Restricted rooted non-separable planar maps

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Abstract

Tutte founded the theory of enumeration of planar maps in a series of papers in the 1960s. Rooted non-separable planar maps have connections, for example, to pattern-restricted permutations, and they are in one-to-one correspondence with the $\beta(1,0)$-trees introduced by Cori, Jacquard and Schaeffer in 1997. In this paper we enumerate 2-face-free rooted non-separable planar maps and obtain restrictions on $\beta(1,0)$-trees giving k-face-free rooted non-separable planar maps. Moreover, we discuss multiple-edge-free rooted non-separable planar maps and provide some rough lower bounds for their number using restricted $\beta(1,0)$-trees. Finally, we enumerate so-called primitive rooted non-separable planar maps (which form a basis for generating all rooted non-separable planar maps). We discuss some equinumerous objects such as primitive $\beta(1,0)$-trees and certain pattern-restricted permutations.

1 Introduction

A *map* is a partition of a compact oriented surface into three finite sets: a set of *vertices* (points), a set of *edges* and a set of *faces* (disjoint simply connected domains). Maps are considered up to orientation-preserving homeomorphisms of the surface. In this paper we deal with classical *planar maps* considered, for example, by Tutte [13] who founded the theory of enumeration of planar maps in a series of papers in the 1960s (see [5] for references). The maps considered by us are *rooted*, meaning that a directed edge is distinguished as the root, which defines a *root-vertex* and the *root-face* (the face located to the right while traversing the root following its direction). For convenience we assume that the root-face of a map coincides with the map's outer face.

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A *cut vertex* in a map M is a vertex v such that there exists a partition of the edge set of M into two subsets such that v is the unique vertex which is incident with edges of the two subsets. A planar map is *non-separable* if it has no loops and no cut vertices. The study of these maps is interesting, in particular, due to their connections to pattern-restricted permutations (see, for example, Kitaev [8]) and due to their connections to theoretical physics, where they are used as a discrete model for 2D quantum gravity, see e.g., Schaeffer and Zinn-Justin [11]. Recent results on rooted non-separable planar maps include those in Kitaev et al. [9] where self-dual rooted non-separable planar maps are enumerated.

From here on we will use *maps* to mean *rooted non-separable planar maps* for the sake of brevity.

All maps on four edges are given in Figure 1.

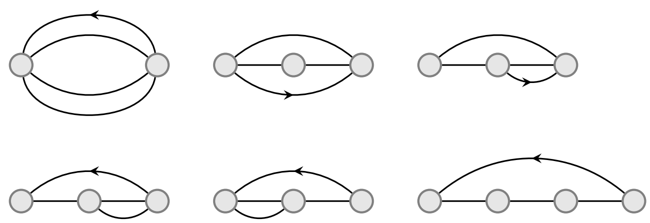


Figure 1: All rooted non-separable planar maps on four edges

The number of rooted non-separable planar maps on $n+1$ edges was first determined by Tutte [12] and is given by

$$\frac{4(3n)!}{n!(2n+2)!}.$$

This enumeration was also proved differently by Brown [2]. Moreover, the generating function $A(x)$ for the maps is given by

$$A(x) = 2 + xB(x),$$

where

$$B(x) = 1 - 8x + 2x(5-6x)B(x) - 2x^2(1+3x)B^2(x) - x^4B^3(x).$$

Alternatively, we can express this in terms of a *hypergeometric function*. Let $\mathbf{a} = (a_1, \ldots, a_p)$, $\mathbf{b} = (b_1, \ldots, b_q)$ be vectors of real numbers. Then hypergeom$(\mathbf{a}, \mathbf{b}, z)$ is the function whose power series is

$$\sum_{k\geq 0} \frac{a_1^{\bar{k}} \cdots a_p^{\bar{k}}}{b_1^{\bar{k}} \cdots b_q^{\bar{k}}} \frac{z^k}{k!},$$

and $a^{\bar{k}}$ is the rising factorial $a(a+1)\cdots(a+k-1)$. Then the generating function for rooted non-separable planar maps is given by

$$A(x) = \frac{2}{3x}\left(\text{hypergeom}\left(\left[-\frac{2}{3}, -\frac{1}{3}\right], \left[\frac{1}{2}\right], \frac{27x}{4}\right) - 1\right),$$

see [10, A000139].

Cori et al. [6] introduced *description trees*, thus placing under one roof several classes of planar maps. In particular, $\beta(1,0)$-trees (defined below) are in one-to-one correspondence with rooted non-separable planar maps. Description trees may be useful in dealing with the corresponding planar maps. For example, Claesson et al. [3] obtained non-trivial equidistribution results on so-called *bicubic maps* using $\beta(0,1)$-trees. In this paper, our main focus is understanding the structure of $\beta(1,0)$-trees corresponding to certain restricted maps and deriving some corollaries of that structure. We believe that our studies will be of help in future research on rooted non-separable planar maps, for example, from an enumerative point of view.

