Pattern-avoiding alternating words

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Abstract. A word $w = w_1w_2\cdots w_n$ is alternating if either $w_1 < w_2 > w_3 < w_4 > \cdots$ (when the word is up-down) or $w_1 > w_2 < w_3 > w_4 < \cdots$ (when the word is down-up). In this paper, we initiate the study of (pattern-avoiding) alternating words. We enumerate up-down (equivalently, down-up) words via finding a bijection with order ideals of a certain poset. Further, we show that the number of 123-avoiding up-down words of even length is given by the Narayana numbers, which is also the case, shown by us bijectively, with 132-avoiding up-down words of even length. We also give formulas for enumerating all other cases of avoidance of a permutation pattern of length 3 on alternating words.

Keywords: alternating permutations, up-down permutations, down-up permutations, pattern-avoidance, Narayana numbers, Fibonacci numbers, order ideals, bijection

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1 Introduction

A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is called *up-down* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \pi_5 < \cdots$. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is called *down-up* if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \pi_5 > \cdots$. A famous result of André is saying that if E_n is the number of up-down (equivalently, down-up) permutations of $1, 2, \ldots, n$, then

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Some aspects of up-down and down-up permutations, also called *reverse alternating* and *alternating*, respectively, are surveyed in [9]. Slightly abusing these definitions, we refer to *alternating permutations* as the union of up-down and down-up permutations. This union is known as the set of *zigzag permutations*.

In this paper, we extend the study of alternating permutations to that of *alternating words*. These words, also called *zigzag words*, are the union of up-down and down-up words, which are defined in a similar way to the definition of up-down and down-up permutations, respectively. For example, 1214, 2413, 2424 and 3434 are examples of up-down words of length 4 over the alphabet $\{1, 2, 3, 4\}$.

Section 2 is dedicated to the enumeration of up-down words, which is equivalent to enumerating down-up words by applying the operation of *complement*. For a word $w = w_1w_2\cdots w_n$ over the alphabet $\{1, 2, \ldots, k\}$ its complement w^c is the word $c_1c_2\cdots c_n$, where for each $i = 1, 2, \ldots, n$, $c_i = k + 1 - w_i$. For example, the complement of the word 24265 over the alphabet $\{1, 2, \ldots, 6\}$ is 53512. Our enumeration in Section 2 is done by linking bijectively up-down words to order ideals of certain posets and using known results.

A (*permutation*) *pattern* is a permutation $\tau = \tau_1\tau_2\cdots\tau_k$. We say that a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ *contains an occurrence* of τ if there are $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic to τ. If π does not contain an occurrence of τ, we say that π *avoids* τ. For example, the permutation 315267 contains several occurrences of the pattern 123, for example, the subsequences 356 and 157, while this permutation avoids the pattern 321. Occurrences of a pattern in words are defined similarly as subsequences order-isomorphic to a given word called pattern (the only difference with permutation patterns is that word patterns can contain repetitive letters, which is not in the scope of this paper).

A comprehensive introduction to the theory of patterns in permutations and words can be found in [4]. In particular, Section 6.1.8 in [4] discusses known results on pattern-avoiding alternating permutations, and Section 7.1.6 discusses results on permutations avoiding patterns in a more general sense.

In this paper we initiate the study of pattern-avoiding alternating words. In Section 3 we enumerate up-down words over k-letter alphabet avoiding the pattern 123. In particular, we show that in the case of even length, the answer is given by the *Narayana numbers* counting, for example, *Dyck paths* with a specified number of peaks (see Theorem 3.2). Interestingly, the number of 132-avoiding words over k-letter alphabet of even length is also given by the Narayana numbers, which we establish bijectively in Section 4. In Section 5 we provide a (non-closed form) formula for the number of 132-avoiding words over k-letter alphabet of odd length. In Section 6 we show that the enumeration of 312-avoiding up-down words is equivalent to that of 123-avoiding up-down words. Further, a classification of all cases of avoiding a length 3 permutation pattern on up-down words is discussed in Section 7. Finally, some concluding remarks are given in Section 8.

In what follows, $[k] = \{1, 2, \ldots, k\}$.

2 Enumeration of up-down words

In this section, we consider the enumeration of up-down words. We shall show that this problem is the same as that of enumerating order ideals of a certain poset. Since up-down words are in one-to-one-correspondence with down-up words by using the complement operation, we consider only down-up words throughout this section.

Table 1 provides the number $N_{k,\ell}$ of down-up words of length ℓ over the alphabet $[k]$ for small values of k and ℓ indicating connections to the *Online Encyclopedia of Integer Sequences* (*OEIS*) [7].

$k \backslash \ell$	0	1	2	3	4	5	6	7	8	9	10	OEIS
2	1	2	1	1	1	1	1	1	1	1	1	trivial
3	1	3	3	5	8	13	21	34	55	89	144	A000045
4	1	4	6	14	31	70	157	353	793	1782	4004	A006356
5	1	5	10	30	85	246	707	2037	5864	16886	48620	A006357
6	1	6	15	55	190	671	2353	8272	29056	102091	358671	A006358
7	1	7	21	91	371	1547	6405	26585	110254	457379	1897214	A006359

Table 1: The number $N_{k,\ell}$ of down-up words on $[k]$ of length ℓ for small values of k and ℓ.

We assume the reader is familiar with the notion of a partially ordered set (poset) and some basic properties of posets; e.g. see [8]. A partially ordered set P is a set together with a binary relation denoted by $\leq_P$ that satisfies the properties of reflexivity, antisymmetry and transitivity. An order ideal of P is a subset I of P such that if $x \in I$ and $y \leq x$ then $y \in I$. We denote $J(P)$ the set of all order ideals of P.

