately after 3 in C_9 since the number of non- μ -matches in row 10 is 2. Then, we create a cycle $C_{11} \in \mathcal{C}_{11}$ by inserting 11 immediately after 4 in C_{10} since the number of non- μ -matches in row 11 is 3. Finally, we create a cycle $C_{12} = C_\lambda \in \mathcal{C}_{12}$ by inserting 12 immediately after 4 in C_{11} since the number of non- μ -matches in row 12 is 3. Thus,

$$\begin{aligned} C_9 &= (1, 8, 7, 6, 5, 4, 3, 2, 9), \\ C_{10} &= (1, 8, 7, 6, 5, 4, 3, 10, 2, 9), \\ C_{11} &= (1, 8, 7, 6, 5, 4, 11, 3, 10, 2, 9), \text{ and} \\ C_{12} &= (1, 8, 7, 6, 5, 4, 12, 11, 3, 10, 2, 9). \end{aligned}$$

Then we have that

$$\begin{aligned} \mathcal{NM}_\mu((1, 8, 7, 6, 5, 4, 12, 11, 3, 10, 2, 9)) = &\langle 1, 12 \rangle, \langle 2, 12 \rangle, \langle 3, 12 \rangle, \\ &\langle 1, 11 \rangle, \langle 2, 11 \rangle, \langle 3, 11 \rangle, \\ &\langle 1, 10 \rangle, \langle 2, 10 \rangle, \\ &\langle 1, 9 \rangle. \end{aligned}$$

Now we can prove Theorem 11.

Proof. Suppose that $k \leq n - 2$. Let

$$\begin{aligned} FD_n(k) &= \{C \in \mathcal{C}_n : NMplot(C) = FD_n(\lambda) \text{ for some } \lambda \vdash k\} \text{ and} \\ NMP_n(k) &= \{C \in \mathcal{C}_n : NM_\mu(C) = k\}. \end{aligned}$$

Theorem 14 shows that $NMp_n(k) \subseteq FD_n(k)$ and Theorem 15 shows that $FD_n(k) \subseteq NMp_n(k)$. Thus, $NM_{n,\mu}(q)|_{q^k} = |NMp_n(k)| = |FD_n(k)|$. Hence, $NM_{n,\mu}(q)|_{q^k}$ equals the number of partitions of k . \square

Now, we will show that for $k < n - 2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$, where $a(k)$ is the number of partitions of k . First we shall show that if $C \in \mathcal{C}_n$ has fewer than $n - 2$ non-matches, then C must be incontractible.

Lemma 16. *If $C \in \mathcal{C}_n$ has fewer than $n - 2$ non-matches, then C must be incontractible.*

Proof. If $i + 1$ immediately follows i in an n -cycle $C \in \mathcal{C}_n$, then, clearly, $\langle j, i + 1 \rangle$ are non- μ -matches for $1 \leq j < i$ and $\langle i, k \rangle$ is a non- μ -match for $i + 2 \leq k \leq n$. This gives $n - 2$ non- μ -matches. \square

Corollary 17. *For $k < n - 2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$ where $a(k)$ is the number of partitions of k .*

Proof. By definition, $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NM_{n,\mu}(q)|_{q^k}$. Thus, by Theorem 11, for $k < n - 2$, $NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$.

By Lemma 16, we have that if a cycle C has k non- μ -matches where $k < n - 2$, then C is incontractible. It follows that if a cycle has $\binom{n-1}{2} - k$ non-trivial μ -matches, then it is incontractible. Thus, for $k < n - 2$, $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = NT_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$. And so $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}} = a(k)$ for $k < n - 2$. \square

5. CONCLUSIONS AND DIRECTION FOR FURTHER RESEARCH

In this paper, we studied the polynomials $NT_{n,\mu}(q) = \sum_{C \in \mathcal{C}_n} q^{NT_\mu(C)}$ and $NTI_{n,\mu}(q) = \sum_{C \in \mathcal{IC}_n} q^{NT_\mu(C)}$. We showed that $NTI_{n,\mu}(1)$ is the number of derangements of S_{n-1} . Thus, the polynomial $NTI_{n,\mu}(q)$ is a q -analogue of the derangement number D_{n-1} . There are several q -analogues of the derangement numbers that have been studied in the literature, see the papers by Garsia and Remmel [3] and Wachs[9]. Our q -analogue of D_{n-1} is different from either the Garsia-Remmel q -analogue or the Wachs q -analogue of the derangement numbers. Moreover, we proved that

$$NT_{n,\mu}(q)|_{q^k} = \sum_{s=1}^{2k+1} \sum_{\substack{A \in \mathcal{IC}_s, \\ NT_\mu(A)=k}} \binom{n-1}{s-1}$$

so that the coefficients of the polynomial $NT_{n,\mu}(q)$ can be expressed in terms of the coefficients of the polynomials $NTI_{j,\mu}(q)$. We also showed that $NTI_{n,\mu}(q)|_{q^{\binom{n-1}{2}-k}}$ equals the number of partition of k for $k < n - 2$.

The main open question is to find some sort of recursion or generating function that would allow us to compute $NTI_{n,\mu}(q)$. Note that several of the sequences $(NT_{n,\mu}(q)|_{q^k})_{n \geq 2}$ appear in the OEIS [8]. For example, we showed that $NT_{n,\mu}(q)|_{q^2} = \binom{n-1}{3} + 2\binom{n-1}{4}$ so

that the sequence $(NT_{n,\mu}(q)|_{q^2})_{n \geq 4}$ starts out 1, 6, 20, 50, 105, 196, 336, 540, 825, \dots . The n th term of this sequence has several combinatorial interpretations including being the number of $\sigma \in S_n$ which are 132-avoiding and have exactly two descents, the number of Dyck paths on length $2n + 2$ with $n - 1$ peaks, and the number of squares with corners on the $n \times n$ grid. Thus, it would be interesting to find bijections from these objects to our $(n + 4)$ -cycles $C \in \mathcal{C}_n$ such that $NT_\mu(C) = 2$. We also showed that $NT_{n,\mu}(q)|_{q^3} = \binom{n-1}{3} + 3\binom{n-1}{4} + 6\binom{n-1}{5} + 5\binom{n-1}{6}$ so that the sequence $(NT_{n,\mu}(q)|_{q^3})_{n \geq 4}$ starts out

$$1, 7, 31, 102, 273, 630, 1302, 2472, 4389 \dots$$

This sequences does not appear in the OEIS. Similarly, the sequence $(NT_{n,\mu}(q)|_{q^4})_{n \geq 5}$ starts out 2, 23, 135, 561, 1870, 5328, 13476, \dots and it does not appear in the OEIS.

Finally, it would be interesting to characterize the charge graphs of those cycles $C \in \mathcal{IC}_{2n}$ for which $NM_\mu(C) = n$. One can see from our tables that sequence $(NTI_{2n,\mu}(q)|_{q^2})_{n \geq 2}$ starts out 1, 6, 29, 130, \dots . Moreover, we have computed the $NTI_{12,\mu}(q)|_{q^6} = 562$. This suggests that this sequence is sequence A008549 in the OEIS. If so, this would mean that $NTI_{2n,\mu}(q)|_{q^n} = \sum_{i=0}^{n-2} \binom{2n-1}{i}$ for $n \geq 2$.

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