Recall that a $\beta(1,0)$*-tree* is a rooted planar tree labeled with positive integers such that the leaves have label 1, the root has label equal to the sum of its children's labels and any other node has label no greater than the sum of its children's labels.

All $\beta(1,0)$-trees on three edges are presented in Figure 2.

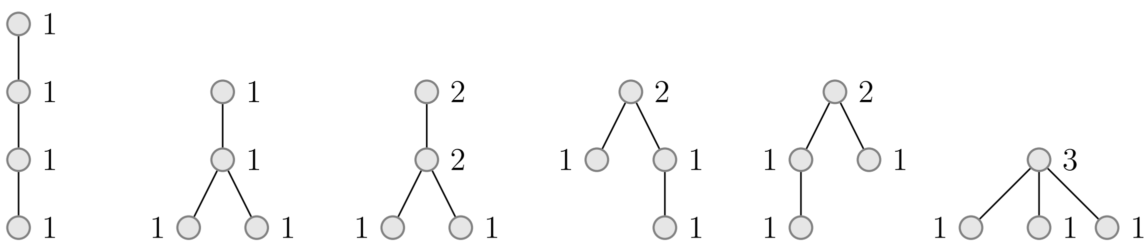


Figure 2: All $\beta(1,0)$-trees on three edges

We refer to [6] for more details on the bijection between $\beta(1,0)$-trees and the maps in question. Here we provide a short description illustrated by the example in Figure 3. The basic strategy is to define maps corresponding to leaves, and to find the map corresponding to an internal node once all the maps corresponding to its children are determined; at the last step, we find the map corresponding to the root, once the maps corresponding to the root's children are found. More precisely, begin with assigning to the leaves of a given tree the maps as in Figure 3, where R indicates the root node and the star is an auxiliary node mark that will disappear at the end of the procedure.

Table 1: Statistics preserved by the bijection

maps	trees
# edges	# vertices
# vertices	# leaves +1
# faces	# internal nodes +1
root-face degree	root-label +1

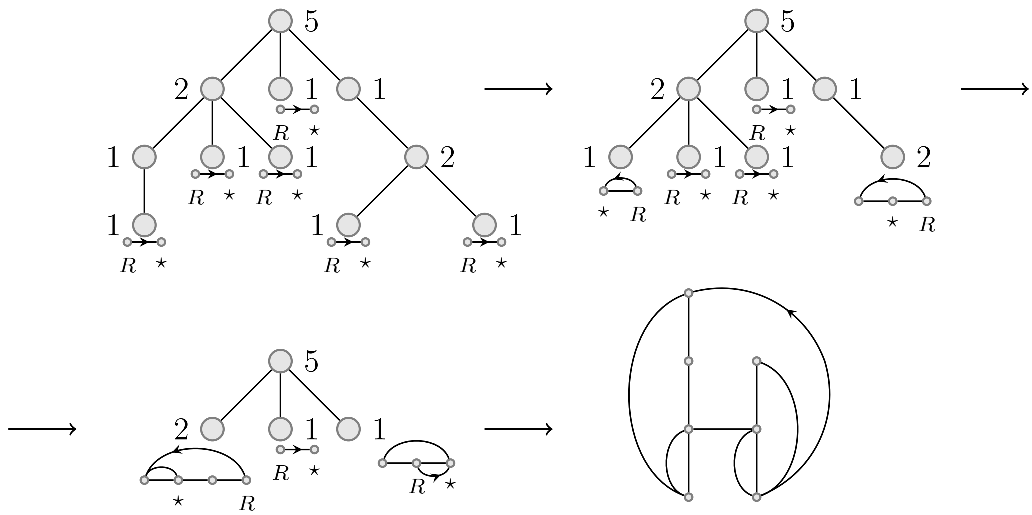


Figure 3: An example of mapping a $\beta(1,0)$-tree to a rooted non-separable planar map

Assuming all the children of an internal node labelled i are assigned maps numbered 1 to m from left-to-right (with two nodes on the outer face of each map labelled with R and a star), take the star node of map j and glue it with the root node of map $j+1$, for $j = 1, 2, \ldots, m-1$, and make the star node of map m the new root pointing at the R node of map 1 (this requires adding a new edge, the new root edge). Finally, in the obtained map, mark by star the i-th node counting from (but not including) the root node in the counterclockwise direction on the outer face. The only difference in performing the operation for the root of the given tree is that there is no need to put a star (which would otherwise be placed next to the root node in clockwise direction). It is not difficult to verify the correspondence of the statistics listed in Table 1.

The reader can check his/her understanding of the bijection by looking at Figures 1 and 2: the trees considered there correspond to the maps in the same order under the bijection.