Let $\mathbf{n}$ be the poset on $[n]$ with its usual order ($\mathbf{n}$ is a linearly ordered set). The *m-element zigzag poset*, denoted Z_m, is shown schematically in Figure 1. Note that the order $<_{Z_m}$ in Z_m is $1 < 2 > 3 < 4 > 5 < \cdots$. The definition of the order $\leq_{Z_m}$ is self-explanatory.

Figure 1: The zigzag poset Z_m.

The poset $Z_m \times \mathbf{n}$ is as shown in Figure 2. Elements of $Z_m \times \mathbf{n}$ are pairs (i, j), where $i \in Z_m$ and $j \in [n]$, and the order is defined as follows:

$$(i, j) \leq (k, \ell) \text{ if and only if } i \leq_{Z_m} k \text{ and } j \leq \ell.$$

It is known that, for $m \geq 2$, the size of $J(Z_m)$ equals to the Fibonacci number F_{m+2}, which is defined recursively as $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for any $n \geq 2$; see Stanley [8, Ch. 3 Ex. 23.a]. The enumeration of $J(Z_m \times \mathbf{n})$ was studied by Berman and Köhler [1]. The following theorem reveals their connection with the enumeration of

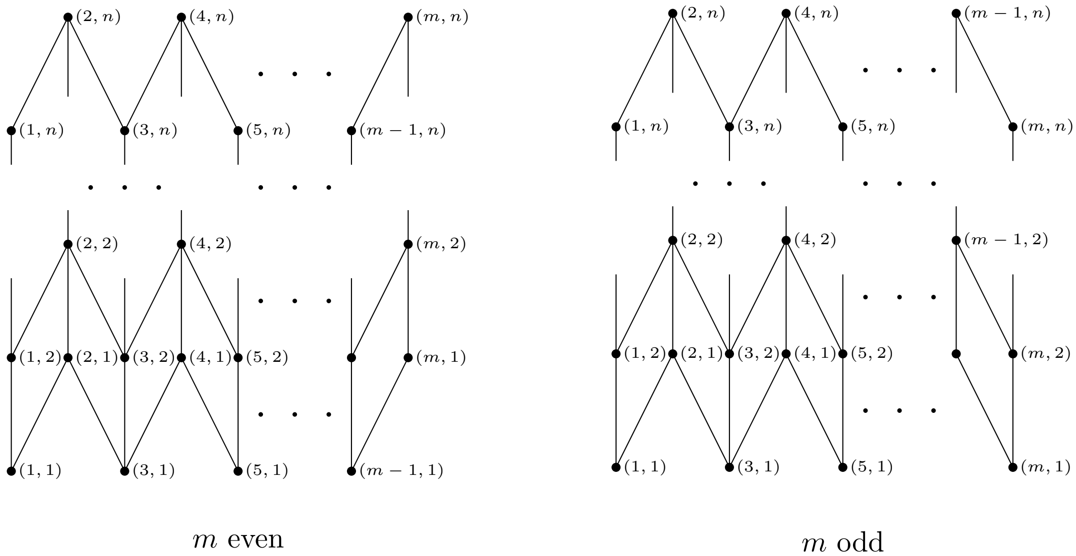

Figure 2: The poset $Z_m \times \mathbf{n}$.

alternating words. We shall give two proofs of it here, a bijective proof and an enumerative proof.

Theorem 2.1. *For any $k \geq 2$ and $\ell \geq 2$, the number $N_{k,\ell}$ of down-up words over $[k]$ of length ℓ is equal to the number of order ideals of $Z_\ell \times (\mathbf{k-2})$.*

Bijective Proof. Let $\mathcal{W}_{k,\ell}$ denote the set of down-up words over $[k]$ of length ℓ. We shall build a bijection between $\mathcal{W}_{k,\ell}$ and $J(Z_\ell \times (\mathbf{k-2}))$.

We first define a map $\Phi : \mathcal{W}_{k,\ell} \to J(Z_\ell \times (\mathbf{k-2}))$. Given a down-up word $w = w_1 w_2 \cdots w_\ell$, we define the word $\alpha = \alpha_1 \alpha_2 \cdots \alpha_\ell$ as follows:

$$\alpha_i = \begin{cases} w_i - 2, & \text{if } i \text{ is odd,} \\ w_i - 1, & \text{if } i \text{ is even,} \end{cases}$$

where $1 \leq i \leq \ell$. Then let

$$\Phi(w) = \{(i, \beta_j) : 1 \leq i \leq \ell,\ 1 \leq \beta_j \leq \alpha_i\}.$$

For example, let $k = 4$ and $\ell = 7$, and consider the word $w = 3241423$. Then, $\alpha = 1120211$ and thus $\Phi(w) = \{(1,1), (2,1), (3,1), (3,2), (5,1), (5,2), (6,1), (7,1)\}$, which is an order ideal of $Z_7 \times \mathbf{2}$.

We need to show that this map is well defined. It suffices to prove that $\Phi(w)$ is an order ideal of $Z_\ell \times (\mathbf{k-2})$, that is to say that, if $(i', j') \leq (i, j)$ and $(i, j) \in \Phi(w)$ then $(i', j') \in \Phi(w)$. From the definition of the order of $Z_\ell \times (\mathbf{k-2})$, we have that $i' \leq_{Z_\ell} i$ and $j' \leq j$. Now, we divide the situation into two cases: $i' = i$ and $i' <_{Z_\ell} i$. For the case $i' = i$, the argument is obviously true from the construction of $\Phi(w)$. We just need to consider the case $i' <_{Z_\ell} i$. At this time, i must be even, and i' can only be $i-1$ or $i+1$. Since $(i,j) \in \Phi(w)$, we have that $\alpha_i \geq j$ and thus $w_i \geq j+1$. From the fact that w is a down-up word, it follows that $w_{i'} > w_i$. Hence, $w_{i'} \geq j+2$ and thus $\alpha_{i'} \geq j$. From the construction of $\Phi(w)$, we obtain that $(i', j') \in \Phi(w)$ for all $j' \leq j$, as desired.