2 Primitive maps and primitive $\beta(1,0)$-trees

In this section we define *primitive maps* as internal 2-face-free maps. Note that the root-face is allowed to be a 2-face. The second and the last maps in Figure 1 are primitive and the map constructed in Figure 3 is non-primitive.

Any map can be constructed from a primitive map just by adding the appropriate multiplicities of edges, thus creating 2-faces.

A $\beta(1,0)$-tree is said to be *primitive* if it corresponds to a primitive map under the bijection above.

Proposition 1. *A primitive $\beta(1,0)$-tree has no vertex which has a single child with maximum label.*

Proof. Consider a node v in a $\beta(1,0)$-tree which has a single child u with maximum label. Because u has maximum label, the map corresponding to the subtree rooted at u has its star node and root node separated by a single edge. Now, when we construct the map corresponding to the subtree rooted at v we simply add a new edge from the star node to the root node of the previous map. These two nodes were already connected so the new edge completes a 2-face. □

The word "primitive" is chosen in analogy with *primitive* (2+2)*-free posets* studied by Dukes et al. [7]. We can mimic the steps in that paper and use the generating function for all rooted non-separable planar maps $A(x)$ provided above to enumerate primitive maps. Namely, let $P(x) = \sum_{n\geq 0} p_n x^n$ be the generating function in question, where p_n is the number of primitive maps on n edges. Then the following result holds.

Theorem 2. *We have*

$$P(x) = A\left(\frac{x}{x+1}\right) = \frac{2(x+1)}{3x}\left(\text{hypergeom}\left(\left[-\frac{2}{3}, -\frac{1}{3}\right], \left[\frac{1}{2}\right], \frac{27x}{4(x+1)}\right) - 1\right).$$

Proof. A primitive map having at least one edge gives rise to an infinite number of rooted non-separable planar maps by choosing multiplicities of edges, which can be recorded in terms of generating functions as follows:

$$A(x) = \sum_{n\geq 0} p_n (x + x^2 + \cdots)^n = \sum_{n\geq 0} p_n \left(\frac{x}{1-x}\right)^n = P\left(\frac{x}{1-x}\right).$$

By substituting $x/(1-x)$ with x we obtain the desired result. □

Remark 3. *We note that through the bijection, multiple edges in a map do not necessarily correspond to a single connected (in graph theoretical sense) place in the corresponding $\beta(1,0)$-tree.*

3 Maps restricted in certain ways

In this section we explore k-face-free maps and multiple-edge-free maps.

3.1 k-face-free maps

Maps that are 2-face-free can be easily obtained from primitive maps by imposing the extra restriction on the corresponding $\beta(1,0)$-trees that they do not have root-label 1. Such trees are necessarily *irreducible*, that is the root has only one child. Therefore the generating function for 2-face-free maps is

$$P(x) - xP(x) = \frac{2(1-x^2)}{3x}\left(\text{hypergeom}\left(\left[-\frac{2}{3}, -\frac{1}{3}\right], \left[\frac{1}{2}\right], \frac{27x}{4(x+1)}\right) - 1\right).$$

The forbidden subtrees that define 2-face-free maps are shown in Figure 4.

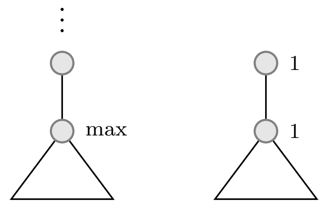


Figure 4: Forbidden subtrees defining $\beta(1,0)$-trees corresponding to 2-face-free maps

There is a natural extension, shown in Figure 5, of these forbidden subtrees that can be used to characterize $\beta(1,0)$-trees corresponding to 3-face-free maps. A justification of the extension is straightforwad and it is based on the bijection between $\beta(1,0)$-trees and planar maps (similarly to the proof of Proposition 1). Indeed, thinking of the recursive way to build the bijection, one needs to make sure that adding a new edge on planar maps does not create an internal 3-face, and also, at the very end, the outer face cannot be a triangle. The two left-most subtrees in Figure 5 guarantee that no internal 3-face emerges (while passing from the nodes containing the "max" symbol to their parents), while two right-most (sub)trees guarantee that the outer-face is not a triangle.

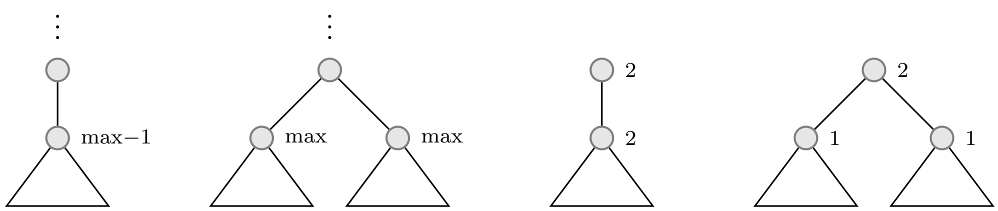


Figure 5: Forbidden subtrees defining $\beta(1,0)$-trees corresponding to 3-face-free maps

Similarly, one can list forbidden subtrees corresponding to the 4-face-free case, which is shown in Figure 6. Moreover, based on the cases $k = 2, 3, 4$, it is not difficult to see how to characterize the forbidden subtrees giving k-face-free maps: it contains all possible combinations for a node having m, $1 \leq m \leq k-1$, children with the sum of their labels $k-m-1$ less than the maximum possible sum, and all $\beta(1,0)$-trees trees with root labels equal k.