Next, we define a map $\Psi : J(Z_\ell \times (\mathbf{k-2})) \to \mathcal{W}_{k,\ell}$. Given an order ideal I of $Z_\ell \times (\mathbf{k-2})$, we define a word $\gamma = \gamma_1\gamma_2\cdots\gamma_\ell$ as follows. For each $1 \leq i \leq \ell$, if there exists at least one j such that $(i,j) \in I$, then let γ_i be the maximum j. Otherwise, we let γ_i=0. The corresponding word $\Psi(I)$ is defined as $(2+\gamma_1)(1+\gamma_2)(2+\gamma_3)(1+\gamma_4)\cdots$. For exmaple, if $I = \{(1,1),(2,1),(3,1),(3,2),(5,1),(5,2),(6,1),(7,1)\}$, then $\gamma = 1120211$ and thus $w = 3241423$.

It is easy to see that, for any even integer i, we have $\gamma_i \leq \gamma_{i+1}$ and $\gamma_i \leq \gamma_{i-1}$, since I is an order ideal. From the construction of $\Psi(I)$, we see that it is a down-up word.

Finally, it is not difficult to prove that $\Psi \circ \Phi = id$ and $\Phi \circ \Psi = id$. Hence Φ is a bijection. This completes our bijective proof.

Enumerative Proof. We first prove that the numbers in question satisfy the following recurrence relation, for $k \geq 3$ and $\ell \geq 2$,

$$N_{k,\ell} = N_{k-1,\ell} + \sum_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} N_{k-1,2i} N_{k,\ell-2i-1} - \delta_{\ell \text{ is even}} N_{k-1,\ell-2}, \tag{1}$$

with the initial conditions $N_{k,0} = 1$, $N_{k,1} = k$ for $k \geq 2$, and $N_{2,\ell} = 1$ for $\ell \geq 2$. To this end, we note that any down-up word w over $[k]$ of length ℓ belongs to one of the following two cases.

Case 1: w does not contain the letter k. Then the number we count is that of down-up words over the alphabet $[k-1]$ of length ℓ, which is $N_{k-1,\ell}$. This corresponds to the first term on the righthand side of (1).

Case 2: w is of the form $w_1 k w_2$, where w_1 is a down-up word of even length over $[k-1]$, and w_2 is an up-down word over $[k]$. Note that the number of up-down words equals to that of down-up words, as mentioned above. This corresponds to the second term on the right hand side of (1). The only exception occurs when the subword after the leftmost letter k is of length one. It can be any letter in $[k-1]$, but $N_{k,1} = k$. So, an additional term occurs, which fixes this. In these cases, $\ell - 2i - 1$ equals 1, which means that ℓ is even. This completes the proof of (1).

Now, let us denote the number of order ideals of $Z_\ell \times \mathbf{k}$ by $M_{k,\ell}$. We note that Berman and Köhler [1, Example 2.3] studied a similar recurrence for $M_{k,\ell}$, which is, for $k \geq 1$ and

$\ell \geq 1$,

$$M_{k,\ell} = M_{k-1,\ell} + \sum_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} M_{k-1,2i} M_{k,\ell-2i-1},$$

with the initial conditions $M_{k,0} = 1$ for $k \geq 0$ and $M_{0,\ell} = 1$ for $\ell \geq 1$.

Owing to their akin recurrence relations, we made a minor change to the number $N_{k,\ell}$ to complete the proof. We let $\widetilde{N}_{k,\ell}$ be $N_{k,\ell}$ except $\widetilde{N}_{k,1} = k - 1$. One can easily check that, for $k \geq 3$ and $\ell \geq 1$,

$$\widetilde{N}_{k,\ell} = \widetilde{N}_{k-1,\ell} + \sum_{i=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \widetilde{N}_{k-1,2i} \widetilde{N}_{k,\ell-2i-1},$$

with the initial conditions $\widetilde{N}_{k,0} = 1$ for $k \geq 2$ and $\widetilde{N}_{2,\ell} = 1$ for $\ell \geq 1$. It follows immediately that, for $k \geq 2$ and $\ell \geq 0$,

$$\widetilde{N}_{k,\ell} = M_{k-2,\ell},$$

since they have the same initial conditions and recurrence relations. Together with the fact $N_{k,\ell} = \widetilde{N}_{k,\ell}$ except $\ell = 1$, we obtain that

$$N_{k,\ell} = M_{k-2,\ell}$$

for $k \geq 2$ and $\ell \geq 2$. This completes our enumerative proof. □

As an immediate corollary of Theorem 2.1, we have the following statement.

Theorem 2.2. *For $k \geq 3$ and $\ell \geq 2$, the numbers $N_{k,\ell}$ of down-up (equivalently, up-down) words of length ℓ over $[k]$ satisfy (1) with the initial conditions $N_{k,0} = 1$, $N_{k,1} = k$ for $k \geq 2$, and $N_{2,\ell} = 1$ for $\ell \geq 2$.*

Note that the Fibonacci numbers have the following recurrence relations [10, pp. 5–6]:

$$F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}, \quad F_{2n+1} = 1 + \sum_{i=1}^{n} F_{2i}.$$

Using (1) and the fact that $N_{2,\ell} = 1$ for $\ell \geq 2$, one can prove the following statement.

Theorem 2.3. *For $\ell \geq 2$, $N_{3,\ell} = F_{\ell+2}$, the $(\ell+2)$th Fibonacci number.*

3 Enumeration of 123-avoiding up-down words

In this section, we consider the enumeration of 123-avoiding up-down words. Denote $A_{k,\ell}$ the number of 123-avoiding up-down words of length ℓ over the alphabet $[k]$, and $A^j_{k,\ell}$ the number of those words counted by $A_{k,\ell}$ that end with j.

3.1 Explicit enumeration

It is easy to see that

$$A_{k,2i} = \sum_{j=2}^{k} A^j_{k,2i}. \tag{2}$$

Next, we deal with the enumeration of $A^j_{k,2i}$. In what follows, for a word w, we have $\{w\}^* = \{\epsilon, w, ww, www, \ldots\}$, where ϵ is the empty word, and $\{w\}^+ = \{w, ww, www, \ldots\}$.

Lemma 3.1. *For $k \geq 3$ and $2 \leq j \leq k$, the numbers $A^j_{k,2i}$ satisfy the following recurrence relation,*

$$A^j_{k,2i} = \sum_{i'=1}^{i} \left(A^{j-1}_{k-1,2i'} - A^j_{k-1,2i'} + A^{j+1}_{k,2i'} \right), \tag{3}$$

with the boundary condition $A^k_{k,2i} = \binom{i+k-2}{i}$. Furthermore, an explicit formula for $A^j_{k,2i}$ is

$$A^j_{k,2i} = \frac{j-1}{k-1} \binom{i+k-2}{i} \binom{i+k-j-1}{i-1}. \tag{4}$$

Proof. We first check the boundary condition. When $j = k$, the words must be of the form

$$\{(k-1)k\}^* \{(k-2)k\}^* \ldots \{2k\}^* \{1k\}^*.$$

The structure is dictated by the presence of the rightmost k; violating the structure, we will be forced to have an occurrence of the pattern 123. Therefore, $A^k_{k,2i} = \binom{i+k-2}{i}$, where we applied the well known formula for the number of solutions of the equation $x_1 + \cdots + x_{k-1} = i$ with $x_i \geq 0$ for $1 \leq i \leq k-1$.

Now we proceed to deduce the recurrence relation (3). All the legal words of length $2i$ ending with j can be divided into the following cases according to the occurrence of the letter 1:

Case 1: For the legal words that contain the letter 1, the letter 1 must appear in the second last position, since otherwise it would lead to a 123 pattern. We now divide all the legal words ending with $1j$ into the following subcases:

Case 1.1: There is only one word of the form $\{1j\}^+$.

Case 1.2: We deal with the words of the form $w\{j'j\}^+\{1j\}^+$, where w is a legal word and $2 \leq j' \leq j-1$. Note that w cannot contain 1 because of an occurrence of $j'j$. Thus, we consider a legal word over the alphabet set $[2, k]$ of even length which ends with $j'j$. By subtracting 1 from each letter of this word, we obtain a legal word over $[k-1]$ ending with $(j'-1)(j-1)$. Thus, the number of all words in this case is equal to that of all words over $[k-1]$ ending with $j-1$, which is $\sum_{i'=1}^{i-1} A^{j-1}_{k-1,2i'}$.

Case 1.3: The others are the words of the form $wj'^{+}\{1j\}^{+}$, where w is a legal word and $j' \geq j+1$. Clearly, the number of such words in this case is $\sum_{i'=1}^{i-1}\sum_{j'=j+1}^{k} A_{k,2i'}^{j'}$.

Case 2: We next deal with the legal words ending with j over the alphabet $[k]\setminus\{1\} = \{2,3,\ldots,k\}$. In this case, it has the same enumeration as that of the legal words over $[k-1]$ ending with $j-1$. The number of such words is $A_{k-1,2i}^{j-1}$.

Thus, we have the following recurrence relation

$$A_{k,2i}^{j} = 1 + \sum_{i'=1}^{i} A_{k-1,2i'}^{j-1} + \sum_{i'=1}^{i-1}\sum_{j'=j+1}^{k} A_{k,2i'}^{j'}. \tag{5}$$

From (5), we have that

$$A_{k,2i}^{j} - A_{k,2i}^{j+1} = \sum_{i'=1}^{i}\left(A_{k-1,2i'}^{j-1} - A_{k-1,2i'}^{j}\right) + \sum_{i'=1}^{i-1} A_{k,2i'}^{j+1},$$

and therefore the recurrence (3) follows.

Now we deduce the formula (4) for $A_{k,2i}^{j}$. Let

$$A'(k,i,j) = \frac{j-1}{k-1}\binom{i+k-2}{i}\binom{i+k-j-1}{i-1}.$$

We next prove that $A_{k,2i}^{j} = A'(k,i,j)$ by induction on $k-j$ and k. We shall show that these numbers have the same base case and satisfy the same recursion. Indeed, for $k=j\geq 2$, this fact is obviously true, since $A'(k,i,k) = A_{k,2i}^{k}$. We will now check that $A'(k,i,j)$ satisfy the following recurrence relation:

$$A'(k,i,j) = \sum_{i'=1}^{i}\left(A'(k-1,i',j-1) - A'(k-1,i',j) + A'(k,i',j+1)\right). \tag{6}$$

Indeed, (6) is true if and only if

$$A'(k,i,j) - A'(k,i-1,j) = A'(k-1,i,j-1) - A'(k-1,i,j) + A'(k,i,j+1),$$

while the later equation is easy to check to be true. This completes the proof. □

Further, the number of 123-avoiding up-down words of length $2i$ over $[k]$ is

$$A_{k,2i} = \sum_{j=2}^{k}\frac{j-1}{k-1}\binom{i+k-2}{i}\binom{i+k-j-1}{i-1} = \frac{1}{i+1}\binom{i+k-2}{i}\binom{i+k-1}{i}.$$

The last equality can be deduced from the Gosper algorithm [5].