3.2 Multiple-edge-free maps

Note that 2-face-free maps can have multiple edges, as illustrated by the second map in Figure 1. However, to be a 2-face-free map is a necessary condition for a map to be multiple-edge-free, and thus the forbidden subtrees in Figure 6 are necessary conditions for a $\beta(1,0)$-tree to correspond to a multiple-edge-free map. These necessary conditions can be extended by one more forbidden structure when an internal node has a single

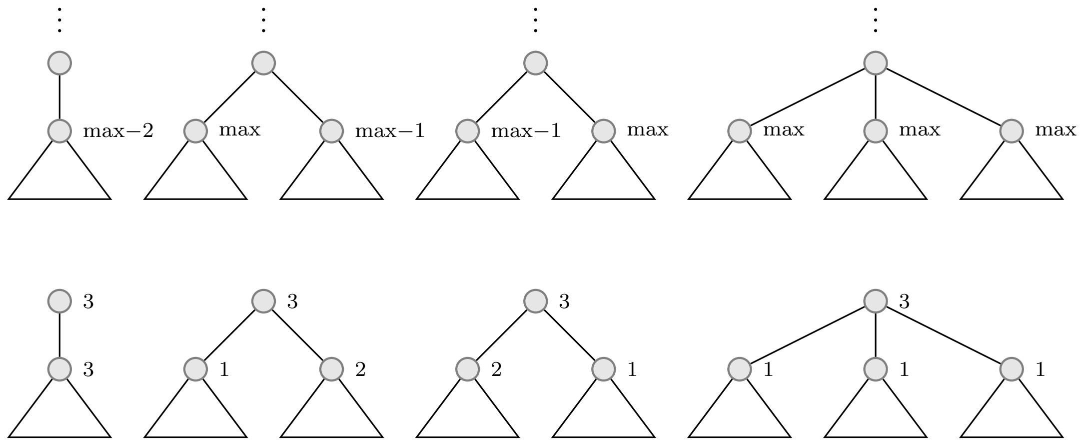


Figure 6: Forbidden subtrees defining $\beta(1,0)$-trees corresponding to 4-face-free maps

child whose label is 1. Indeed, in this case, when adding in the map the directed edge corresponding to such an internal node with the beginning at the starred node and the end at the previous root, we would be creating a multiple edge. Thus we have three forbidden structures giving necessary (but not sufficient) conditions on $\beta(1,0)$-trees to correspond to multiple-edge-free maps.

On the other hand, Lemma 4 below gives some sufficient conditions for a map to be multiple-edge-free.

Lemma 4. *If each internal node in a $\beta(1,0)$-tree has a sibling then the corresponding map is multiple-edge-free.*

Proof. Following the recursive way to build the map corresponding to a $\beta(1,0)$-tree where each node has a sibling, we see that any edge we add to build the map has its endpoints u and v with the property that before adding the edge, any path from u to v goes through a cut-vertex. Thus, the edge (u,v) cannot be a multiple edge. □

Based on the sufficient condition in Lemma 4, one can get a rough lower bound on the number of multiple-edge-free maps. Indeed, assume that all labels, possibly except for the root, are 1. Then we can erase all the labels and use the known generating function for rooted trees where each node has a sibling (see `https://oeis.org/A005043`):

$$\frac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)}.$$

(Actual lower bounds can be obtained by expanding the generating function above.)

We can improve the lower bound above by deriving an explicit generating function for the number of $\beta(1,0)$-trees where each internal node has a sibling and all labels, possibly except for the root, are at most 2 (such trees still correspond to multiple-edge-free maps). Let $B_2(x)$ be the generating function for these trees (x is responsible for the number of nodes; we define the coefficient to x^0 corresponding to the empty tree to be 0). Then $B_2 = B_2(x)$ satisfies the equation:

$$B_2 = x + B_2(B_2 - x) + (B_2 - x)(B_2 - x) + x(B_2 + (B_2 - x))^2.$$

This can be seen by considering what a tree of this kind can be (see Figure 7) and using the basic generating functions techniques. Indeed, either a tree is a single node (corresponding to x) or its root has degree 2 or more. When the degree of the root is more than 2 (see the middle two trees in Figure 7), the root's leftmost child, say u, has either label 1 or 2. In the first case, the generating function for the number of such trees is given by $B_2(B_2 - x)$ since for the subtree starting at u we can pick any valid $\beta(1,0)$-tree and rewrite its root to be 1 (this gives the factor of B_2), and independently, the remaining part of the tree must be a valid $\beta(1,0)$-tree different from a single node (otherwise u would have no sibling; this gives the factor of $B_2 - x$). Finally, if the root's degree is 2 (the rightmost tree in Figure 7) then its children, independently from each other, can either have label 1 or 2. In the case when the label is 1, the number of choices for the corresponding subtree is given by the generating function B_2, while in the case when the label is 2, we have that our choices are given by $B_2 - x$ (the single node tree must be excluded, since it will not give the root label 2, while any other legal tree has root label at least 2 which can be rewritten to be 2). Thus, for each of the two subtrees we have independently $B_2 + (B_2 - x)$ choices, and we add the factor of x to take into account the tree's root.

We can easily solve the last equation, which gives us

$$B_2(x) = \frac{1 + 3x + 4x^2 - \sqrt{1 - 2x - 7x^2}}{4 + 8x}.$$

The first few numbers that appear as coefficients (starting from the coefficient to x) are

$$1, 0, 1, 5, 11, 39, 113, 377, 1207, 4043, 13509, 45957, 157171, \ldots.$$

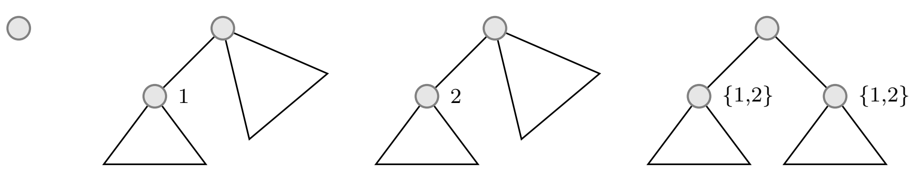


Figure 7: The structure of $\beta(1,0)$-trees with labels at most 2, excluding the root

Even though it becomes quickly cumbersome and difficult to deal with, one can try to handle some of larger cases getting a better approximation for the number of multiple-edge-free maps. For example, assume that all labels, possibly except for the root are at most 3 and let $B_3(x)$ be the generating function for these trees. Then looking at

Figure 8 and applying analysis similar to that applied to Figure 7, one can conclude that $B_3 = B_3(x)$ satisfies the functional equation of degree 4:

$$0 = (x + 2x^2 + 4x^3) - (1 + 5x + 12x^2)B_3 + (3 + x^2 + 9x + 4x^3)B_3^2 - (x + 6x^2)B_3^3 + x^3 B_3^4.$$

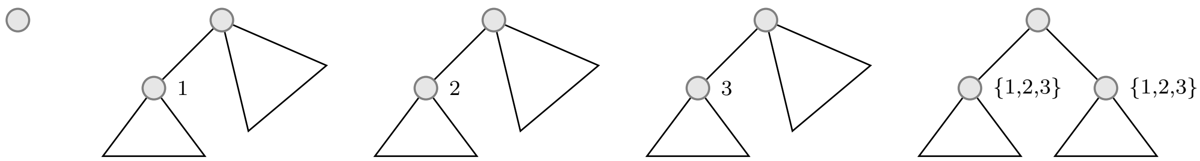


Figure 8: The structure of $\beta(1,0)$-trees with labels at most 3, excluding the root

4 Connections to the theory of permutation patterns

In this section we provide a link between our studies and the theory of pattern-avoiding permutations (see [8] for a comprehensive reference book on the area). We need the following definitions.

A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ *avoids the classical pattern* 3142 if there are no indices $i_1 < i_2 < i_3 < i_4$ such that $\pi_{i_2} < \pi_{i_4} < \pi_{i_1} < \pi_{i_3}$. If a permutation fails to avoid a pattern, we say that it *contains* the pattern. For example, 32541 avoids 3142, while 462531 contains one occurrence of the pattern 3142, namely, the subsequence 4253. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ *avoids the vincular pattern* $2\underline{41}3$ if there are no indices $i_1 < i_2 < i_4$ such that $\pi_{i_2+1} < \pi_{i_1} < \pi_{i_4} < \pi_{i_2}$. For example, the permutation 253164 avoids the pattern $2\underline{41}3$, while 365241 contains one occurrence of the pattern, namely, the subsequence 3524. We let $\mathrm{Av}(3142, 2\underline{41}3)$ denote the set of all permutations avoiding the patterns 3142 and $2\underline{41}3$ simultaneously. Claesson et al. [4] proved that permutations in $\mathrm{Av}(3142, 2\underline{41}3)$ of length n are in one-to-one correspondence with $\beta(1,0)$-trees on n edges (and thus with rooted non-separable planar maps on $n+1$ edges; this is the reason the set of permutations $\mathrm{Av}(3142, 2\underline{41}3)$ is called the set of *non-separable permutations*). Claesson et al. [4] actually provided two bijective proofs of this fact. We are interested in the following bijection (that turns out to be less powerful in terms of statistics preserved but sufficient for our purposes). We now provide a description of the bijection, illustrated by Figures 9–12. We skip justifications that all the steps work, in particular, that in dealing with permutations we do not create a prohibited pattern.