Now we consider legal words of odd length. For any legal word of length $2i$ ($i\geq 1$) ending with j ($2\leq j\leq k$), we can adjoin any letter in $[j-1]$ at the end to form an up-down word of length $2i+1$ over $[k]$. In fact, such words are necessarily 123-avoiding. So, we obtain that

$$\begin{aligned}
A_{k,2i+1} &= \sum_{j=2}^{k}(j-1)A_{k,2i}^{j} \\
&= \sum_{j=2}^{k}\frac{(j-1)^2}{k-1}\binom{i+k-2}{i}\binom{i+k-j-1}{i-1} \\
&= \frac{i+2k-2}{(i+1)(i+2)}\binom{i+k-2}{i}\binom{i+k-1}{i}.
\end{aligned}$$

Also, the last equality can be deduced from the Gosper algorithm [5].

Hence, we have proved the following theorem.

Theorem 3.2. *For $A_{k,\ell}$, the number of 123-avoiding up-down words of length ℓ over $[k]$, $A_{k,0}=1$, $A_{k,1}=k$, and for $\ell \geq 2$,*

$$A_{k,\ell} = \begin{cases} \frac{1}{i+1}\binom{i+k-2}{i}\binom{i+k-1}{i}, & \text{if } \ell = 2i, \\ \frac{i+2k-2}{(i+1)(i+2)}\binom{i+k-2}{i}\binom{i+k-1}{i}, & \text{if } \ell = 2i+1. \end{cases} \tag{7}$$

3.2 Generating functions

In this subsection, an expression for the generating function for the numbers $A_{k,i}$ of 123-avoiding up-down words of length i over $[k]$ is given. We adopt the notation of *Narayana polynomials*, which are defined as $N_0(x) = 1$ and, for $n \geq 1$,

$$N_n(x) = \sum_{i=0}^{n-1}\frac{1}{i+1}\binom{n}{i}\binom{n-1}{i}x^i.$$

Due to Brenti [2] and Reiner and Welker [6, Section 5.2], a remarking generating function for $A_{k,2i}$ can be expressed as follows:

$$\sum_{i\geq 0} A_{k,2i}x^i = \frac{N_{k-2}(x)}{(1-x)^{2k-3}}. \tag{8}$$

On the other hand, by Theorem 3.2, a routine computation leads to the following identity,

$$A_{k,2i-1} = A_{k,2i} - A_{k-1,2i}, \tag{9}$$

for all $i \geq 2$. (Note that we shall also give a combinatorial interpretation of (9) in Section 7.) Thus, together with $A_{k,1} = k$, it follows that

$$\begin{aligned}\sum_{i\geq 1} A_{k,2i-1}x^i &= x + \sum_{i\geq 1} A_{k,2i}x^i - \sum_{i\geq 1} A_{k-1,2i}x^i\\ &= x + \frac{N_{k-2}(x)}{(1-x)^{2k-3}} - \frac{N_{k-3}(x)}{(1-x)^{2k-5}}\\ &= x + \frac{N_{k-2}(x) - (1-x)^2 N_{k-3}(x)}{(1-x)^{2k-3}}.\end{aligned}$$

Hence, we are ready to obtain the main result of this subsection,

$$\begin{aligned}\sum_{i\geq 0} A_{k,i}x^i &= \sum_{i\geq 0} A_{k,2i}x^{2i} + \sum_{i\geq 1} A_{k,2i-1}x^{2i-1}\\ &= \frac{N_{k-2}(x^2)}{(1-x^2)^{2k-3}} + x + \frac{N_{k-2}(x^2) - (1-x^2)^2 N_{k-3}(x^2)}{x(1-x^2)^{2k-3}}\\ &= x + \frac{(1+x)N_{k-2}(x^2) - (1-x^2)^2 N_{k-3}(x^2)}{x(1-x^2)^{2k-3}}.\end{aligned}$$

4 A bijection between $S^{132}_{k,2i}$ and $S^{123}_{k,2i}$

Let p be a pattern and $S^p_{k,\ell}$ be the set of p-avoiding up-down words of length ℓ over $[k]$. In this section, we will build a bijection between $S^{132}_{k,2i}$ and $S^{123}_{k,2i}$.

The idea here is to introduce the notion of irreducible words and show that irreducible words in $S^{132}_{k,2i}$ can be mapped in a 1-to-1 way into irreducible words in $S^{123}_{k,2i}$, while reducible words in these sets can be mapped to each other as well.

Definition 1. *A word w is* reducible, *if $w = w_1w_2$ for some non-empty words w_1 and w_2, and each letter in w_1 is no less than every letter in w_2. The place between w_1 and w_2 in w is called a* cut-place.

For example, the word 242313 is irreducible, while the word 341312 is reducible (it can be cut into 34 and 1312).

Note that in a reducible up-down word, if we have a cut-place, and there are equal elements on both sides of it, then to the left such elements must be bottom elements, and to the right they must be top elements.