The idea here is to show that the objects in question can be generated iteratively in the same way which will induce a bijection sending irreducible objects to irreducible ones and reducible objects to reducible ones. As it was mentioned above, a $\beta(1,0)$-tree is irreducible if the root has only one child; a permutation $\pi_1\pi_2\cdots\pi_n$ is called *irreducible* if there is no i, $1 < i \leq n$, such that $\pi_j < \pi_k$ for all $1 \leq j < i \leq k \leq n$ (in other words, an irreducible permutation does not have a place so that everything to the left of it is smaller than everything to the right of it).

Generating $\beta(1,0)$-trees on n edges. All reducible trees on n-edges can be generated by taking several irreducible trees on a total of n edges and gluing all their roots into a single node whose label is defined to be the sum of its children. This is shown schematically on three irreducible trees (T_1, T_2 and T_3) in Figure 9.

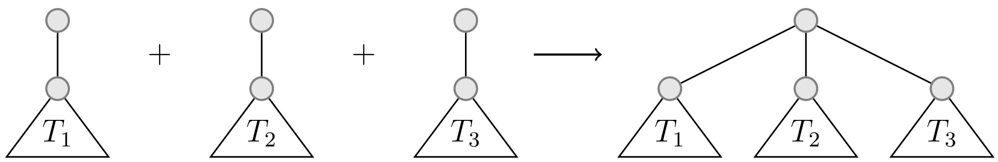


Figure 9: Constructing a reducible $\beta(1,0)$-tree from three irreducible ones

All irreducible trees on n edges can be obtained by taking an arbitrary tree on $n-1$ edges with root label k, and mapping it to k irreducible trees by adding one edge as shown in Figure 10 (that creates the new root having a single child) and picking k different valid options for the old root (that will determine the label of the new root).

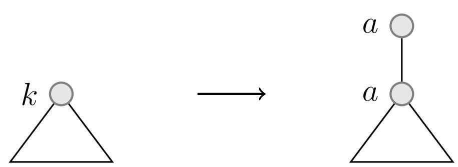


Figure 10: Constructing an irreducible $\beta(1,0)$-tree from an arbitrary tree. Here $1 \leq a \leq k$

Generating $(3142, 2\underline{41}3)$-avoiding permutations of length n. To mimic the steps in the generation of $\beta(1,0)$-trees we do the following. A reducible permutation of length n can be obtained by taking several irreducible $(3142, 2\underline{41}3)$-avoiding permutations of total length n, placing them next to each other and increasing all the letters of the second permutation by the length of the first one, then increasing all the letters of the third permutation by the sum of the lengths of the first and second permutations, etc. This process is shown schematically on three permutations (A_1, A_2 and A_3) in Figure 11.

To create an irreducible permutation of length n, take any $(3142, 2\underline{41}3)$-avoiding permutation of length $n-1$ with k left-to-right maxima (a letter is a *left-to-right maximum* if there is nothing larger than it to the left of the letter) and first insert the letter n right in front of a left-to-right maximum. In the obtained permutation, locate the possibly empty factors A, B and C as shown in Figure 12 so that BnC substitutes the rightmost irreducible permutation (all letters in A, a possibly reducible permutation, are smaller than any letter in BnC). Keep the same relative order inside A, B and C and rearrange the factors, as shown in Figure 12, to build the permutation $\tilde{B}\tilde{A}n\tilde{C}$ with any letter in $\tilde{A}$ larger than any letter in $\tilde{B}$ and $\tilde{C}$. The claim is that $\tilde{B}\tilde{A}n\tilde{C}$ is an irreducible $(3142, 2\underline{41}3)$-avoiding permutation which follows from a lemma proved in [4] that a $(3142, 2\underline{41}3)$-avoiding permutation is irreducible if and only if n (the largest letter) precedes 1 (the smallest letter).

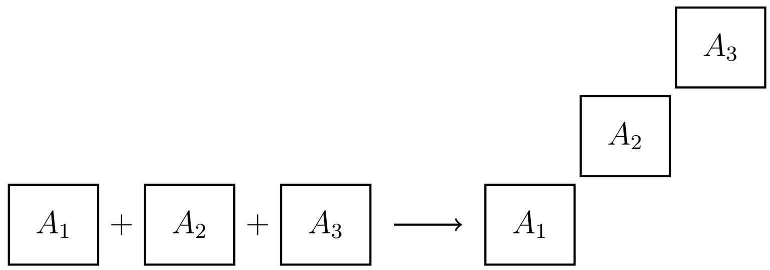


Figure 11: Constructing a reducible $(3142, 2\underline{41}3)$-avoiding permutation from irreducible $(3142, 2\underline{41}3)$-permutations

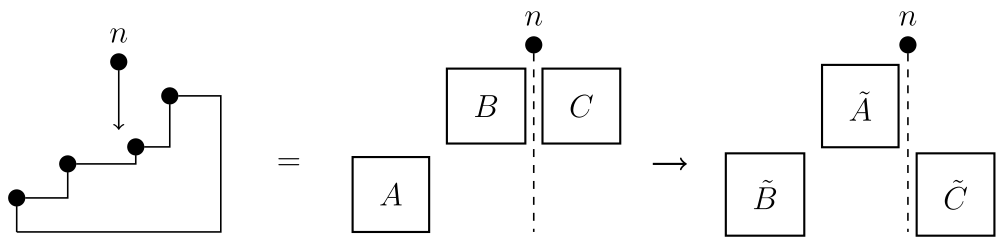


Figure 12: Constructing an irreducible $(3142, 2\underline{41}3)$-avoiding permutation from any $(3142, 2\underline{41}3)$-permutation

Example 5. *Consider the permutation* 12*. We have two possible locations in which to insert* $n = 3$*: in front of the* 1 *or in front of the* 2*. If we choose the latter then* $A = \{1\}$*,* $B = \emptyset$ *and* $C = \{2\}$*. Thus we get the permutation* 231 *out of the procedure. We can further enlarge this permutation by adding a* 4 *in front of the* 3*, giving us the permutation* 4231*. On trees the corresponding procedures are shown in Figure 13.*

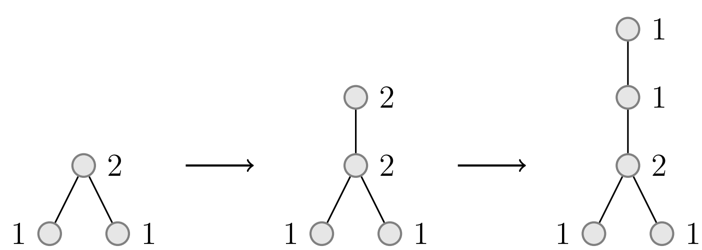
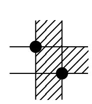
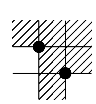


Figure 13: Growing the tree corresponding to the permutation 4231

To state our results, we need the notion of the *mesh patterns* below.

$M =$  , $M' =$ 

A permutation π contains the mesh pattern M if there is an index i such that $\pi_i > \pi_{i+1}$, with the additional requirements that everything to the right of π_{i+1} is either larger than π_i or smaller than π_{i+1}. A permutation π contains the mesh pattern M' if we additionally require that π_i is the largest element in the permutation.

Mesh patterns were introduced by Brändén and Claesson [1] to to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns.

The main result in this section is the following theorem enumerating the set of permutations avoiding one classical, one vincular and one mesh pattern.

Theorem 6. *Primitive maps, and primitive $\beta(1,0)$-trees, are in one-to-one correspondence with permutations in* $\mathrm{Av}(3142, 2\underline{41}3)$ *that avoid the mesh pattern M.*

This implies that the generating function for the set of permutations $\mathrm{Av}(3142, 2\underline{41}3, M)$ is

$$P(x) = \frac{2(x+1)}{3x}\left(\text{hypergeom}\left(\left[-\frac{2}{3}, -\frac{1}{3}\right], \left[\frac{1}{2}\right], \frac{27x}{4(x+1)}\right) - 1\right).$$

We need a lemma.

Lemma 7. *Let T be an irreducible tree. Then the label of the child of the root is the maximum possible if and only if T corresponds to a permutation π containing an occurrence of M'.*

Proof. Suppose the label of the child of the root is a maximum. This means that n was put in right before the last left-to-right-maximum in the permutation. But the last left-to-right maximum is $n-1$ and this will yield the claimed occurrence. Now consider a permutation π with an occurrence of M'. The right element of the pattern must be the largest point in $\tilde{C}$, which also has to be the rightmost element, and it corresponds to $n-1$ in C. This implies that n was put in right before the last left-to-right maximum which means that the child of the root was labeled with the maximum possible value. □

Proof of Theorem 6. Suppose the permutation π corresponds to a map that is not primitive. Then the corresponding tree T has at least one vertex that is a single child and has maximum label. Consider the subtree that contains this vertex and has the vertex parent as its root. This subtree is irreducible and by Lemma 7 corresponds to a permutation containing the pattern M'. As the entire tree T is built out of this subtree and others, the occurrence of the pattern M' is either maintained (in the case when the permutation becomes the rightmost component) or becomes an occurrence of M (since the descent will stay a descent no matter what you do).