Lemma 4.1. *A word w in $S^{132}_{k,2i}$ is irreducible if and only if $w = w_1xy$, where w_1 is a word in $S^{132}_{k,2i-2}$, x is the minimum letter in w (possibly, there are other copies of x in w) and y is the maximum letter in w (possibly, there are other copies of y in w).*

Proof. If x is not the minimum element in w, then the element right before it, and the minimum element in w will form the pattern 132. Since w is irreducible, y is forced to be no less than the minimum element in w_1. On the other hand, if y is not the maximum element in w, then the maximum one in w_1 and the element just preceding it will form the pattern 132. This completes the proof. □

Now, given a word w in $S^{132}_{k,2i}$, we can count in how many ways it can be extended to an irreducible word in $S^{132}_{k,2i+2}$. Suppose that a and b are the minimum and the maximum elements in w, respectively. Then the number of extensions of w in $S^{132}_{k,2i+2}$ is $a\cdot(k-b+1)$, since there are a choices of the next to last element and $k-b+1$ choices of the last element.

Next, we discuss a procedure of turning any word w in $S^{123}_{k,2i}$ into an irreducible word in $S^{123}_{k,2i+2}$. From this procedure, it would be clear that the number of choices is $a\cdot(k-b+1)$, where a and b are the minimum and the maximum elements in w, respectively.

Suppose that $w = b_1t_1b_2t_2\cdots b_it_i$, where b_j's and t_j's stand for bottom and top elements, respectively. To obtain the desired word, we inserting a new top element x, shift the bottom elements one position to the left, and then insert one more bottom element y. Then, the extension is of the form

$$w' = b_1xb_2t_1b_3t_2\cdots b_it_{i-1}yt_i,$$

where $x \geq b$ and $y \leq a$, and again, a and b are the minimum and the maximum elements in w, respectively. For example, if $w = 242313 \in S^{123}_{5,6}$, $a = 1$ and $b = 4$, then w' can be 24241313 or 25241313.

To see that the resulting word w' is an up-down word. In fact, it is sufficient to show that $b_{j+1} < t_j$ for $1 \leq j \leq i-1$ and $t_j > b_{j+2}$ for $1 \leq j \leq i-2$. The first inequality follows from the fact that w is an up-down word, while the second one is true, because otherwise $t_j \leq b_{j+2} < t_{j+1}$ and thus $b_jt_jt_{j+1}$ would form a 123 pattern. We also claim that w' belongs to $S^{123}_{k,2i+2}$. An equivalent condition an up-down w is 123 avoiding is that w satisfy $b_1 \geq b_2 \geq \cdots \geq b_i$ and $t_1 \geq t_2 \geq \cdots \geq t_i$. From the construction of w', we obtain that w' is also 123 avoiding. Besides, w' is irreducible, since $b_j < t_j$ for $1 \leq j \leq i$.

Now, a bijection between $S^{132}_{k,2i}$ and $S^{123}_{k,2i}$ is straightforward to set recursively, with a trivial base case of words of length 2 mapped to themselves. Indeed, if we assume that we can map all words in $S^{132}_{k,2i}$ to all words in $S^{123}_{k,2i}$, then applying the same choice of x and y, we can map all irreducible words in $S^{132}_{k,2i+2}$ to all irreducible words in $S^{123}_{k,2i+2}$. Finally, each reducible word w is of the form w_1w_2, where w_1 is irreducible word with maximum possible even length. But then w_1 and w_2 are of smaller lengths than w, and we can map them recursively.

For example, $w = 3435121213 \in S^{132}_{5,10}$ is reducible, since it can be cut into 3435 and 121213. We calculate that $\phi(3435)$ by two steps: first $\phi(34) = 34$ and the second $\phi(3435) = 3534$. Similarly, $\phi(121213) = 131212$. It follows that $\phi(3435121213) =$

$S_{4,4}^{132}$	$S_{4,4}^{123}$	$S_{4,4}^{132}$	$S_{4,4}^{123}$
1212	1212	2312	2312
2412	2412	3412	3412
1213	1312	1313	1313
2313	2313	3413	3413
2323	2323	3423	3423
1214	1412	1314	1413
2314	2413	1414	1414
2414	2414	3414	3414
2324	2423	2424	2424
3424	3424	3434	3434

Table 2: The bijection $\phi : S_{4,4}^{132} \to S_{4,4}^{123}$.

3534131212, i.e. the word 3435121213 in $S_{5,10}^{132}$ is mapped to 3534131212 in $S_{5,10}^{123}$. Also, see Table 2 showing images of all words in $S_{4,4}^{132}$.

5 Enumeration of 132-avoiding up-down words

In this section, we consider the enumeration of 132-avoiding up-down words. Denote $B_{k,\ell}$ the number of 132-avoiding up-down words of length ℓ over the alphabet $[k]$, and $B_{k,\ell}^{j}$ the number of those words counted by $B_{k,\ell}$ whose letter in the second last position is j.