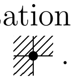
Now suppose that the permutation π contains the pattern M and consider a particular occurrence xy of the pattern M. It must lie in a single irreducible component of π. If x is the largest letter in its component then the occurrence of M is actually an occurrence of M' inside the component (all the letters of π, if any, that are larger than x can be ignored). If that is the case then this component corresponds to a tree that is not primitive by Lemma 7. This would then imply that the entire tree corresponding to π is not primitive, completing the proof. However, if x is not the largest letter in the component, we can reverse the construction of π (removing largest elements and, possibly, components) until x becomes the largest letter in its component; it then remains to note that reversing the construction keeps descents and does not introduce any elements to the right of a descent that lie between the descent letters. Thus, xy will correspond to an occurrence of M' in its component at some step of constructing π, which, by Lemma 7, corresponds to a non-primitive tree that will make the entire tree corresponding to π non-primitive. □

Lemma 8. *Let π be a permutation in* $\mathrm{Av}(3142, 2\underline{41}3)$ *of length $n-1$ with m occurrences of the mesh pattern M. Then inserting n before the last right-to-left maximum in π produces a permutation with $m+1$ occurrences of M. Inserting n before any other right-to-left maximum produces a permutation with m occurrences of M.*

Proof. Suppose π contains the pattern M. Note that the pattern cannot be split between A and B since the underlying classical pattern is 21, and it can neither be split between B and C since the left-most element in C is greater than everything in B. If the pattern lies entirely in A, B or C it is easy to see that it will be preserved. Consider, for example, the case when it is in B. Then since the relative order of elements in B and C is maintained the pattern will still exist in $\tilde{B}$. □

Theorem 9. *The number of occurrences of M in a permutation π in* $\mathrm{Av}(3142, 2\underline{41}3)$ *equals the number of 2-faces (excluding the root-face) in the corresponding map, and the number of vertices in the corresponding $\beta(1,0)$-tree that are a single child with maximum label.*

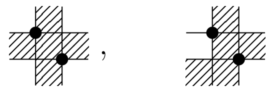
Proof. Since an occurrence of the pattern M lies entirely in an irreducible component of a permutation π we see that the number of occurrences of M equals the sum of the numbers of occurrences in each component. The theorem now follows from Lemma 8, since the number of occurrences of M can only be increased by inserting a new largest element in a component in front of the last left-to-right maximum in that component. This corresponds exactly to increasing the number of vertices that are a single child with maximum label. □

A permutation does not start with the largest element if and only if it avoids the mesh pattern $N =$ . This gives the following corollary to Theorem 9. Indeed, a permutation starts with the largest element in the case when we have $a = 1$ in Figure 10, and this is the prohibited structure to the right in Figure 4.

Corollary 10. *Permutations in* $\mathrm{Av}(3142, 2\underline{41}3, M, N)$ *correspond to 2-face-free planar maps.*

Since the permutations in $\mathrm{Av}(3142, 2\underline{41}3, M)$ are in one-to-one correspondence with primitive planar maps and primitive $\beta(1,0)$-trees, it is natural to call these permutations *primitive* among the permutations in $\mathrm{Av}(3142, 2\underline{41}3)$. Thus, one can raise a question of generating all permutations in $\mathrm{Av}(3142, 2\underline{41}3)$ from primitive ones, which can be answered as follows.

Given a primitive permutation π in $\mathrm{Av}(3142, 2\underline{41}3)$ and any letter x in π, a smaller letter y can be inserted adjacent to x, creating a descent xy, as long as this descent is an occurrence of M that is not involved in an occurrence of the pattern 3142. Equivalently we can require that xy is an occurrence of at least one of the mesh patterns below.



Using the procedure above we will create all permutations in $\mathrm{Av}(3142, 2\underline{41}3)$. Indeed, given a permutation in $\mathrm{Av}(3142, 2\underline{41}3)$, we can destroy all occurrences of the pattern M, one by one, by removing the smaller element in each occurrence (it is easy to see that this operation will not create an occurrence of 3142 or $2\underline{41}3$) and ending up with a primitive permutation. Consider for example the permutation 25314 in $\mathrm{Av}(3142, 2\underline{41}3)$. This permutation contains the mesh pattern M in the descent 31. If we remove the 1 we end up with a primitive permutation in $\mathrm{Av}(3142, 2\underline{41}3)$.

5 Open questions

We end with a few open questions.

1. West-2-stack-sortable permutations are equinumerous to the permutations in the set $\mathrm{Av}(3142, 2\underline{41}3)$ studied above. It would be interesting to try to understand what describes the primitive permutations (equinumerous to $\mathrm{Av}(3142, 2\underline{41}3, M)$) there.

2. In Remark 3, we noted that we cannot see with the bijection given exactly how multiple edges in a map are manifested in the corresponding $\beta(1,0)$-tree. Is this possible, perhaps with a different bijection?

3. Similarly, can we understand what structure multiple edges in planar maps correspond to in $(3142, 2\underline{41}3)$-avoiding permutations?

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