From the bijection, it follows that

$$B_{k,2i} = A_{k,2i} = \frac{1}{i+1}\binom{i+k-2}{i}\binom{i+k-1}{i}. \tag{10}$$

For any legal word of length $2i$ ($i \geq 1$) whose letter in the second last position is j ($1 \leq j \leq k-1$), the minimum letter in this word is also j, since it is 132-avoiding. By subtracting $j-1$ from each letter of this word, we obtain a legal word over $[k-j+1]$ whose letter in the second last position is 1. Thus, we have that

$$B_{k,2i}^{j} = B_{k-j+1,2i}^{1}.$$

Similarly, we obtain that

$$B_{k,2i}^{1} = B_{k,2i} - \sum_{j=2}^{k-1} B_{k,2i}^{j} = B_{k,2i} - B_{k-1,2i} = \frac{i+2k-3}{(i+1)(k-1)}\binom{i+k-3}{i-1}\binom{i+k-2}{i}.$$

Now, we consider legal words of odd length. For any legal word of length $2i$ ($i \geq 1$) whose letter in the second last position is j ($1 \leq j \leq k-1$), the minimum letter in the word is also j and hence, we can adjoin any letter in $[j]$ at the end to form an up-down word on $[k]$ of length $2i+1$. In fact, such words are necessarily 132-avoiding. So, we obtain that for $i \geq 1$

$$\begin{aligned}
B_{k,2i+1} &= \sum_{j=1}^{k-1} j B_{k,2i}^{j} \\
&= \sum_{j=1}^{k-1} j B_{k-j+1,2i}^{1} \\
&= \sum_{j=1}^{k-1} j \, \frac{i+2(k-j)-1}{(i+1)(k-j)} \binom{i+k-j-2}{i-1} \binom{i+k-j-1}{i} \\
&= \sum_{j=1}^{k-1} \frac{(k-j)(i+2j-1)}{j(i+1)} \binom{i+j-2}{i-1} \binom{i+j-1}{i}.
\end{aligned}$$

Unfortunately, we were not able to find a closed form formula for $B_{k,2i+1}$. We conclude this section with listing expressions for $B_{k,2i+1}$ for $k = 3,4,5,6$ and $i \geq 1$:

$$\begin{aligned}
B_{3,2i+1} &= \frac{1}{2}(i^2+3i+4), \\
B_{4,2i+1} &= \frac{1}{12}(i^4+8i^3+29i^2+46i+36), \\
B_{5,2i+1} &= \frac{1}{144}(i^6+15i^5+103i^4+381i^3+832i^2+972i+576), \\
B_{6,2i+1} &= \frac{1}{2880}(i^8+24i^7+266i^6+1704i^5+6929i^4+18096i^3+30244i^2+29136i+14400).
\end{aligned}$$

6 Enumeration of 312-avoiding up-down words

In this section, we consider the enumeration of 312-avoiding up-down words. Denote $C_{k,\ell}$ the number of 312-avoiding up-down words of length ℓ over the alphabet $[k]$, and $C_{k,\ell}^{j}$ the number of those words counted by $C_{k,\ell}$ that end with j.

Recall the definition of the complement w^c of a word w given in Section 1. Also, for a word $w = w_1 w_2 \cdots w_n$, its *reverse word* w^r is given by $w^r = w_\ell w_{\ell-1} \cdots w_1$. It is clear that the operations of reverse and complement are both bijections on alternating words.

Let p be a pattern and let $\widehat{S}_{k,\ell}^{p}$ denote the set of all p-avoiding down-up words of length ℓ over an alphabet $[k]$.

We first consider the words of odd length.

Proposition 6.1. *For all $k \geq 1$ and $i \geq 0$, we have that the number of 312-avoiding up-down words of length $2i+1$ on $[k]$ is the same as that of 123-avoiding up-down words on $[k]$ of the same length. Namely,*

$$C_{k,2i+1} = A_{k,2i+1}.$$

Proof. We shall prove this theorem by establishing a bijection between $S^{312}_{k,2i+1}$ and $S^{123}_{k,2i+1}$. Applying the complement operation to the former of these sets, and reverse and complement to the latter set, it suffices to show that there exists a bijection between $\widehat{S}^{132}_{k,2i+1}$ and $\widehat{S}^{123}_{k,2i+1}$.

The map $\psi(w) : \widehat{S}^{132}_{k,2i+1} \to \widehat{S}^{123}_{k,2i+1}$ is defined as follows. For any $w = w_1 w_2 \cdots w_{2i+1} \in \widehat{S}^{132}_{k,2i+1}$, let w' be $w_2 \cdots w_{2i+1}$. It is clear that $w' \in S^{132}_{k,2i}$. Thus let

$$\psi(w) = w_1 \phi(w'),$$

where the map $\phi : S^{132}_{k,2i} \to S^{123}_{k,2i}$ is described in Section 4.

We need to show that ψ is well-defined. From the construction of ϕ, we see that ϕ preserves the first letter, i.e. $\phi(w')_1 = w'_1 = w_2$. Therefore, it follows that $\psi(w) \in \widehat{S}^{123}_{k,2i+1}$. Hence, the map ψ is well-defined.

It is not difficult to see by construction that ψ is a bijection. Hence, we get a bijection between $\widehat{S}^{132}_{k,2i+1}$ and $\widehat{S}^{123}_{k,2i+1}$. This completes the proof. □

Now let us consider the words of even length. Note that

$$C_{k,2i} = \sum_{j=2}^{k} C^{j}_{k,2i}.$$

For any word $w \in S^{312}_{k,2i}$ whose last letter is j, the maximum letter of w is also j, since the word is 312-avoiding and having $j' > j$ in w would lead to an occurrence of three letters $j' w_{2i-1} j$ forming the pattern 312. Thus, we have that

$$C^{j}_{k,2i} = C^{j}_{j,2i},$$

where $2 \leq j \leq k$.

Moreover, for any word in $S^{312}_{j,2i}$ ending with j, we can remove j to form a word of length $2i-1$, which is also 312-avoiding. On the other hand, for any word $S^{312}_{j,2i-1}$, we can adjoin a letter j at the end to form a 312-avoiding word of length $2i$. Thus,

$$C^{j}_{j,2i} = C_{j,2i-1}.$$

So, we obtain that

$$\begin{aligned}
C_{k,2i} &= \sum_{j=2}^{k} C_{j,2i-1} \\
&= \sum_{j=2}^{k} \frac{i-3+2j}{i(i+1)} \binom{i+j-3}{i-1} \binom{i+j-2}{i-1} \\
&= \frac{1}{i+1} \binom{i+k-2}{i} \binom{i+k-1}{i} \\
&= A_{k,2i},
\end{aligned}$$

where the second last equality can be deduced from the Gosper algorithm [5].

We have just obtained the main result in this section.

Theorem 6.2. *The sets of* 312*-avoiding up-down words and* 123*-avoiding up-down words are equinumerous, that is,*

$$C_{k,\ell} = A_{k,\ell}$$

for all $k \geq 1$ *and* $\ell \geq 0$.

7 Enumeration of other pattern-avoiding up-down words

In this section, we consider the enumeration of other pattern-avoiding up-down words. In order to avoid confusion, let $N_{k,\ell}(p)$ denote the number of p-avoiding up-down words of length ℓ over the alphabet $[k]$.

We first focus on all six length 3 permutation patterns to be avoided on up-down words of odd length.

Theorem 7.1. *For all* $k \geq 1$ *and* $i \geq 0$*, we have*

$$N_{k,2i+1}(123) = N_{k,2i+1}(312) = N_{k,2i+1}(213) = N_{k,2i+1}(321)$$

and

$$N_{k,2i+1}(132) = N_{k,2i+1}(231).$$

Proof. Through the reverse operation, the following equations hold:

$$N_{k,2i+1}(132) = N_{k,2i+1}(231),$$

$$N_{k,2i+1}(123) = N_{k,2i+1}(321),$$

and

$$N_{k,2i+1}(312) = N_{k,2i+1}(213).$$

Combing with Theorem 6.2, the proof is complete. □

Next, we obtain the following result for the case of the even length.

Theorem 7.2. *For all $k \geq 1$ and $i \geq 1$, there is*

$$N_{k,2i}(123) = N_{k,2i}(132) = N_{k,2i}(312) = N_{k,2i}(213) = N_{k,2i}(231).$$

Proof. Through the complement and reverse operations, it follows that

$$N_{k,2i}(132) = N_{k,2i}(213),$$

and

$$N_{k,2i}(231) = N_{k,2i}(312).$$

From Section 4, we have that

$$N_{k,2i}(132) = N_{k,2i}(123).$$

Together with Theorem 6.2, we complete the proof. $\square$

In the rest of this section, we deal with the only remaining case, 321-avoiding up-down words of even length. Our approach is based on deriving the desired from an alternative enumeration of 123-avoiding up-down words.

All 123-avoiding up-down words of length ℓ over $[k]$, for $\ell \geq 4$, can be divided into the following two cases:

- Legal words containing no k in them. These words are counted by $A_{k-1,\ell}$.
- Legal words that contain at least one k. Such words $w = w_1w_2\cdots w_\ell$ are necessarily of the form $w_1kw_3\cdots w_\ell$, since otherwise w_1w_2k would be an occurrence of the pattern 123. Clearly, $w_1 \geq w_3$ (otherwise, $w_1w_3w_4$ would form the pattern 123). We let w' be $kw_3\cdots w_\ell$ if $w_1 = w_3$ and $w_1w_3\cdots w_\ell$ if $w_1 > w_3$. Clearly, this is a reversible procedure and the obtained words w' are 123-avoiding down-up words. By applying the complement operation, we obtain 321-avoiding up-down words over $[k]$ of length $\ell - 1$.

It follows that for $\ell \geq 4$,

$$A_{k,\ell} = A_{k-1,\ell} + N_{k,\ell-1}(321),$$

and thus

$$N_{k,\ell-1}(321) = A_{k,\ell} - A_{k-1,\ell}.$$

Hence, by Theorem 3.2, we are ready to obtain an expression for $N_{k,\ell}(321)$.

Theorem 7.3. *For the number of 321-avoiding up-down words of length ℓ over $[k]$, $N_{k,0}(321) = 1$, $N_{k,1}(321) = k$, $N_{k,2}(321) = \binom{k}{2}$, and for $\ell \geq 3$,*

$$N_{k,\ell}(321) = \begin{cases} \frac{i(i+2k-3)(i+2k-2)+2(k-2)(k-1)}{(i+1)(i+2)(k-2)(k-1)}\binom{i+k-2}{i}\binom{i+k-3}{i}, & \text{if } \ell = 2i, \\ \frac{i+2k-2}{(i+1)(i+2)}\binom{i+k-2}{i}\binom{i+k-1}{i}, & \text{if } \ell = 2i+1. \end{cases}$$

Since $N_{k,2i+1}(123) = N_{k,2i+1}(321)$, we actually give another approach to deal with 321-avoiding up-down words of odd length.

8 Concluding remarks

In this paper we initiated the study of (pattern-avoiding) alternating words. In particular, we have shown that 123-avoiding up-down words of even length are given by the Narayana numbers. Thus, alternating words can be used, for example, for encoding Dyck paths with a specified number of peaks [3]. To our surprise, the enumeration of 123-avoiding up-down words turned out to be easier than that of 132-avoiding up-down words, as opposed to similar studies for permutations, when the structure of 132-avoiding permutations is easier than that of 123-avoiding permutations.

Above, we gave a complete classification of avoidance of permutation patterns of length 3 on alternating words. We state it as an open direction of research to study avoidance of longer patterns and/or patterns of different types (see [4]) on alternating (up-down or down-up) words.

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